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*Dedicated to Prof. Hong-Kun Xu on the occasion of his $60^{th}$ anniversary*

# Weak sharpness for solutions of nonsmooth variational inequalities and applications

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ABSTRACT. In this paper, we first give some new characterizations of weak sharpness of the solution set of nonsmooth variational inequalities in terms of partial subdiferentials/Gâteaux derivatives of involving bifunctions. As applications, we use a new characterization to establish sufficient conditions for guaranteeing finite termination of an arbitrary algorithm solving nonsmooth variational inequalities under the weak sharpness assumption.

## 1. INTRODUCTION

In [10], Ferris generalized the notion of sharp minima due to Polyak [23] (or strongly unique local minima due to Cromme [9]) to include the case of a non-singleton solution set and introduced the notion of weak sharp minima for a convex optimization problem. This notion plays an important role in sensitivity analysis, error bounds and (finite) convergence analysis of a large number of optimization algorithms and has been extensively established by many authors (see, e.g., [11, 8, 5, 6, 7] and the references therein). Extending this notion, Patricksson [22] introduced the concept of weak sharp solutions for variational inequalities and studied the finite convergence of approximation algorithms for solving monotone variational inequalities under the weak sharpness of the solution sets. Marcotte and Zhu [18] derived the necessary and sufficient condition for a solution set to be weakly sharp in term of its dual gap function and also studied finite convergence of sequences generated by some algorithms for solving variational inequalities whose the solution set is weakly sharp. Later, weak sharpness of solutions and its applications to the finite convergence property of various algorithms for finding solutions of variational inequalities have been investigated by many authors (see, e.g., [26, 25, 14, 19, 20, 17, 3, 16, 27] and references therein). Some authors extended and studied the concept of weak sharp solutions and its applications to general variational inequalities, e.g., set-valued variational inequalities [1, 24], variational-type inequalities [15], mixed variational inequalities [13] and nonsmooth variational inequalities [2, 21].

It is well-known that a variational inequality provides the first order necessary and sufficient optimality conditions for a solution of a convex and differentiable minimization problem. However, if the objective function of a convex minimization problem is not necessarily differentiable but has some kind of directional derivative, e.g., Dini directional derivative, Clarke directional derivative, etc., then the optimization problem can be solved by using a nonsmooth variational inequality. We refer the reader to [4, Chapter 6] for a comprehensive study of nonsmooth variational inequalities and their applications.

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Recently, authors [2] introduced the concept of weakly sharp solutions for nonsmooth variational inequalities and gave a characterization of the weak sharpness of the solutions set in terms of error bound of the dual gap function. An application to the finite convergence of the gradient projection method for solving nonsmooth variational inequalities is also provided. Other characterizations of weak sharpness of solution set of nonsmooth variational inequalities and its applications to finite convergence analysis were established in [21]. The aim of this paper is to provide some new characterizations of the weak sharpness of the solutions set for nonsmooth variational inequalities in connections with partial subdifferentials/Gâteaux derivatives of involving bifunctions. Applications to the finite convergence of algorithms for solving nonsmooth variational inequalities are also presented.

## 2. Preliminaries

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$, $||\cdot||$, respectively. We denote by $\mathbf{0}$ the zero vector of the corresponding vector space. For a subset $C$ of $H$, we denote by $\mathrm{co}C$ the convex hull of $C$, by $\mathrm{cl}C$, or $\overline{C}$, the closure of $C$. The support function of $C$ is defined by $\rho(x,C) := \sup\{\langle x,c\rangle : c\in C\}$ for $x\in H$. The polar $C^\circ$ of $C$ is defined by

$$C^\circ := \{x^*\in H : \langle x^*,x\rangle \le 0 \text{ for all } x\in C\}.$$

