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Dedicated to Prof. Hong-Kun Xu on the occasion of his 60^{th} anniversary

An iterative method for solving multiple-set split feasibility problems in Banach spaces

S. AL-HOMIDAN[1], B. ALI [2] and Y. I. SULEIMAN[3]

ABSTRACT. In this paper, we study generalized multiple-set split feasibility problems (in short, GMSSFP) in the frame work of p-uniformly convex real Banach spaces which are also uniformly smooth. We construct an iterative algorithm which is free from an operator norm and prove its strong convergence to a solution of GMSSFP, that is, a solution of convex problem and a common fixed point of a countable family of Bregman asymptotically quasi-nonexpansive mappings without requirement for semi-compactness on the mappings. We illustrate our algorithm and convergence result by a numerical example.

1. INTRODUCTION

During the last two decades, split type problems have been extensively studied in the literature because of their applications in many real life problems, namely, image processing, data compression, magnetic resonance imaging, image reconstruction, intensity-modulated radiation therapy, neural networks, graph matching, etc., see, for example, [2, 3, 4, 5, 6, 7, 9, 10] and the references therein. One of spilt type problems is the following generalized multiple-set split feasibility problem (in short, GMSSFP):

$$\text{find } x^* \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \text{ such that } f(x^*) = 0, \tag{1.1}$$

where K is a nonempty closed and convex subset of a Banach space E, $\{T_i\}_{i\in\mathbb{N}} : K \to K$ is a countable family of Bregman asymptotically quasi-nonexpansive mappings [14] such that $\bigcap_{i=1}^{\infty} \text{Fix}(T_i) \neq \emptyset$, $\text{Fix}(T_i)$ denotes the set of common fixed points of T_i, and $f : E \to \mathbb{R}$ is a lower semicontinuous convex function. We denote by Ω the solution set of the problem (1.1). It is considered and studied by Giang et al. [16] in the setting of Hilbert spaces. A more general version of this problem is introduced and studied by Al-Homidan et al. [2].

If E_1, E_2 and E_3 are Banach spaces, $K_1 \subseteq E_1$, $K_2 \subseteq E_2$, $E = E_1 \times E_2$, and $f : E \to \mathbb{R}$ and $T_i = T : K_1 \times K_2 \to E$, $i \in \mathbb{N}$, are given, respectively, by

$$f(x,y) := \frac{1}{2}\|Ax - By\|^2 \text{ and } T(x,y) := \left(S_i^1 x, S_i^2 y\right), \quad \forall (x,y) \in K_1 \times K_2,$$

where $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators, $S_i^1 : K_1 \to K_1$ and $S_i^2 : K_2 \to K_2$ are Bregman asymptotically quasi-nonexpansive mappings for each

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Corresponding author: S. Al-Homidan; homidan@kfupm.edu.sa

$i = 1, 2, \dots, m$, then problem (1.1) coincides with the following multiple-set split equality feasibility problem (MSSEFP) defined as:

$$\begin{cases} \text{find } x^* \in \bigcap_{i=1}^m \text{Fix}(S_i^1) \text{ and } y^* \in \bigcap_{i=1}^m \text{Fix}(S_i^2) \\ \text{such that } Ax^* = By^*. \end{cases} \tag{1.2}$$

Furthermore, if $B = I$ the identity map and $E_2 = E_3$, then the problem (1.1) reduces to the following multiple-set split feasibility problem:

$$\begin{cases} \text{find } x^* \in \bigcap_{i=1}^m \text{Fix}(S_i^1) \\ \text{such that } Ax^* = y^* \in \bigcap_{i=1}^m \text{Fix}(S_i^2), \end{cases} \tag{1.3}$$

where $S_i^1 : E_1 \to E_1$ and $S_i^2 : E_2 \to E_2$ are asymptotically quasi-nonexpansive mappings such that $\bigcap_{i=1}^m \text{Fix}(S_i^j) \neq \emptyset$, $j = 1, 2$. It is considered and studied in [11, 20] in the setting of Hilbert spaces. A particular case of problem (1.3) is studied by Moudafi [18] for quasi-nonexpansive mappings defined on Hilbert spaces.

Qin et al. [20] constructed an iterative scheme which requires prior knowledge of operator norms and proved strong convergence theorem for approximating the solutions of problem (1.3) under the assumption of semi-compactness on the asymptotically nonexpansive mappings. In practical, most of the operators do not posses this property. Therefore, the following question can be raised.

Question. *Can we obtain an iterative algorithm which converges strongly to a solution of problem* (1.3) *for a class of asymptotically quasi-nonexpansive and / or more general than asymptotically quasi-nonexpansive mappings in Banach spaces and without the assumption of semi-compactness or without prior knowledge of operator norms?*

Over the last decade, several attempts were made to provide answers to this question, see, for example, [12, 23, 27]. Giang et al. [16] proved a strong convergence theorem for approximating the solutions of problem (1.3) without prior knowledge of operator norms, but for quasi-nonexpansive mappings in Hilbert spaces.

This fact has motivated us to consider generalized multiple-set split feasibility problem (GMSSFP) in Banach spaces, and to provide the affirmative answer to the above question. We also present a numerical example to demonstrate the validity of our results.

In the next section, we highlight some well-known definitions, notations and results which will be needed in the proof of main results of this paper. In Section 3, we propose an iterative algorithm and prove its strong convergence to a solution of the problem (1.1) in the setting of p-uniformly convex real Banach spaces which are also uniformly smooth, but without semi-compactness assumption of the mappings and without prior knowledge of operator norms.

