CARPATHIAN J. MATH. 36 (2020), No. 1, 119 - 126
Online version at https://www.carpathian.cunbm.utcluj.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Dedicated to Prof. Hong-Kun Xu on the occasion of his 60^{th} *anniversary*

Solving split generalized mixed equality equilibrium problems and split equality fixed point problems for nonexpansive-type maps

M. O. NNAKWE

ABSTRACT. Let X be a 2-uniformly convex and uniformly smooth real Banach space. In this paper, an iterative algorithm of *Krasnosel'skii-type* is constructed and used to approximate a common solution of *split generalized mixed equality equilibrium problems* ($SGMEEP$) and *split equality fixed point problems* ($SEFPP$) for *quasi-ψ-nonexpansive maps*. A strong convergence theorem of the sequence generated by this algorithm is proved without imposing any compactness-type condition on either the operators or the space considered. The theorem proved improves and complements important recent results in the literature.

1. INTRODUCTION

Let K be a nonempty closed convex subset of real Banach space X with dual space, X^*. Let χ be a map from K to $\mathbb{R}$. Let f be a bifunctional from $K \times K$ to $\mathbb{R}$ and $\mathcal{A}$ from K to X^*. The *generalized mixed equilibrium problem*, is a problem of finding:

$$u \in K \text{ such that } f(u,y) + \chi(y) - \chi(u) + \langle \mathcal{A}u, y-u \rangle \geq 0, \ \forall\, y \in K. \tag{1.1}$$

The set of solutions of inequality (1.1), denoted by $GMEP$, is given by

$$GMEP = \{u \in K : f(u,y) + \chi(y) - \chi(u) + \langle \mathcal{A}u, y-u \rangle \geq 0, \ \forall\, y \in K\}.$$

It is known that this class of problems contains the class of equilibrium problems, variational inequality problems e.t.c., which many problems in physics, optimization, economics and other applied sciences can be reduced to particular cases of ($GMEP$). These problems have been studied extensively by many authors in the setting of Hilbert spaces and Bancach spaces (see e.g., Blum and Oettli [4], Chang *et al* [11], Chidume and Monday [12], and the references contained in them).

Recently, Moudafi [19] studied the *split equality fixed point problem (SEFPP)* in a Hilbert space. This problem is a problem of finding:

$$u \in F(\mathcal{Y}), \ \ v \in F(\mathcal{R}) \ \text{ such that } \ \mathcal{A}u = \mathcal{B}v,$$

where $\mathcal{Y} : H^1 \to H^1$, $\mathcal{R} : H^2 \to H^2$ are nonlinear maps with nonempty fixed point sets, $\mathcal{A} : H_1 \to H_3$ and $\mathcal{B} : H_2 \to H_3$ are bounded linear maps. The set of solutions of ($SEFPP$) is denoted by

$$\text{SEFPP} \ = \ \{(u,v) \in K \times M : u \in F(\mathcal{Y}),\ v \in F(\mathcal{R}) \ \text{ and } \ \mathcal{A}u = \mathcal{B}v\}. \tag{1.2}$$

This problem has recently attracted the attention of numerous researchers due to its diver applications, for example, applications in game theory, intensity-modulated radiation

Received: 01.03.2019. In revised form: 28.07.2019. Accepted: 05.08.2019
2010 *Mathematics Subject Classification.* 47H09, 47J25.
Key words and phrases. *Nonexpansive-type maps, split fixed point problems, split generalized equaity equilibrium problems, strong convergence.*

therapy, decomposition methods for partial differential equations, application in fully discretized models of inverse problems which arise from phase retrievals and in medical image reconstruction (see, e.g., Censor *et al.* [8], Attouch *et al.* [3], Byrne [6, 7] and the references therein). If $H^2 = H^3$ and $\mathcal{B}$ is the identity map on H^2, then, the $(SEFPP)$ reduces to the *split common fixed point problem* $(SCFPP)$ introduced by Censor and Segal [9]. For more results on $(SEFPP)$, see e.g., Zhao [22], Chidume *et al.* [14], Chang *et al.* [10], Giang *et al.* [15] and the references therein.