The distance from a given point $x\in H$ to $C$ is defined by

$$\mathrm{dist}(x,C) := \inf_{y\in C}\|y-x\|,$$

and the projection of $x$ onto $C$ is defined by

$$P_C(x) := \{y\in C : \|y-x\| = \mathrm{dist}(x,C)\}.$$

It is well-known that $P_C(x)$ is a singleton set if $C$ is nonempty, closed and convex. In this case, $P_C$ is a nonexpansive mapping, that is,

$$\|P_C(x)-P_C(y)\| \le \|x-y\|, \quad \text{for all } x,y\in H.$$

Let $X$ be a nonempty closed convex subset of $H$. The tangent cone to $X$ at a point $x\in X$ is defined as

$$T_X(x) := \mathrm{cl}\left(\bigcup_{\lambda>0}\frac{X-x}{\lambda}\right).$$

The normal cone to $X$ at $x\in X$ is defined by $N_X(x) := [T_X(x)]^\circ$. In other words,

$$N_X(x) = \{x^*\in H : \langle x^*, y-x\rangle \le 0 \text{ for all } y\in X\}.$$

**Definition 2.1.** Let $\Lambda$ be a subset of $[0,+\infty)$. A mapping $f : H\to(-\infty,\infty]$ is said to be

(a) positively homogeneous if $f(\lambda x) = \lambda f(x)$ for all $x\in H$ and $\lambda\ge 0$;
(b) $\Lambda$-subhomogeneous if $f(\lambda x)\le\lambda f(x)$ for all $x\in H$ and $\lambda\in\Lambda$;
(c) subadditive if $f(x+y)\le f(x)+f(y)$ for all $x,y\in H$.

Let $f : H\to(-\infty,\infty]$ be a function. The domain of $f$ is the set $\mathrm{dom}(f) := \{x\in H : f(x)<\infty\}$. We say that $f$ is convex if it is convex on its domain, and proper if its domain is nonempty. Let $f : H\to(-\infty,\infty]$ be a proper, convex function and $x\in\mathrm{dom}(f)$. The subdifferential of $f$ at $x$ is the set

$$\partial f(x) := \{v\in H : f(y)-f(x)\ge\langle v,y-x\rangle \text{ for all } y\in H\}.$$

For $v \in H$ and $x \in \mathrm{dom}(f)$, the directional derivative of $f$ at $x$ in the direction $v$ is defined by

$$f'(x)(v) := \lim_{t\to 0+} \frac{f(x+tv)-f(x)}{t}.$$

The function $f$ is said to be Gâteaux differentiable at $x$ if there exists $\zeta \in H$ such that $f'(x)(v) = \langle \zeta, v\rangle$ for all $v \in H$. The vector $\zeta$ is called the Gâteaux derivative of $f$ at $x$ and denoted by $\nabla f(x)$.

For a bifunction $f : H \times H \to (-\infty, \infty]$ and $x, y, d \in H$, we denote by $\partial_2 f(x; y)$, $f_2'(x; y)(d)$ and $\nabla_2 f(x; y)$ the subdifferential, the directional derivative in the direction $d$ and the Gâteaux derivative of $f(x; \cdot)$ at $y$ (when they do exist).

From now on, let $X$ be a nonempty, closed and convex subset of $H$ and $h : X \times H \to (-\infty, \infty]$ be a bifunction such that $h(x; \mathbf{0}) = 0$ for each $x \in X$. The nonsmooth variational inequality problem (in short, NVIP) is to find $x^* \in X$ such that

(2.1) $$h(x^*; y - x^*) \geq 0, \quad \text{for all } y \in X.$$

A problem closely to NVIP (2.1) is the following Minty type nonsmooth variational inequality problem (in short, MNVIP): Find $x^* \in X$ such that

(2.2) $$h(y; x^* - y) \leq 0, \quad \text{for all } y \in X.$$

We denote by $X^*$ and $X_*$ the solution sets of NVIP (2.1) and MNVIP (2.2), respectively.

When $h(x; y - x) = \langle F(x), y - x\rangle$ for all $x, y \in X$, where $F : X \to H$, then (2.1) reduces to the following classical variational inequality problem (in short, VIP): Find $x^* \in X$ such that

(2.3) $$\langle F(x^*), y - x^*\rangle \geq 0, \quad \text{for all } y \in X.$$

For a comprehensive study of nonsmooth variational inequalities and their applications to nonsmooth optimization, we refer [4].