2. Preliminaries

Let E be a smooth, strictly convex and reflexive Banach space with its topological dual E^*, and K be a non-empty, closed and convex subset of E. For the geometry of Banach spaces, we refer [15, 26]. Let $f_p : E \to \mathbb{R}$ be given by $f_p := \frac{1}{p}\|x\|^p$ where $1 < p < \infty$. The Bregman distance [8] $\Delta_p : E \times E \to [0, \infty)$ with respect to f_p is defined by

$$\Delta_p(x, y) := \frac{1}{q}\|x\|^p - \langle J_p x, y\rangle + \frac{1}{p}\|y\|^p, \tag{2.4}$$

where $q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, J_p is the generalized duality mapping from E into 2^{E^*} defined by

$$J_p(x) := \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\|_* = \|x\|^{p-1}\}, \quad \forall x \in E.$$

Alber [1] defined the V_p-mapping $V_p : E^* \times E \to [0, \infty)$ associated with f_p as

$$V_p(x^*, x) := \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p, \quad \forall x \in E \text{ and } x^* \in E^*. \tag{2.5}$$

Clearly,

$$V_p(x^*, x) = \Delta_p(J_p^{-1}(x^*), x), \quad \forall x \in E \text{ and } x^* \in E^*, \tag{2.6}$$

which implies that

$$V_p(x^*, x) + \langle y^*, J_p^{-I}(x^*) - x\rangle \leq V_p(x^* + y^*, x), \quad \forall x \in E \text{ and } x^*, y^* \in E^*. \tag{2.7}$$

Moreover, V_p is convex in the first variable. So, for all $z, x_i \in E$ and $\lambda_i \in (0, 1)$ with $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$\Delta_p\left(J_p^{-1}\left(\sum_{i=1}^{n} \lambda_i J_p x_i\right), z\right) = V_p\left(\sum_{i=1}^{n} \lambda_i J_p x_i, z\right) \leq \sum_{i=1}^{n} \lambda_i \Delta_p(x_i, z). \tag{2.8}$$

Bregman [8] defined the generalized projection $\Pi_K : E \to K$ as

$$\Pi_K x = x^* \quad \text{if and only if} \quad \Delta_p(x^*, x) := \inf_{y \in K} \Delta_p(y, x).$$

In Hilbert spaces, the Bregman projection Π_K coincides with the metric projection P_K onto K. Further, for a given $x \in E$,

$$\langle J_p x - J_p(\Pi_K x), y - \Pi_K x\rangle \leq 0, \quad \forall y \in K. \tag{2.9}$$

$$\text{and} \quad \Delta_p(\Pi_K x, y) + \Delta_p(x, \Pi_K x) \leq \Delta_p(x, y), \quad \forall y \in K. \tag{2.10}$$

We know that if a Banach space E is $q-$*uniformly smooth* then there exist a constant $C_q > 0$ and a real number $q > 1$ such that

$$\|x - y\|^q \leq \|x\|^q - q\langle J_q x, y\rangle + C_q\|y\|^q, \quad \forall\, x, y \in E; \tag{2.11}$$

Lemma 2.1. [19] *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in E. If $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$ and either $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ is bounded, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

A mapping $T : K \to E$ is said to be:

(a) Bregman nonexpansive [21] if $\Delta_p(Tx, Ty) \leq \Delta_p(x, y), \forall x, y \in K$;
(b) Bregman quasi-nonexpansive [21] if Fix$(T) \neq \emptyset$ and $\Delta_p(Tx, x^*) \leq \Delta_p(x, x^*), \forall x \in K, \forall x^* \in$ Fix(T).
(c) Bregman asymptotically quasi-nonexpansive [14] if there exists a sequence $\{k_n\}_{n=1}^{\infty} \subset [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$ such that for each $n \in \mathbb{N}$, we have

$$\Delta_p(T^n x, T^n x^*) \leq k_n \Delta_p(x, x^*), \quad \forall x \in K,\ x^* \in \text{Fix}(T).$$

Lemma 2.2. [22] *Let E be a reflexive Banach space and K be a nonempty, closed and convex subset of E. Let $T : K \to K$ be a closed Bregman asymptotically quasi-nonexpansive mapping with the sequence $\{k_n\}_{n=1}^{\infty} \subset [1, +\infty)$ such that $\lim_{n\to\infty} k_n = 1$. Then Fix$(T)$ is closed and convex.*

Lemma 2.3. [25] *Let $q > 1$ and $r > 0$ be fixed real numbers. A Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given* *sequence* $\{\alpha_n\}_{n=1}^{\infty}$ *of positive numbers with* $\sum_{n=1}^{\infty}\alpha_n = 1$ *and for any positive integers* i, j *with* $i < j$,

$$\left\|\sum_{n=1}^{\infty}\alpha_n x_n\right\|^q \leq \sum_{n=1}^{\infty}\alpha_n\|x_n\|^q - \alpha_i\alpha_j g(\|x_i - x_j\|).$$

Lemma 2.4. [17] *Let* $\{\Gamma_n\}_{n=1}^{\infty}$ *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{\Gamma_{n_j}\}_{j\geq 0}^{\infty}$ *of* $\{\Gamma_n\}_{n=1}^{\infty}$ *such that* $\Gamma_{n_j} < \Gamma_{n_{j+1}}$ *for all* $j \geq 0$. *Also consider the sequence of integers* $\{\zeta(n)\}_{n\geq 0}^{\infty}$ *defined by* $\zeta(n) := \max\{r \leq n : \Gamma_r < \Gamma_{r+1}\}$. *Then* $\{\zeta(n)\}_{n\geq n_0}^{\infty}$ *is a decreasing sequence verifying* $\lim_{n\to\infty}\zeta(n) = 0$, *and for all* $n \geq n_0$, *the following two estimates hold;*

$$\Gamma_{\zeta(n)} < \Gamma_{\zeta(n)+1} \quad \text{and} \quad \Gamma_n < \Gamma_{\zeta(n)+1}.$$

Lemma 2.5. [24] *Let* $\{\gamma_n\}_{n=1}^{\infty}$ *be a sequences in* $(0,1)$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *be in* $\mathbb{R}$ *satisfying* $\sum_{n=1}^{\infty}\gamma_n = \infty$ *and* $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\gamma_n\delta_n| < \infty$. *If* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of nonnegative real number such that* $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n$, $\forall n \geq 0$, *then* $\lim_{n\to\infty}a_n = 0$.