Motivated by the result of Moudafi [19], Bnouchachem [5], introduced the following split equilibrium problem in Hilbert spaces.

Let $f : K \times K \to \mathbb{R}$ and $g : M \times M \to \mathbb{R}$ be bifunctionals, where K and M are closed convex subsets of H_1 and H_2, respectively, and $\mathcal{A} : H_1 \to H_2$ is a bounded linear map.

The *split equilibrium problem (SEQP)* is the problem of finding $u \in K$ such that

$$f(u, y) \geq 0, \ \forall \, y \in K, \text{ such that } v = \mathcal{A}(u) \in M \text{ solves } g(v, z) \geq 0, \ \forall \, z \in M. \tag{1.3}$$

Motivated by the result of Bnouchachem [5], Zhaoli *et al.* [17], considered the following *split equality equilibrium problem (SEEP)*, which is a problem of finding $(u, v) \in K \times M$ such that

$$f(u, y) + \chi(y) - \chi(u) \geq 0, \ \forall \, y \in K, \ g(v, z) + \varphi(z) - \varphi(v) \geq 0 \text{ and } Au = Bv, \tag{1.4}$$

where $f : K \times K \to \mathbb{R}$ and $g : M \times M \to \mathbb{R}$ are bifunctionals, $\chi : K \to \mathbb{R} \cup \{\infty\}$ and $g : M \times M \to \mathbb{R} \cup \{\infty\}$ are proper lower semi-continuous functions, $\mathcal{A} : K \subset H_1 \to H_3$ and $\mathcal{B} : M \subset H_2 \to H_3$ are bounded linear maps. They proved in a real Hilbert space that the sequence generated by the following algorithm with $(x_1, y_1) \in K \times M$, given by

$$\begin{cases} f(u_n, u) + \chi(u) - \chi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n \rangle \geq 0, \ \forall \, u \in K, \\ g(v_n, v) + \chi(v) - \chi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n \rangle \geq 0, \ \forall \, v \in M, \\ x_{n+1} = a_n u_n + (1 - a_n) T(u_n - \gamma_n \mathcal{A}^*(\mathcal{A}u_n - \mathcal{B}v_n)), \ \forall \, n \geq 1, \\ y_{n+1} = a_n v_n + (1 - a_n) S(v_n - \gamma_n \mathcal{B}^*(\mathcal{A}u_n - \mathcal{B}v_n)), \ \forall \, n \geq 1, \end{cases} \tag{1.5}$$

converges weakly to a solution of $F(T) \cap F(S) \cap (SEFPP)$, where $\{a_n\}$, $\{r_n\}$ and $\{\gamma_n\}$ are positive real sequences satisfying certain conditions. The authors obtained strong convergence by further assuming that T and S are semi-compact.

We remark that in order to obtain strong convergence of the sequence generated by algorithm (1.5), compactness-type condition was imposed.

Motivated by the results above, we study the following split generalized mixed equality equilibrium problem in real Banach spaces. Let X^1, X^2 and X^3 be real Banach spaces and K, M be nonempty closed and convex subsets of X^1 and X^2, respectively. Let $f : K \times K \to \mathbb{R}$ and $g : M \times M \to \mathbb{R}$ be bifunctionals, $\chi : K \to \mathbb{R} \cup \{\infty\}$ and $\varphi : M \to \mathbb{R} \cup \{\infty\}$ be proper lower semi-continuous and convex functions. Let $\mathcal{A} : K \to {X^*}^1$ and $\mathcal{B} : M \to {X^*}^2$ be continuous and monotone maps, and $U : X^1 \to X^3$, $V : X^2 \to X^3$ be bounded linear maps.