**Definition 2.2.** Let $K \subseteq H$ be a nonempty convex set. A bifunction $h : K \times H \to (-\infty, \infty]$ is said to be upper sign continuous if for all $x, y \in K$,

$$h(y + \lambda(x - y); x - y) \leq 0 \text{ for all } \lambda \in (0, 1) \quad \Rightarrow \quad h(x; y - x) \geq 0.$$

**Definition 2.3.** Let $F : H \to H$ be a mapping and $K$ be a subset of $H$. The mapping $F$ is said to be

(i) monotone on $K$ if

$$\langle F(x) - F(y), x - y\rangle \geq 0 \quad \forall x, y \in K;$$

(ii) pseudomonotone on $K$ if for all $x, y \in K$,

$$\langle F(x), y - x\rangle \geq 0 \;\Rightarrow\; \langle F(y), y - x\rangle \geq 0;$$

The monotonicity is extended for bifunctions as follows.

**Definition 2.4.** Let $K \subset H$ be a nonempty set and $h : K \times H \to (-\infty, \infty]$ be a bifunction. The bifunction $h$ is said to be

(a) monotone on $K$ if for all $x, y \in K$,

$$h(x; y - x) + h(y; x - y) \leq 0;$$

(b) pseudomonotone at $x \in K$ if for each $y \in K$,

$$h(x; y - x) \geq 0 \quad \Rightarrow \quad h(y; x - y) \leq 0;$$

**Remark 2.1.** If $h$ is monotone on $K$, then it is pseudomonotone on $K$. The inverse is not true.

The following lemma provides sufficient conditions for convexity and closedness of the solution set of NVIP (2.1).

**Lemma 2.1.** [21] *Assume that $h$ is positively homogeneous in the second argument, upper sign continuous and pseudomonotone. If $x \mapsto h(y; x-y)$ is convex on $X$ for each $y \in X$, then $X^*$ is convex. If $x \mapsto h(y; x-y)$ is lower semicontinuous on $X$ for each $y \in X$, then $X^*$ is closed.*

From now on we always assume that the solution set $X^*$ of NVIP (2.1) is nonempty, closed and convex.

## 3. Characterizations of weak sharpness and applications

We first recall the definition of weak sharp solutions for NVIP (2.1) which was introduced in [2].

**Definition 3.5.** The solution set $X^*$ of NVIP (2.1) is said to be weakly sharp if there exists a constant $\alpha > 0$ such that, for all $x^* \in X^*$,

$$h(x^*; d) \geq \alpha||d||, \qquad \text{for all } d \in T_X(x^*) \cap N_{X^*}(x^*).$$

The constant $\alpha$ is called the modulus of weak sharpness of the solution set $X^*$.

**Example 3.1.** Let $X = [0,1]^2$ and $h : X \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$h(x; d) = \frac{x_1 d_1}{|d_2| + 1} + (x_1 + x_2)(d_1^3 + d_2^3) + d_1 + d_2$$

for all $x = (x_1, x_2) \in X$ and $d = (d_1, d_2) \in \mathbb{R}^2$. Assume that $x^* = (x_1^*, x_2^*) \in X^*$ is a solution of NVIP (2.1). Then,

$$\frac{x_1^*(y_1 - x_1^*)^3}{|y_2 - x_2^*| + 1} + (x_1^* + x_2^*)((y_1 - x_1^*)^3 + (y_2 - x_2^*)^3) + (y_1 - x_1^*) + (y_2 - x_2^*) \geq 0$$

for all $(y_1, y_2) \in [0,1]^2$. So, $x^*$ must be $(0,0)$. Thus, $X^* = \{(0,0)\}$. We have $T_X(0,0) = \mathbb{R}^2_+$ and $N_{X^*}(0,0) = \mathbb{R}^2$. Hence,

$$T_X(0,0) \cap N_{X^*}(0,0) = \mathbb{R}^2_+.$$

For any $d = (d_1, d_2) \in T_X(0,0) \cap N_{X^*}(0,0)$, one has

$$h(0; d) = d_1 + d_2 \geq \sqrt{d_1^2 + d_2^2} = ||d||.$$

Thus, $X^*$ is weakly sharp with modulus 1.