Lemma 2.6. [13] *Let* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{c_n\}_{n=1}^{\infty}$ *be sequences of non-negative real numbers such that* $\sum_{n=1}^{\infty}b_n < \infty$ *and* $\sum_{n=1}^{\infty}c_n < \infty$. *If* $a_{n+1} \leq (1+b_n)a_n + c_n$, $\forall n \geq 1$, *then* $\lim_{n\to\infty}a_n = 0$.

3. Main results

Throughout this section, unless otherwise specified, we assume that E is a p-uniformly convex real Banach space which is also uniformly smooth for $1 < p < \infty$ with its dual E^* and a constant C_q in (2.11). Assume that the problem (1.1) is consistent and $f : E \to \mathbb{R}$ is a non-negative, weakly lower semicontinuous convex function with search direction ζ_n and step length

$$\tau_n = \begin{cases} \rho_n \frac{f(x_n)^{p-1}}{\|\zeta_n\|^p}, & \text{if } \zeta_n \neq 0; \\ 0, & \text{otherwise,} \end{cases}$$

satisfying the following conditions with $\{\rho_n\}_{n=1}^{\infty} \subset \left(0, \left(\frac{q}{C_q}\right)^{\frac{1}{q-1}}\right)$:

(B1) $\langle \zeta_n, x_n - x^*\rangle \geq f(x_n), \forall x^* \in \Omega$;
(B2) $0 \leq a \leq \tau_n \leq \overline{a}$ for some $a, \overline{a} \in \mathbb{R}$ and $\forall n \in \Gamma = \{n \in \mathbb{N} : \zeta_n \neq 0\}$;
(B3) $\inf_{n\in\Gamma}\left(\rho_n\left(1 - \frac{\rho_n^{q-1}C_q}{q}\right)\right) > 0$.

Theorem 3.1. *Let* K *be a non-empty, closed and convex subset of* E. *Let* $\{T_i\}_{i\in\mathbb{N}} : K \to K$ *be a countable family of closed Bregman asymptotically quasi-nonexpansive mappings with the sequence* $\{k_{i,n}\} \subset [1,\infty)$ *such that* $\lim_{n\to\infty}k_{i,n} = 1$ *and* $(I - T_i)$ *is demiclosed at zero for each* $i \in \mathbb{N}$. *For arbitrary* $u \in K$ *and for the initial choice* $x_0 \in K$, *define iterative algorithm by*

$$(3.12)\quad \begin{cases} u_n = \prod_K J_p^{-1}(J_p x_n - \tau_n\zeta_n), \\ x_{n+1} = \prod_K J_p^{-1}\left(\alpha_n J_p u + (1-\alpha_n)\left(\beta_{n,0}J_p u_n + \sum_{i=1}^{\infty}\beta_{n,i}J_p T_i^n u_n\right)\right), \quad n \geq 0, \end{cases}$$

where α_n*,* $\beta_{n,0}$*,* $\beta_{n,i} \in [\epsilon, 1-\epsilon]$*,* $\epsilon \in (0,1)$ *satisfying* $\beta_{n,0} + \sum_{i=0}^{\infty} \beta_{n,i} = 1$*;* $\lim_{n\to\infty} \alpha_n = 0$*,* $\sum_{i=1}^{\infty} \alpha_n = +\infty$*,* $k_n = \sup\{k_{i,n} : i \geq 1\}$*;* $(1-\beta_{n,0})k_n \leq 1$*. Then the sequence* $\{x_n\}$ *converges strongly to* $x^* = \prod_{\Omega} u$*.*

Proof. Let $x^* \in \Omega$ and $z_n = J_p^{-1}\left(\beta_{n,0}J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n\right)$. Using (3.1), we get

$$
\begin{aligned}
\Delta_p(z_n, x^*) =& \Delta_p\left(J_p^{-1}(\beta_{n,0}J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n), x^*\right) \\
=& V_p\left(\beta_{n,0}J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n, x^*\right) \\
=& \frac{1}{q}\left\|\beta_{n,0}J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n\right\|^q \\
& - \left\langle \beta_{n,0}J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n, x^* \right\rangle + \frac{1}{p}\left\|x^*\right\|^p .
\end{aligned}
$$