The *split generalized mixed equality equilibrium problem* is a problem of finding $(u, v) \in K \times M$ such that

$$\begin{aligned} & f(u, y) + \psi(y) - \psi(u) + \langle \mathcal{A}u, y - u \rangle \geq 0, \quad \text{for all } \ y \in K, \\ & g(v, z) + \varphi(z) - \varphi(v) + \langle \mathcal{B}v, z - v \rangle \geq 0, \quad \text{for all } \ z \in M, \\ & \ \text{and} \quad Uu = Vv. \end{aligned}$$

The set of solutions of split generalized mixed equality equilibrium problem shall be denoted by:

$$\begin{aligned} SGMEEP \;=\; & \big\{(u,v)\in K\times M : f(u,y)+\chi(y)-\chi(u)+\langle \mathcal{A}u, y-u\rangle \geq 0,\ \forall\, y\in K, \\ & g(v,z)+\varphi(z)-\varphi(v)+\langle \mathcal{B}v, z-v\rangle \geq 0,\ \forall\, z\in M \ \text{ and } \ \mathcal{A}u=\mathcal{B}v\big\}. \end{aligned} \tag{1.6}$$

We study the following *Krasnosel'skii-type* algorithm given by

$$\begin{cases} (x^1,y^1)\in X^1\times X^2,\ K^1=X^1,\ M^1=X^2,\ e^n\in J_{X^3}(Uu^n-Vv^n), \\ u^n=Q_rx^n,\ \ v^n=Q_ry^n,\ \ \theta^n=J_{X^1}^{-1}(J_{X^1}u^n-\mu U^*e^n), \\ \delta^n=J_{X^2}^{-1}(J_{X^2}v^n+\mu V^*e^n),\ \ z^n=J_{X^1}^{-1}(\beta J_{X^1}x^n+(1-\beta)J_{X^1}\mathcal{Y}\theta_n), \\ w^n=J_{X^2}^{-1}(\beta J_{X^2}y_n+(1-\beta)J_{X^2}\mathcal{R}\delta_n),\ \ K^{n+1}=\{p\in K^n:\psi(p,z^n)\leq\psi(p,x^n)\}, \\ M^{n+1}=\{q\in M^n:\psi(p,w^n)\leq\psi(q,y^n)\}, \\ x^{n+1}=\Pi_{K^{n+1}}x^1,\quad y^{n+1}=\Pi_{M^{n+1}}y^1,\ n\geq 1, \end{cases} \tag{1.7}$$

where X^1 and X^2 are 2-uniformly convex and uniformly smooth real Banach spaces, X^3 is a real Banach space, $\mathcal{Y}$ and $\mathcal{R}$ are closed quasi-ψ-nonexpansive maps, U and V are bounded linear maps, $\beta\in(0,1)$ and μ are some positive constants satisfying appropriate mild conditions. Then, the sequence generated by Algorithm (1.7) converges strongly to some point in the solution set. The theorem proved, in particular, improves and complements the results of Chidume *et al.* [13] and Zhaoli *et al.* [18, 17], which themselves are important generalization of some recent results in the literature (see e.g., Chidume *et al.* [13] and Zhaoli *et al.* [18, 17]).

2. PRELIMINARIES

Let X be a real normed space with with dual space X^*. Consider a map $\psi: X\times X\to\mathbb{R}$ defined by $\psi(u,y)=||u||^2-2\langle u,Jv\rangle+||v||^2$, for all $u,v \in X$. This map which was introduced by Alber [1] will play a central role in the sequel.

The following lemmas will be needed in the sequel.