**Example 3.2.** Let $X = [1,2] \times [0,1]$ and $h : X \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $h(x; d) = x_1 d_2 + x_2 d_2^3$ for all $x = (x_1, x_2) \in X$ and $d = (d_1, d_2) \in \mathbb{R}^2$. If $x^* = (x_1^*, x_2^*) \in X$ is a solution of (2.1), then

$$x_1^*(x_2 - x_2^*) + x_2^*(x_2 - x_2^*)^3 \geq 0, \quad \forall x_2 \in [0,1].$$

Thus, $x_2^* = 0$ and $x_1^* \in [1,2]$ and then $X^* = [1,2] \times \{0\}$ is the solution set of (2.1). For $x^* \in X^*$, we have $T_X(x^*) = \mathbb{R} \times \mathbb{R}_+$ and $N_{X^*}(x^*) = \{0\} \times \mathbb{R}$. Hence, $T_X(x^*) \cap N_{X^*}(x^*) = \{0\} \times \mathbb{R}_+$. Let $x^* = (x_1^*, x_2^*) \in X^*$ and $d = (d_1, d_2) \in T_X(x^*) \cap N_{X^*}(x^*)$, we have

$$h(x^*; d) = x_1^* d_2 + x_2^* d_2 = x_1^* d_2 \geq d_2 = ||d||.$$

Thus, $X^*$ is weakly sharp with modulus 1.

Some characterizations of weak sharpness for the solution set $X^*$ of nonsmooth variational inequalities are presented in [2, 21]. Our aim is to give some new characterizations for weak sharpness of the solution set $X^*$. From now on, for our purpose, we assume that the function $h$ is locally Lipschitz at $(x^*, \mathbf{0})$ for each $x^* \in X^*$. The first result is stated as follows.

**Theorem 3.1.** *Assume that $h(x;\cdot)$ is convex for each $x \in X$. The following statements are equivalent:*

(i) *$X^*$ is weakly sharp.*

(ii) *There exists $\alpha > 0$ such that*

$$h(P_{X^*}(x); x - P_{X^*}(x)) \geq \alpha \text{dist}(x, X^*), \quad \forall x \in X. \tag{3.4}$$

(iii) *There exists $\gamma > 0$ such that*

$$\gamma\overline{\mathbb{B}} \subset \partial_2 h(x^*; \mathbf{0}) + [T_X(x^*) \cap N_{X^*}(x^*)]^\circ, \quad \forall x^* \in X^*. \tag{3.5}$$

*Proof.* $(i) \Rightarrow (ii)$. This was proved in [21]. We present the proof here for the reader's convenience. For any $x \in X$, we have $||x - P_{X^*}(x)|| = \text{dist}(x, X^*)$, and

$$x - P_{X^*}(x) \in T_X(P_{X^*}(x)) \cap N_{X^*}(P_{X^*}(x)).$$

Thus, by weak sharpness of $X^*$, one has

$$h(P_{X^*}(x); x - P_{X^*}(x)) \geq \alpha||x - P_{X^*}(x)|| = \alpha\text{dist}(x, X^*).$$

$(ii) \Rightarrow (iii)$. We follow some lines of the proof of [2, Theorem 4.1]. Let $x^* \in X^*$. If $T_X(x^*) \cap N_{X^*}(x^*) = \{\mathbf{0}\}$, then (3.5) holds. Assume now that $T_X(x^*) \cap N_{X^*}(x^*) \neq \{\mathbf{0}\}$. Let $\mathbf{0} \neq d \in T_X(x^*) \cap N_{X^*}(x^*)$. Then, $\langle d, d\rangle > 0$ and