This implies

$$
\begin{aligned}
\Delta_p(z_n, x^*) \leq & \frac{1}{q}\left(\beta_{n,0}\|u_n\|^p + \sum_{i=1}^{\infty} \beta_{n,i}\|T_i^n u_n\|^q\right) - \beta_{n,0}\langle J_p u_n, x^*\rangle \\
& - \sum_{i=1}^{\infty} \beta_{n,i}\langle J_p T_i^n u_n, x^*\rangle + \frac{1}{p}\|x^*\|^p \\
& = \frac{1}{q}\beta_{n,0}\|u_n\|^q - \beta_{n,0}\langle J_p u_n, x^*\rangle + \frac{1}{p}\beta_{n,0}\|x^*\|^p + \frac{1}{q}\sum_{i=1}^{\infty} \beta_{n,i}\|T_i^n u_n\|^q \\
& - \sum_{i=1}^{\infty} \beta_{n,i}\langle J_p T_i^n u_n, x^*\rangle + \frac{1}{p}\sum_{i=1}^{\infty} \beta_{n,i}\|x^*\|^p \\
& = \beta_{n,0}\Delta_p(u_n, x^*) + \sum_{i=1}^{\infty} \beta_{n,i}\Delta_p(T_i^n u_n, x^*) \\
& \leq \beta_{n,0}\Delta_p(u_n, x^*) + \sum_{i=1}^{\infty} \beta_{n,i}k_{n,i}\Delta_p(u_n, x^*) \\
& \leq \beta_{n,0}\Delta_p(u_n, x^*) + (1-\beta_{n,0})k_n\Delta_p(u_n, x^*) \\
& = \left(\beta_{n,0} + (1-\beta_{n,0})k_n\right)\Delta_p(u_n, x^*) \\
& \leq \left(\beta_{n,0} + 1\right)\Delta_p(u_n, x^*).
\end{aligned} \tag{3.13}
$$

We also compute

$$\begin{aligned}\Delta_p(u_n,x^*) =&\Delta_p(\Pi_K J_p^{-1}(J_px_n-\tau_n\zeta_n),x^*)\\ \le&\Delta_p(J_p^{-1}(J_px_n-\tau_n\zeta_n),x^*)\\ =&V_p(J_px_n-\tau_n\zeta_n,x^*)\\ \le&\frac{1}{q}\,\|J_px_n-\tau_n\zeta_n\|^q-\langle J_px_n-\tau_n\zeta_n,x^*\rangle+\frac{1}{p}\|x^*\|^p\\ \le&\frac{1}{q}\|J_px_n\|^q-\tau_n\langle\zeta_n,x_n\rangle+\frac{C_q\tau_n^q}{q}\|\zeta_n\|^q-\langle J_px_n,x^*\rangle+\tau_n\langle\zeta_n,x^*\rangle+\frac{1}{p}\|x^*\|^p\\ \le&\frac{1}{q}\|x_n\|^q-\langle J_px_n,x^*\rangle+\frac{1}{p}\|x^*\|^p-\tau_n\langle\zeta_n,x_n-x^*\rangle+\frac{C_q\;\tau_n^q}{q}\|\zeta_n\|^q\\ \le&\Delta_p(x_n,x^*)-\tau_nf(x_n)+\frac{C_q\;\tau_n^q}{q}\|\zeta_n\|^q.\end{aligned}$$

This implies

$$\begin{aligned}\Delta_p(u_n,x^*)\le&\Delta_p(x_n,x^*)-\frac{\rho_nf(x_n)^p}{\|\zeta_n\|^p}+\frac{\rho_n^qf(x_n)^{q(p-1)}C_q\;\tau_n^q}{q\|\zeta_n\|^{pq}}\|\zeta_n\|^q\\ \le&\Delta_p(x_n,x^*)-\rho_n\left(1-\frac{C_q\rho_n^{q-1}}{q}\right)\frac{f(x_n)^p}{\|\zeta_n\|^p}\\ =&\Delta_p(x_n,x^*)-\rho_n\left(1-\frac{C_q\rho_n^{q-1}}{q}\right)\frac{f(x_n)^p}{\|\zeta_n\|^p}\le\Delta_p(x_n,x^*).\end{aligned}\tag{3.14}$$

It follows from (3.13) and (3.14) that

(3.15)

$$\begin{aligned}\Delta_p(x_{n+1},x^*)=&\Delta_p\left(\Pi_KJ_p^{-1}(\alpha_nJ_pu+(1-\alpha_n)J_pz_n),x^*\right)\\ \le&\Delta_p\left(J_p^{-1}(\alpha_nJ_pu+(1-\alpha_n)J_pz_n),x^*\right)\\ =&\Delta_p(\alpha_nu+(1-\alpha_n)z_n,x^*)\\ =&\frac{1}{q}\|\alpha_nu+(1-\alpha_n)z_n\|^p-\alpha_n\langle J_pu,x^*\rangle-(1-\alpha_n)\langle J_pz_n,x^*\rangle+\frac{1}{p}\|x^*\|^p\\ \le&\frac{1}{q}\alpha_n\|u\|^q+(1-\alpha_n)\frac{1}{q}\|z_n\|^q-\alpha_n\langle J_pu,x^*\rangle-(1-\alpha_n)\langle J_pz_n,x^*\rangle+\frac{1}{p}\|x^*\|^p\\ =&\alpha_n\frac{1}{q}\|u\|^q-\alpha_n\langle J_pu,x^*\rangle+\alpha_n\frac{1}{p}\|x^*\|^p+(1-\alpha_n)\frac{1}{q}\|z_n\|^q\\ &-(1-\alpha_n)\langle J_pz_n,x^*\rangle+(1-\alpha_n)\frac{1}{p}\|x^*\|^p\\ =&\alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\Delta_p(z_n,x^*)\\ \le&\alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\left(\beta_{n,0}+1\right)\Delta_p(u_n,x^*)\\ \le&\alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\left(\beta_{n,0}+1\right)\Delta_p(x_n,x^*)\\ &-\rho_n\left(1-\frac{C_q\rho_n^{q-1}}{q}\right)\frac{f(x_n)^p}{\|\zeta_n\|^p}(1-\alpha_n)\left(\beta_{n,0}+1\right).\end{aligned}$$

Since $\liminf_{n\in\Gamma}\left(\rho_n(1-\frac{C_q\rho_n^{q-1}}{q})\right)>0$, from assumption (B3), the inequality (3.15) becomes

$$\begin{aligned}\Delta_p(x_{n+1},x^*)\le&\alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\left(\beta_{n,0}+1\right)\Delta_p(x_n,x^*)\\ \le&\left(1+\beta_{n,0}\right)\Delta_p(x_n,x^*)+\alpha_nM,\end{aligned}$$

where $M = \sup\{\Delta_p(u, x^*), n \in \mathbb{N}\}$. By Lemma 2.6, we conclude that the sequence $\{x_n\}$ and so $\{u_n\}$ are bounded in K.