Lemma 2.1. (Alber and Ryazantseva [2]) *Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then,*

$$\psi(u,J^{-1}u^*)=\mathcal{M}(u,u^*)\leq\mathcal{M}(u,u^*+v^*)-2\langle J^{-1}u^*-u,v^*\rangle,\ \forall\, u\in X,\ u^*,v^*\in X^*.$$

Lemma 2.2. (Alber, [1]) *Let K be a nonempty closed and convex subset of a smooth and strictly convex Banach space X. Then,*

$$\psi(u,\Pi v)+\psi(\Pi v,v)\leq\psi(u,v),\ \forall\, u\in K,\ v\in X.$$

Lemma 2.3. (Kamimura and Takahashi, [16]) *Let X be a uniformly convex and uniformly smooth real Banach space and $\{x^n\}$, $\{y^n\}$ be sequences in X such that either $\{x^n\}$ or $\{y^n\}$ is bounded. If $\lim_{n\to\infty}\psi(x^n,y^n)=0$, then, $\lim_{n\to\infty}||x^n-y^n||=0$.*

Lemma 2.4. *Let X be a 2-uniformly convex and smooth real Banach space and $J^{-1}:X^*\to X$ be the normalized duality map. Then, there exists a positive constant a such that*

$$||J^{-1}u-J^{-1}v||\leq\frac{1}{a}||u-v||,\ \forall\, u,v\in X^*$$

Lemma 2.5. (Xu, [20]) *Let X be a uniformly convex real Banach space. Let $r>0$. Then, there exists a strictly increasing continuous and convex function $g:[0,\infty)\to[0,\infty)$ such that g(0)=0 and for all $u,v\in B_r(0):=\{v\in E:||v||\leq r\}$ and $\lambda\in[0,1]$, we have that:*

$$||\lambda u+(1-\lambda)v||^2\leq\lambda||u||^2+(1-\lambda)||v||^2-\lambda((1-\lambda))g(||u-v||).$$

Definition 2.1. Let K be a nonempty, closed and convex subset of a real Banach space, X and $\mathcal{Y} : K \to K$ be a map.

(1) $\mathcal{Y}$ is called *quasi-ψ-nonexpansive* if $F(\mathcal{Y}) \neq \emptyset,\ \psi(p, \mathcal{Y}u) \leq \psi(p, u),\ \forall\, p \in F(\mathcal{Y}),\ u \in K.$

(2) $\mathcal{Y}$ is said to be *closed* if for any sequence $\{x^n\} \subset K$ with $x^n \to x^*$ and $\mathcal{Y}x^n \to y$, then, $y = \mathcal{Y}x^*$.

Basic Assumptions. Let K be a nonempty closed convex subset of a real Banach space X with dual space, X^*. Let $\chi : K \to \mathbb{R}$ be a lower semi-continuous and convex functional. Let $\mathcal{A} : K \to X^*$ be continuous and monotone. For solving the generalized mixed equilibrium problems, (1.1), we assume that the bifunctional $f : K \times K \to \mathbb{R}$ satisfies the following conditions:
(A_1) $f(u, u) = 0,$ for all $u \in K,$
(A_2) f is monotone, i.e. $f(u, v) + f(v, u) \leq 0,$ for all $u, v \in K,$
(A_3) $\limsup_{t \downarrow 0} f(u + t(z - u), v) \leq (u, v),$ for all $u, v, z \in K,$
(A_4) $f(u, \cdot)$ is convex and lower semi-continuous, for all $u \in K.$

3. Main result

Theorem 3.1. *Let X^1, X^2 be 2-uniformly convex and uniformly smooth real Banach spaces, X^3 be a real Banach space. Let K and M be nonempty closed convex subsets of X^1 and X^2, respectively. Let $f : K \times K \to \mathbb{R}$ and $g : M \times M \to \mathbb{R}$ be bifunctionals satisfying conditions (A_1) - (A_4). Let $\chi : K \to \mathbb{R} \cup \{\infty\}$ and $\varphi : M \to \mathbb{R} \cup \{\infty\}$ be proper lower semi-continuous and convex functions. Let $\mathcal{A} : K \to X^{*1}$ and $\mathcal{B} : M \to X^{*2}$ be continuous and monotone maps. Let $U : X^1 \to X^3$ and $V : X^2 \to X^3$ be bounded linear maps with adjoints U^* and V^*, respectively. Let $\mathcal{Y} : X^1 \to X^1$ and $\mathcal{R} : X^2 \to X^2$ be closed quasi-ψ-nonexpansive maps such that $F(\mathcal{Y}) \neq \emptyset$ and $F(\mathcal{R}) \neq \emptyset$. Let $\{(x^n, y^n)\}$ be a sequence in $X^1 \times X^2$ generated iteratively by algorithm (1.7). Assume $\mathcal{F} := SGMEEP \cap SEFPP \neq \emptyset$, $\beta \in (0, 1)$ and μ is such that $0 < \mu < a/(||A||^2 + ||B||^2)$, where a is the constant in Lemma 2.4. Then, $\{(x^n, y^n)\}$ converges strongly to some point $(x^*, y^*) \in \mathcal{F}$.*