$$\langle d, y^* - x^*\rangle \leq 0, \quad \text{for all } y^* \in X^*.$$

Thus, $X^*$ is separated from $x^* + d$ by the hyperplane

$$H_d = \{x \in H : \langle d, x - x^*\rangle = 0\}.$$

Since $d \in T_X(x^*)$, for each sequence of positive numbers $\{t_k\}$ converging to $0$, there exists a sequence $\{d_k\}$ converging to $d$ such that $x^* + t_k d_k \in X$. Thus, $\langle d, d_k\rangle > 0$ for $k$ sufficiently large. So $\langle d, x^* + t_k d_k - x^*\rangle = t_k\langle d, d_k\rangle > 0$ for $k$ large enough, i.e., $x^* + t_k d_k$ belongs to the open set $\{x \in H : \langle d, x - x^*\rangle > 0\}$ which is separated from $X^*$ by $H_d$. Hence,

$$\text{dist}(x^* + t_k d_k, X^*) \geq \text{dist}(x^* + t_k d_k, H_d) = \frac{t_k\langle d, d_k\rangle}{||d||}.$$

By (3.4), for all $k$ large enough, we have

$$h(P_{X^*}(x^* + t_k d_k); x^* + t_k d_k - P_{X^*}(x^* + t_k d_k)) \geq \alpha\frac{t_k\langle d, d_k\rangle}{||d||}$$

By the locally Lipschitz continuity of $h$ at $(x^*, \mathbf{0})$, the Lipschitz continuity of $P_{X^*}$ with having in mind that $x^* = P_{X^*}(x^* + t_k d)$ for all $k$ and $h(x; \mathbf{0}) = 0$, the latter inequality yields

$$\frac{h(x^*; t_k d) - h(x^*; \mathbf{0})}{t_k} \geq \alpha\frac{\langle d, d_k\rangle}{||d||}.$$

Letting $k \to \infty$ in both sides of the latter inequality, we get

$$h_2'(x^*; \mathbf{0})(d) \geq \alpha||d||.$$

It follows that, for all $x^* \in X^*$, we have

$$\sup\{\langle \zeta, d\rangle : \zeta \in \partial_2 h(x^*; \mathbf{0})\} \geq \alpha||d||, \quad \forall d \in T_X(x^*) \cap N_{X^*}(x^*). \tag{3.6}$$

We will show that, for $x^* \in X^*$,

$$\sup\{\zeta, d\rangle : \zeta \in \partial_2 h(x^*; \mathbf{0}) + [T_X(x^*) \cap N_{X^*}(x^*)]^\circ\} \geq \alpha||d||, \quad \forall d \in H. \tag{3.7}$$

Indeed, let $x^* \in X^*$. If $d \notin T_X(x^*) \cap N_{X^*}(x^*)$, then there exists $d^* \in [T_X(x^*) \cap N_{X^*}(x^*)]^\circ$ such that $\langle d, d^*\rangle > 0$. Let $\zeta^* \in \partial_2 h(x^*; \mathbf{0})$ and $\lambda > 0$. We have $\zeta^* + \lambda d^* \in \partial_2 h(x^*; \mathbf{0}) + [T_X(x^*) \cap N_{X^*}(x^*)]^\circ$ and $\langle \zeta^* + \lambda d^*, d\rangle \to \infty$ as $\lambda \to \infty$. Thus, the supremum in (3.7) is infinite and (3.7) holds for $d \notin T_X(x^*) \cap N_{X^*}(x^*)$. Suppose $d \in T_X(x^*) \cap N_{X^*}(x^*)$. Since $\mathbf{0} \in [T_X(x^*) \cap N_{X^*}(x^*)]^\circ$,

$$\sup\{\langle \zeta, d\rangle : \zeta \in \partial_2 h(x^*; \mathbf{0})\} \leq \sup\{\langle \zeta, d\rangle : \zeta \in \partial_2 h(x^*; \mathbf{0}) + [T_X(x^*) \cap N_{X^*}(x^*)]^\circ\}.$$

This, together with (3.6), implies that (3.7) holds. Hence,

$$\rho(d, \alpha\mathbb{B}) \leq \rho\left(d, \partial_2 h(x^*; \mathbf{0}) + [T_X(x^*) \cap N_{X^*}(x^*)]^\circ\right), \quad \forall d \in H.$$

This implies that (see, [7, Theorem A1, part 8])

$$\alpha\mathbb{B} \subset \operatorname{cl}\left(\partial_2 h(x^*; \mathbf{0}) + [T_X(x^*) \cap N_{X^*}(x^*)]^\circ\right).$$

Therefore, (3.5) holds for $0 < \gamma < \alpha$.