Now, using the boundedness of $\{x_n\}$ and Lemma 2.2, we have that Ω is closed and convex thanks to the convexity of f. So, we pick arbitrary element $x^* = \prod_\Omega u$ and set $z_n = J_p^{-1}\left(\beta_{n,0} J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n\right)$. Again by using (3.1), we compute

$$\begin{aligned}
\Delta_p(z_n, x^*) =& \Delta_p\left(J_p^{-1}(\beta_{n,0} J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n), x^*\right)\\
=& V_p\left(\beta_{n,0} J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n, x^*\right)\\
=& \frac{1}{q}\left\|\beta_{n,0} J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n\right\|^q - \left\langle \beta_{n,0} J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n, x^*\right\rangle + \frac{1}{p}\|x^*\|^p.
\end{aligned}$$

Since $\{u_n\}$ is bounded, by Lemma 2.3, we get

(3.16)

$$\begin{aligned}
\Delta_p(z_n, x^*) \leq& \beta_{n,0}\frac{1}{q}\|u_n\|^q + \frac{1}{q}\sum_{i=1}^{\infty} \beta_{n,i}\|T_i^n u_n\|^p - \frac{1}{q}\beta_{n,0}\beta_{n,i} g(\|J_p x_n - J_p T_i^n u_n\|)\\
& - \beta_{n,0}\langle J_p u_n, x^*\rangle - \sum_{i=1}^{\infty} \beta_{n,i}\langle J_p T_i^n u_n, x^*\rangle + \frac{1}{p}\|x^*\|^p\\
=& \beta_{n,0}\frac{1}{q}\|u_n\|^q - \beta_{n,0}\langle J_p u_n, x^*\rangle + \beta_{n,0}\frac{1}{p}\|x^*\|^p + \frac{1}{q}\sum_{i=1}^{\infty} \beta_{n,i}\|T_i^n u_n\|^q\\
& - \sum_{i=1}^{\infty} \beta_{n,i}\langle J_p T_i u_n, x^*\rangle + \frac{1}{p}\sum_{i=1}^{\infty} \beta_{n,i}\|x^*\|^p - \frac{1}{q}\beta_{n,0}\beta_{n,i} g(\|J_p u_n - J_p T_i^n u_n\|)\\
=& \beta_{n,0}\Delta_p(u_n, x^*) + \sum_{i=1}^{\infty} \beta_{n,i}\Delta_p(T_i^n u_n, x^*) - \frac{1}{q}\beta_{n,0}\beta_{n,i} g(\|J_p u_n - J_p T_i^n u_n\|)\\
\leq& \beta_{n,0}\Delta_p(u_n, x^*) + \sum_{i=1}^{\infty} \beta_{n,i}\, k_{n,i}\Delta_p(u_n, x^*) - \frac{1}{q}\beta_{n,0}\beta_{n,i} g(\|J_p u_n - J_p T_i^n u_n\|)\\
\leq& \beta_{n,0}\Delta_p(u_n, x^*) + (1 - \beta_{n,0})(\beta_{n,0} + 1)\Delta_p(u_n, x^*) - \frac{1}{q}\beta_{n,0}\beta_{n,i} g(\|J_p u_n - J_p T_i^n u_n\|)\\
\leq& k_n\Delta_p(u_n, x^*) - \frac{1}{q}\beta_{n,0}\beta_{n,i} g(\|J_p u_n - J_p T_i^n u_n\|).
\end{aligned}$$

Substituting relations (3.14) and (3.16) in (3.15), we obtain

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) =& \Delta_p(\Pi_K J_p^{-1}(\alpha_n J_p u + (1 - \alpha_n) J_p z_n), x^*)\\
\leq& \Delta_p(J^{-1}(\alpha_n J_p u + (1 - \alpha_n) J_p z_n), x^*)\\
=& V_p(\alpha_n J_p u + (1 - \alpha_n) J_p z_n, x^*)\\
=& V_p(\alpha_n J_p u + (1 - \alpha_n) J_p z_n - \alpha_n J_p u + \alpha_n J_p u - \alpha_n J_p x^* + \alpha_n J_p x^*, x^*)\\
=& V_p\big([\alpha_n J_p u + (1 - \alpha_n) J_p z_n - \alpha_n(J_p u - J_p x^*)] + [\alpha_n(J_p u - J_p x^*)], x^*\big)\\
\leq& V_p(\alpha_n J_p u + (1 - \alpha_n) J_p z_n - \alpha_n(J_p u - J_p x^*), x^*) + \alpha_n\langle J_p u - J_p x^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