Proof. We divide the proof into three steps.

Step 1. We show that the sequences $\{x^n\}$, $\{y^n\}$ are well defined; $\mathcal{F} \subset K^n \times M^n$, $\forall\, n \geq 1$. First, we show that K^n and M^n, are closed and convex. Clearly, $K^1 = X^1$ and $M^1 = X^2$ are closed and convex. Assume that K^n and M^n are closed and convex for some $n \geq 1$. Applying the definition of K^{n+1}, we have that: $K^{n+1} = \{p \in K^n : 2\langle p, Jx^n - Jz^n\rangle \leq ||x^n||^2 - ||z^n||^2\}$. Thus, K_{n+1} is closed and convex. Similarly, M^{n+1} is closed and convex. These imply that K^n and M^n are closed and convex. Hence, $\{x^n\}$, $\{y^n\}$ are well defined.

Claim. $\mathcal{F} \subset K^n \times M^n$, $\forall\, n \geq 1$. Clearly, $\mathcal{F} \subset K^1 \times M^1$. Assume that $\mathcal{F} \subset K^n \times M^n$, for some $n \geq 1$. Let $(p, q) \in \mathcal{F}$. Then, by Lemma 2.5 and definition of $\mathcal{Y}$, we have that:

$$\begin{aligned}
\psi(p, z^n) &= \psi(p, J_{X^1}^{-1}(\beta J_{X^1}x^n + (1-\beta)J_{X^1}\mathcal{Y}\theta_n)) \\
&\leq \beta\psi(p, x^n) + (1-\beta)\psi(p, \mathcal{Y}\theta^n) - \beta(1-\beta)G\big(||J_{X^1}x_n - J_{X^1}\mathcal{Y}\theta^n||\big) \\
&\leq \beta\psi(p, x^n) + (1-\beta)\psi(p, \theta^n) - \beta(1-\beta)G\big(||J_{X^1}x_n - J_{X^1}\mathcal{Y}\theta^n||\big). \qquad (3.7)
\end{aligned}$$

Again, using equation (1.7), Lemma 2.1 and a result of Zhang [21], we have that:

$$\begin{aligned}
\psi(p, \theta^n) &= \mathcal{M}(p, J_{X^1}u^n - \mu U^* e^n) \\
&\leq \mathcal{M}(p, J_{X^1}u^n) - 2\mu\langle J_{X^1}^{-1}(J_{X^1}u^n - \mu U^* e^n) - p, U^* e^n\rangle
\end{aligned}$$

$$= \psi(p,u^n) - 2\mu\langle U(\theta^n - p), e^n\rangle \tag{3.8}$$
$$\leq \psi(p,x^n) - 2\mu\langle U(\theta^n - p), e^n\rangle. \tag{3.9}$$

From inequalities (3.7) and (3.9), we get that:

$$\psi(p,z^n) \leq \psi(p,x^n) - 2\mu(1-\beta)\langle U(\theta^n - p), e^n\rangle - \beta(1-\beta)G\big(||J_{X^1}x^n - J_{X^1}\mathcal{Y}\theta^n||\big). \tag{3.10}$$