$(iii) \Rightarrow (i)$. Let $x^* \in X^*$ be any. If $T_X(x^*) \cap N_{X^*}(x^*) = \{\mathbf{0}\}$ then (3.5) holds. Assume $T_X(x^*) \cap N_{X^*}(x^*) \neq \{\mathbf{0}\}$ and let $\mathbf{0} \neq v \in T_X(x^*) \cap N_{X^*}(x^*)$. By (3.5), there exists $\zeta^* \in \partial_2 h(x^*; \mathbf{0})$ such that

$$\alpha\frac{v}{||v||} - \zeta^* \in [T_X(x^*) \cap N_{X^*}(x^*)]^\circ.$$

Thus,

$$\left\langle \alpha\frac{v}{||v||} - \zeta^*, v\right\rangle \leq 0. \tag{3.8}$$

Since $\zeta^* \in \partial_2 h(x^*; \mathbf{0})$, we have

$$\langle \zeta^*, v\rangle = \langle \zeta^*, v - \mathbf{0}\rangle \leq h(x^*; v) - h(x^*; \mathbf{0}) = h(x^*; v).$$

This, together with (3.8), implies that $\alpha||v|| \leq h(x^*; v)$. This ends the proof. □

**Remark 3.2.** In [21], the equivalence between (i) and (ii) is proved under the assumptions that $h$ is $[0, 1]$-subhomogeneous in the second argument and upper semicontinuous in both arguments. The assumpions required in Theorem 3.1 are different to those required in [21].

If $h(x; \cdot)$ is Gâteaux differentiable at $\mathbf{0}$ for each $x \in X$, then we have the following characterization of weak sharpness for $X^*$.

**Theorem 3.2.** *Assume that $h(x; \cdot)$ is Gâteaux differentiable at* **0** *for each $x \in X$ and $X^*$ is weakly sharp. Then, there exists $\alpha > 0$ such that*

$$\alpha\overline{\mathbb{B}} \subset \nabla_2 h(x^*, \mathbf{0}) + [T_X(x^*) \cap N_{X^*}(x^*)]^\circ, \quad \forall x^* \in X^*. \tag{3.9}$$

*Proof.* It follows the lines of the proof of Theorem 3.1 with $\nabla_2 h(x^*; \mathbf{0})$ in the places of $\partial_2 h(x^*; \mathbf{0})$ and $\zeta^*$. □

Moreover, when $h(x; \cdot)$ is convex and it is Gâteaux differentiable at $\mathbf{0}$, we have the following result.

**Theorem 3.3.** *Assume, for each $x \in X$, that $h(x; \cdot)$ is Gâteaux differentiable at* **0** *and is convex. The following statements are equivalent:*

(i) *$X^*$ is weakly sharp with modulus $\alpha > 0$.*
(ii) *There exists $\alpha > 0$ such that (3.4) holds.*
(iii) *There exists $\alpha > 0$ such that (3.9) holds.*

**Example 3.3.** We reconsider Example 3.2. We see that $h(x; \cdot)$ is convex and Gâteaux differentiable for all $x \in X$. For $x = (x_1, x_2) \in X$, $d = (d_1, d_2) \in \mathbb{R}^n$, we have $\nabla_2 h(x; d) = (0, x_1 + 3x_2 d_2)$. Thus, $\nabla_2 h(x; \mathbf{0}) = (0, x_1)$. Since $[T_X(x^*) \cap N_{X^*}(x^*)]^\circ = \mathbb{R} \times \mathbb{R}_-$ for $x^* \in X^*$, we see that

$$\overline{\mathbb{B}} \subset \nabla_2 h(x^*; \mathbf{0}) + [T_X(x^*) \cap N_{X^*}(x^*)]^\circ.$$

Thus, (3.9) holds with $\alpha = 1$.