This implies

$$\begin{aligned}\Delta_p(x_{n+1},x^*) \leq & V_p(\alpha_n J_p x^* + (1-\alpha_n)J_p z_n, x^*) + \alpha_n\langle J_p u - J_p x^*,\ x_{n+1}-x^*\rangle\\ =&\Delta_p(J_p^{-1}(\alpha_n J_p x^* + (1-\alpha_n)J_p z_n), x^*) + \alpha_n\langle J_p u - J_p x^*, x_{n+1}-x^*\rangle\\ \leq&\Delta_p(x^*,x^*) + (1-\alpha_n)\Delta_p(z_n,x^*) + \alpha_n\langle J_p u - J_p x^*, x_{n+1}-x^*\rangle\\ \leq&(1-\alpha_n)k_n\Delta_p(u_n,x^*) + \alpha_n\langle J_p u - J_p x^*, x_{n+1}-x^*\rangle\\ &-\frac{1}{q}(1-\alpha_n)\beta_{n,0}\beta_{n,i}g(\|J_p u_n - J_p T_i^n u_n\|)\\ \leq&(1-\alpha_n)\left(\beta_{n,0}+1\right)\Delta_p(x_n,x^*) + \alpha_n\langle J_p u - J_p x^*, x_{n+1}-x^*\rangle\\ &-(1-\alpha_n)\rho_n\left(1-\frac{C_q\rho_n^{q-1}}{q}\right)\frac{f(x_n)^p}{\|\zeta_n\|^p}\\ &-\frac{1}{q}(1-\alpha_n)\beta_{n,0}\beta_{n,i}g(\|J_p u_n - J_p T_i^n u_n\|).\end{aligned} \tag{3.17}$$

To show that $\{\Delta_p(x_n,x^*)\}$ converges strongly to zero. We consider two possibilities.

CASE 1. Suppose that the sequence $\{\Delta_p(x_n,x^*)\}_{n=1}^{\infty}$ is monotonically decreasing. This implies that $\lim_{n\to\infty}\Delta_p(x_n,x^*)$ exists and $\lim_{n\to\infty}\Delta_p(x_{n+1},x^*) = \lim_{n\to\infty}\Delta_p(x_n,x^*) = 0$. Since $\{x_n\}$ is bounded, $\lim_{n\to\infty}\alpha_n = 0$ and $\beta_{n,0}$, $\beta_{n,i}\in[\epsilon, 1-\epsilon]$, the inequality (3.17) becomes

$$\begin{aligned}\lim_{n\to\infty} g(\|J_p u_n - J_p T_i^n u_n\|) \leq& \lim_{n\to\infty}\left(\left(\beta_{n,0}+1\right)\Delta_p(x_n,x^*) - \Delta_p(x_{n+1},x^*)\right)\\ &-\lim_{n\to\infty}\rho_n\left(1-\frac{C_q\rho_n^{q-1}}{q}\right)\frac{f(x_n)^{p-1}}{\|\zeta_n\|^p}f(x_n).\end{aligned} \tag{3.18}$$

Based on condition (B3), we deduce from (3.18) that

$$\lim_{n\to\infty} g(\|J_p u_n - J_p T_i^n u_n\|) = 0 \text{ and} \tag{3.19}$$

$$\lim_{n\to\infty}\frac{f(x_n)^{p-1}}{\|\zeta_n\|^p}\ f(x_n) = 0. \tag{3.20}$$

Continuity of g implies

$$\lim_{n\to\infty}\|J_p u_n - J_p T_i^n u_n\| = 0. \tag{3.21}$$

Uniform norm to norm continuity of J_p^{-1} gives

$$\lim_{n\to\infty}\|u_n - T_i^n u_n\| = 0. \tag{3.22}$$

From condition (B2) and equation (3.20), we obtain

$$\lim_{n\to\infty} f(x_n) = 0. \tag{3.23}$$

Hence,

$$\lim_{n\to\infty}\|\zeta_n\| = 0. \tag{3.24}$$

We compute

$$\begin{aligned}
&\Delta_p(x_{n+1}, z_n)\\
&= \Delta_p(\Pi_K J_p^{-1}(\alpha_n J_p u + (1-\alpha_n)J_p z_n), z_n)\\
&\leq \Delta_p(J_p^{-1}(\alpha_n J_p u + (1-\alpha_n)J_p z_n), z_n)\\
&= V_p(\alpha_n J_p u + (1-\alpha_n)J_p z_n, z_n)\\
&= \frac{1}{q}\,\|\alpha_n J_p u + (1-\alpha_n)J_p z_n\|^q - \langle \alpha_n J_p u + (1-\alpha_n)J_p z_n, z_n\rangle + \frac{1}{p}\|z_n\|^p\\
&\leq \frac{1}{q}\alpha_n\|u\|^q - \alpha_n\langle J_p u, z_n\rangle + \frac{1}{p}\alpha_n\|z_n\|^p + \frac{1}{q}(1-\alpha_n)\|z_n\|^q\\
&- (1-\alpha_n)\langle J_p z_n, z_n\rangle + \frac{1}{p}(1-\alpha_n)\|z_n\|^p\\
&= \alpha_n\Delta_p(u, z_n) + (1-\alpha_n)\Delta_p(z_n, z_n).
\end{aligned}$$

We immediately get $\lim_{n\to\infty} \Delta_p(x_{n+1}, z_n) = 0$. By Lemma 2.1, we have

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{3.25}$$

Note that

$$\begin{aligned}
\|J_p z_n - J_p u_n\| &= \left\|\beta_{n,0} J_p u_n + \sum_{i=1}^{\infty} \beta_{n,i} J_p T_i^n u_n - J_p u_n\right\|\\
&= \left\|\sum_{i=1}^{\infty} \beta_{n,i}(J_p T_i^n u_n - J_p u_n)\right\|\\
&\leq \sum_{i=1}^{\infty} \beta_{n,i}\|J_p T_i^n u_n - J_p u_n\|.
\end{aligned}$$