We now apply Theorem 3.2 to give a finite convergence result for algorithms solving a nonsmooth variational inequality. Our result is stated as follows.

**Theorem 3.4.** *Let $\{x_k\}$ be a sequence in $X$ and $\{\lambda_k\}$, $\{\gamma_k\}$ be sequences of real numbers. Assume that $h(x;\cdot)$ is Gâteaux differentiable at $\mathbf{0}$ for each $x \in X$, $X^*$ is weakly sharp and $\{x_k\}$ satisfies*

$$\lim_{k\to\infty}[H(x_k) + P_{N_X(x_k)}(\lambda_k x_k + \gamma_k H(x_k))] = 0, \tag{3.10}$$

*where $H : X \to \mathbb{R}^n$ is defined by $H(x) = \nabla_2 h(x;\mathbf{0})$ for $x \in X$. Then, $x_k \in X^*$ for all $k$ sufficiently large, if one of the following conditions holds:*

(i) *$H$ is monotone on $X$;*
(ii) *$H$ is uniformly continuous on $X$ and $\operatorname{dist}(x_k, X^*) \to 0$ as $k \to \infty$.*
(iii) *$H$ is continuous on $X^*$, $\{x_k\}$ is bounded and all accumulation points belong to $X^*$.*

*Proof.* Since $X^*$ is weakly sharp, by Theorem 3.2, there exists $\alpha > 0$ such that for each $x^* \in X^*$

$$\alpha\overline{\mathbb{B}} \subset H(x^*) + [T_X(x^*) \cap N_{X^*}(x^*)]^\circ.$$

Then, for any $z \in \mathbb{B}$, we have $\alpha z - H(x^*) \in [T_X(x^*) \cap N_{X^*}(x^*)]^\circ$. Hence, for any $v \in T_X(x^*) \cap N_{X^*}(x^*)$, one has $\langle \alpha z - H(x^*), v\rangle \le 0$. Taking $z = v/||v||$, we get

$$\langle H(x^*), v\rangle \ge \alpha||v||. \tag{3.11}$$

Now, assume that the conclusion of the Theorem does not hold. Then, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \notin X^*$ for all $i$. For each $i$, set $y_{k_i} = P_{X^*}(x_{k_i})$. Then, $y_{k_i} \in X^*$ and $x_{k_i} - y_{k_i} \in T_X(y_{k_i}) \cap N_{X^*}(y_{k_i})$. By (3.11), we have

$$\alpha||x_{k_i} - y_{k_i}|| \le \langle H(y_{k_i}), x_{k_i} - y_{k_i}\rangle, \quad \forall i.$$

Since $x_{k_i} \notin X^*$, we have $||x_{k_i} - y_{k_i}|| > 0$. Hence,

$$\alpha \le \left\langle -H(y_{k_i}), \frac{y_{k_i} - x_{k_i}}{||y_{k_i} - x_{k_i}||}\right\rangle, \quad \forall i. \tag{3.12}$$

(i) Since $H$ is monotone, by (3.12), we have

$$\begin{aligned}\alpha &\le \left\langle H(x_{k_i}) - H(y_{k_i}), \frac{y_{k_i} - x_{k_i}}{||y_{k_i} - x_{k_i}||}\right\rangle + \left\langle P_{N_X(x_{k_i})}[\lambda_k x_{k_i} + \gamma_{k_i} H(x_{k_i})], \frac{y_{k_i} - x_{k_i}}{||y_{k_i} - x_{k_i}||}\right\rangle \\ &\quad + \left\langle -H(x_{k_i}) - P_{N_X(x_{k_i})}[\lambda_k x_{k_i} + \gamma_{k_i} H(x_{k_i})], \frac{y_{k_i} - x_{k_i}}{||y_{k_i} - x_{k_i}||}\right\rangle \\ &\le ||H(x_{k_i}) + P_{N_X(x_{k_i})}[\lambda_k x_{k_i} + \gamma_{k_i} H(x_{k_i})]||\end{aligned}$$

Letting $i \to \infty$, we obtain $\alpha \le 0$ which a contradiction. Thus, $x_k \in X^*$ for $k$ large enough.
(ii) Since $H$ is uniformly continuous on $X$, there exists $\delta > 0$ such that $||H(x) - H(y)|| < \alpha/4$ for all $x, y \in X$ with $||x - y|| < \delta$.

Moreover, since $||x_{k_i} - y_{k_i}|| = \operatorname{dist}(x_{k_i}, X^*) \to 0$ as $i \to \infty$, there exists $i_1 \ge 1$ such that $||x_{k_i} - y_{k_i}|| < \delta$ for all $i \ge i_1$. Thus, $||F(x_{k_i}) - F(y_{k_i})|| < \alpha/4$ for all $i \ge i_1$. By (3.10), there exists $i_0 \ge i_1$ such that $||H(x_{k_i}) + P_{N_X(x_{k_i})}[x_{k_i} - H(x_{k_i}]|| < \alpha/4$ for all $i \ge i_0$. By (3.12), for all $i \ge i_0$, we have

$$\begin{aligned}\alpha &\le \left\langle H(x_{k_i}) - H(y_{k_i}), \frac{y_{k_i} - x_{k_i}}{||y_{k_i} - x_{k_i}||}\right\rangle + \left\langle P_{N_X(x_{k_i})}[\lambda_k x_{k_i} + \gamma_{k_i} H(x_{k_i})], \frac{y_{k_i} - x_{k_i}}{||y_{k_i} - x_{k_i}||}\right\rangle \\ &\quad + \left\langle -H(x_{k_i}) - P_{N_X(x_{k_i})}[\lambda_k x_{k_i} + \gamma_{k_i} H(x_{k_i})], \frac{y_{k_i} - x_{k_i}}{||y_{k_i} - x_{k_i}||}\right\rangle \\ &\le ||H(x_{k_i}) - H(y_{k_i})|| + ||H(x_{k_i}) + P_{N_X(x_{k_i})}[\lambda_k x_{k_i} + \gamma_{k_i} H(x_{k_i})]|| \\ &< \; < \alpha/4 + \alpha/4 = \alpha/2.\end{aligned}$$

This is a contradiction. Thus, $x_k \in X^*$ for $k$ large enough.
(iii) Since $\{x_k\}$ is bounded, we may assume that $\{x_{k_i}\}$ converges to some $x^* \in X$. The continuity of the projection mapping $P_{X^*}$ implies that the sequences $\{y_{k_i}\}$ converges to $x^*$. Thus, by continuity of $H$, we have $||H(y_{k_i}) - H(x_{k_i})|| \to 0$ as $i \to \infty$. Progressing as in the proof of (ii), we conclude that $x_k \in X^*$ for sufficiently large $k$. □

**Remark 3.3.** By Moreau decomposition (see, [12, Theorem 3.2.5]), we have, for all $k$, that

$$\lambda_k x_k + \gamma_k H(x_k) = P_{T_X(x_k)}(\lambda_k x_k + \gamma_k H(x_k)) + P_{N_X(x_k)}(\lambda_k x_k + \gamma_k H(x_k)),$$

or, equivalently,

$$H(x_k) + P_{N_X(x_k)}(\lambda_k x_k + \gamma_k H(x_k)) = \lambda_k x_k + (\gamma_k + 1)H(x_k) - P_{T_X(x_k)}(\lambda_k x_k + \gamma_k H(x_k)).$$

Thus, (3.4) is equivalent to

$$\lim_{k\to\infty}[\lambda_k x_k + (\gamma_k + 1)H(x_k) - P_{T_X(x_k)}(\lambda_k x_k + \gamma_k H(x_k))] = 0. \tag{3.13}$$

If we choose $\lambda_k = 0$ and $\gamma_k = -1$ for all $k$, then (3.13) becomes

$$\lim_{k\to\infty} P_{T_X(x_k)}(-H(x_k)) = 0. \tag{3.14}$$

Note that the condition (3.14) is analogous to a well-known condition ensuring finite termination of a sequences for solving optimization problems, variational inequalities under weak sharpness of solution sets (see, e.g., [8, 18]).

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