Using (3.21), we get $\lim_{n\to\infty} \|J_p z_n - J_p u_n\| = 0$ which implies

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{3.26}$$

Further, $\|J_p c_n - J_p x_n\| = \|J_p x_n - \tau_n\zeta_n - J_p x_n\| = \tau_n\|\zeta_n\|$, where $c_n = J_p^{-1}(J_p x_n - \tau_n\zeta_n)$. It follows from (3.24) that

$$\lim_{n\to\infty} \|J_p c_n - J_p x_n\| = 0.$$

This implies that

$$\lim_{n\to\infty} \|c_n - x_n\| = 0. \tag{3.27}$$

By Lemma 2.1 and relation (2.10), we get

$$\Delta_p(u_n, x_n) \leq \Delta_p(c_n, x_n) - \Delta_p(c_n, u_n) \leq \Delta_p(c_n, x_n).$$

By Lemma 2.1 again, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.28}$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.29}$$

Since $\{x_n\}$ is bounded, E is reflexive, we can find a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to \overline{u}$ and

$$\limsup_{n\to\infty}\langle J_p u - J_p x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle J_p u - J_p x^*, x_{n_i} - x^*\rangle. \tag{3.30}$$

Therefore,

$$u_{n_i} \rightharpoonup \overline{u} \text{ as } i \to \infty. \tag{3.31}$$

It follows from the demiclosedness of $(I - T_i)$ at zero for each $i \in \mathbb{N}$, using (3.31) and (3.22), that $\overline{u} \in \bigcap_{i=1}^{\infty} \text{Fix}\,(T_i)$. In addition, using (3.23) and the fact that f is nonnegative weakly lower semicontinuous, we obtain

$$0 \leq f(\overline{u}) \leq \liminf_{i\to\infty} f(x_{n_i}) = 0. \text{ Hence } \overline{u} \in \Omega.$$

We next demonstrate that $\lim_{n\to\infty} \Delta_p(x_n, x^*) = 0$, where $x^* = \Pi_\Omega u$.

Applying (3.30) and (3.29), we get

$$\limsup_{n\to\infty} \langle J_p u - J_p x^*, x_{n+1} - x^* \rangle \leq \limsup_{n\to\infty} \langle J_p u - J_p x^*, x_n - x^* \rangle = \langle J_p u - J_p x^*, \overline{u} - x^* \rangle \leq 0. \tag{3.32}$$

Note that

$$\Delta_p(x_{n+1}, x^*) \leq (1-\alpha_n)(\beta_{n,0}+1)\Delta_p(x_n, x^*) + \alpha_n \langle J_p u - J_p x^*, x_{n+1} - x^* \rangle. \tag{3.33}$$

We write inequalities (3.33) as

$$\begin{aligned}
\Gamma_{n+1} &\leq (1-\alpha_n\beta_{n,0})\Gamma_n + N(\beta_{n,0} - \alpha_n) + \alpha_n\gamma_n \\
&\leq (1-\alpha_n\beta_{n,0})\Gamma_n + \alpha_n\beta_{n,0}\left(\frac{N(\beta_{n,0}-\alpha_n)}{\alpha_n\beta_{n,0}} + \frac{\gamma_n}{\beta_{n,0}}\right) \\
&\leq (1-\omega_n)\Gamma_n + \omega_n\delta_n,
\end{aligned} \tag{3.34}$$

where $\omega_n = \alpha_n\beta_{n,0}$, $N = \sup\{\Delta_p(x_n, x^*) : n \geq 0\}$, $\gamma_n = \langle J_p u - J_p x^*, x_{n+1} - x^* \rangle$ and $\delta_n = \left(\frac{N(\beta_{n,0}-\alpha_n)}{\alpha_n\beta_{n,0}} + \frac{\gamma_n}{\beta_{n,0}}\right)$ satisfying $\omega_n \in (0,1)$, $\lim_{n\to\infty} \omega_n = 0$, $\sum_{n=1}^{\infty} \omega_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. By Lemma 2.5, we conclude that

$$\lim_{n\to\infty} \Delta_p(x_n, x^*) = 0.$$

CASE 2. Suppose that the sequence $\{\Delta_p(x_n, x^*)\}_{n=1}^{\infty}$ is not monotonically decreasing. Set $\Gamma_n = \Delta_p(x_n, x^*)$, $\forall n \geq 1$ and let $\Gamma : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ for some n_0 large enough by

$$\nu(n) := \max\{r \in \mathbb{N} : r \leq n, \Gamma_r < \Gamma_{r+1}\}.$$

Clearly, Γ is non-decreasing sequence such that $\Gamma(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\nu n} < \Gamma_{\nu n+1}, \quad \forall n \geq n_0. \tag{3.35}$$

Since $\{x_{\nu n}\}$ is bounded, repeating similar steps in Case 1, we deduce that

$$\Gamma_{\nu n+1} \leq (1-\omega_{\nu n})\Gamma_{\nu n} + \omega_{\nu n}\delta_{\nu n}, \tag{3.36}$$

where $\lim_{n\to\infty} \omega_{\nu n} = 0$ and $\limsup_{n\to\infty} \delta_{\nu n} \leq 0$. Substituting (3.35) in (3.36) gives

$$\Gamma_{\nu n} \leq (1-\omega_{\nu n})\Gamma_{\nu n} + \omega_{\nu n}\delta_{\nu n}.$$

This implies $\Gamma_{\nu n} \leq \delta_{\nu n}$, $\limsup_{n\to\infty} \Gamma_{\nu n} \leq 0$, $\lim_{n\to\infty} \Gamma_{\nu n} = 0$, $\limsup_{n\to\infty} \Gamma_{\nu(n+1)} \leq \limsup_{n\to\infty} \Gamma_{\nu n}$ and $\lim_{n\to\infty} \Gamma_{\nu(n+1)} = 0$. Therefore

$$0 \leq \Gamma_{\nu n} \leq \max\{\Gamma_{\nu n}, \Gamma_{\nu(n+1)}\} \leq \Gamma_{\nu(n+1)}.$$

Applying Lemma 2.4, we conclude that

$$\lim_{n\to\infty} \Delta_p(x_n, x^*) = 0.$$

Hence, in both cases, the sequence $\{x_n\}$ converge strongly to $x^* = \Pi_\Omega u$ where $x^* \in \Omega$. □

If $E = H$ a Hilbert space in Theorem 3.1, then $\Delta_p(x, y) = \|x - y\|^2$, and so, $\{T_i\}_{i=1}^n$ is a finite sequence of asymptotically quasi-nonexpansive mappings.

Corollary 3.1. *Let K be a non-empty, closed, convex subset of a real Hilbert space H. Let $\{T_i\}_i^n : K \to K$ be a finite sequence of asymptotically quasi-nonexpansive mappings with the sequence $\{k_{i,n}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_{i,n} = 1$ and $(I - T_i)$ is demiclosed at zero for each $i \in \mathbb{N}$. For arbitrary $u \in K$ and for the initial choice $x_0 \in K$, define iterative algorithm by*

$$\begin{cases} u_n = P_K(x_n - \tau_n \zeta_n) \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)\left(\beta_{n,0} u_n + \sum_{i=1}^{\infty} \beta_{n,i} T_i^n u_n\right), \quad n \geq 0, \end{cases} \tag{3.37}$$

where α_n, $\beta_{n,0}$, $\beta_{n,i} \in [\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ satisfying $\beta_{n,0} + \sum_{i=0}^{\infty} \beta_{n,i} = 1$; $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{i=1}^{\infty} \alpha_n = +\infty$, $k_n = \sup\{k_{i,n} : i \geq 1\}$; $(1 - \beta_{n,0})k_n \leq 1$. Then the sequences $\{x_n\}$ converge strongly to $x^* = P_\Omega u$.

Finally, we illustrate our algorithm and convergence result by the following example.

Example 3.1. Let K be the unit ball in $E = \ell^2$. For each $i \in \mathbb{N}$, let $T_i : K \to K$ be defined by

$$T_i(x) = \left(0, \frac{1}{2^i} x_1^2, a_2 x_2, a_3 x_3, \ldots\right), \quad \forall x = (x_1, x_2, x_3, \ldots) \in K,$$

where $\{a_j\}$ is a sequence in $(0, 1)$ such that $\prod_{j=2}^{\infty} a_j = \frac{1}{2}$. It is clear that

(i) $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) = \{x = (0, 0, 0, \ldots)\}$,
(ii) $\|T_i x - T_i y\| \leq 2\|x - y\|, \forall x, y \in K$,
(iii) $T_i^n = \left(0, 0, \frac{1}{2^i} a_2^{n-1} x_1^2, a_3^{n-1} a_2 x_2, a_4^{n-1} a_3 x_3, \ldots\right)$ for $n \geq 2$, and
(iv) $\|T_i^n x - T_i^n y\| \leq 2 \prod_{j=2}^{n} a_j \|x - y\|, \forall x, y \in K$, $n \geq 2$.

Let $k_1^{\frac{1}{2}} = 2$ and $k_n^{\frac{1}{2}} = 2 \prod_{j=2}^{n} a_j$ for $n \geq 2$. Then $\lim_{n\to\infty} k_n = \lim_{n\to\infty} \left(2 \prod_{j=2}^{n} a_j\right)^2 = 1$. In view of (i), (ii) and (iv), we have

$$\Delta_p\left(T_i^n x, T_i^n x^*\right) \leq k_n \Delta_p(x, x^*), \quad \forall x \in K, \; x^* \in \mathrm{Fix}(T_i).$$

Therefore each T_i is Bregman asymptotically quasi-nonexpansive which is not quasi-nonexpansive and $(I - T_i)$ is demiclosed at 0 for each $i \in \mathbb{N}$. Consider a functional $f : E \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{2}\|x\|^2, \quad \forall x \in E.$$

Clearly, f is a non-negative lower semi-continuous convex function. In fact, $\nabla f(x) = x$ and

$$P_K(x) = \begin{cases} x, & x \in K; \\ \frac{x}{\|x\|}, & \text{otherwise.} \end{cases}$$

Take $a_j = \frac{1}{j}$, $\zeta_n = \nabla f(x_n)$, so that $\tau_n = \frac{\rho_n}{2}$ and we obtain

$$\begin{cases} u_n = P_K\left(x_n - \frac{\rho_n}{2} x_n\right), \\ x_{n+1} = \frac{1}{2n+1} u + \frac{2n}{2n+1}\left[\frac{n}{2n+1} u_n + \sum_{i=1}^{\infty} \frac{n+1}{2^i(2n+1)} \left(0, 0, \frac{1}{2^i} \frac{x_1^2}{2^{n-1}}, \frac{x_2}{2 \times 3^{n-1}}, \frac{x_3}{3 \times 4^{n-1}}, \ldots\right)\right], \; n \geq 2. \end{cases}$$

Finally, all the hypothesis of Theorem 3.1 are satisfied with $x^* = (0,0,0,...) \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$ satisfying $f(x^*) = 0$. Therefore $\Omega = \{(0,0,0,0,...)\}$.

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[1]DEPARTMENT OF MATHEMATICS AND STATISTICS
KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN, SAUDI ARABIA
E-mail address: homidan@kfupm.edu.sa

[2]DEPARTMENT OF MATHEMATICAL SCIENCES
BAYERO UNIVERSITY, KANO
KANO, NIGERIA
E-mail address: bashiralik@yahoo.com

[3]DEPARTMENT OF MATHEMATICS
KANO UNIVERSITY OF SCIENCE AND TECHNOLOGY
P.M.B. 3042, WUDIL, KANO, NIGERIA
E-mail address: yubram@yahoo.com