Similarly, we obtain that

$$\psi(q,w^n) \leq \psi(q,y^n) + 2\mu(1-\beta)\langle V(\delta^n - q), e^n\rangle - \beta(1-\beta)G\big(||J_{X^2}y^n - J_{X^2}\mathcal{R}\delta^n||\big). \tag{3.11}$$

From inequalities (3.10), (3.11) and the fact that $Up = Vq$, we get that:

$$\begin{aligned} \psi(p,z^n) + \psi(q,w^n) &\leq \psi(p,x^n) + \psi(q,y^n) - 2\mu(1-\beta)\langle U\theta^n - V\delta^n, e^n\rangle \\ &\quad -\beta(1-\beta)\big[G\big(||J_{X^1}x^n - J_{X^1}\mathcal{Y}\theta^n|| + G\big(||J_{X^2}y^n - J_{X^2}\mathcal{R}\delta^n||\big)\big] \end{aligned} \tag{3.12}$$

Furthermore, from equation (1.7), Lemma (2.4) and inequality (3.12), we have that:

$$\begin{aligned} -2\mu(1-\beta)\langle U\theta^n - V\delta^n, e^n\rangle &= -2\mu(1-\beta)||Uu^n - Vv^n||^2 + 2\mu(1-\beta)\langle U(u^n - \theta^n), e^n\rangle \\ &\quad +2\mu(1-\beta)\langle V(\delta^n - v^n), e^n\rangle \\ &\leq -2\mu(1-\beta)||Uu^n - Vv^n||^2 \\ &\quad + 2\frac{\mu^2(1-\beta)}{a}(||U||^2 + ||V||^2)||Uu^n - Vv^n||^2 \\ &= -2\mu(1-\beta)\big[1 - \frac{\mu}{a}(||U||^2 + ||V||^2)\big]||Uu^n - Vv^n||^2. \end{aligned} \tag{3.13}$$

From inequalities (3.12), (3.13) and $\omega := 2\mu(1-\beta)\big[1 - \frac{\mu}{a}(||U||^2 + ||V||^2)\big] > 0$, we have that:

$$\begin{aligned} \psi(p,z^n) + \psi(q,w^n) &\leq \psi(p,x^n) + \psi(q,y^n) - \omega||Uu^n - Vv^n||^2 \\ &\quad -\beta(1-\beta)\big[G\big(||J_{X^1}x^n - J_{X^1}\mathcal{Y}\theta^n||\big) + G\big(||J_{X^2}y^n - J_{X^2}\mathcal{R}\delta^n||\big)\big] \quad (3.14) \\ &\leq \psi(p,x^n) + \psi(q,y^n). \quad (3.15) \end{aligned}$$

This implies that $(p,q) \in K^{n+1} \times M^{n+1}$. Hence, $\mathcal{F} \subset K^n \times M^n$, for all $n \geq 1$.

Step 2. We prove that the sequences $\{x^n\}$ and $\{y^n\}$ are convergent.

First, we prove that $\{x^n\}$ and $\{y^n\}$ are bounded. From the definition of $\{x^n\}$ and Lemma 2.2, we have that $\psi(x^n,x^1) \leq \psi(p,x^1) - \psi\big(p,x^n\big) \leq \psi(p,x^1)$, $\forall\,(p,q) \in \mathcal{F} \subset K^n \times M^n$. This implies that $\{\psi(x^n,x^1)\}$ is bounded. Hence, $\{x^n\}$ is bounded. Since $x^{n+1} = \Pi_{K^{n+1}}x^1 \in K^{n+1} \subset K^n$ and $x^n = \Pi_{K^n}x^1$, we have that $\psi(x^n,x^1) \leq \psi(x^{n+1},x^1)$ and this implies that $\{\psi(x^n,x^1)\}$ is nondecreasing. Hence, $\lim\limits_{n\to\infty}\phi(x^n,x^1)$ exists. Furthermore, for $m \geq n$, we have that:

$$\begin{aligned} \phi(x^m,x^n) = \phi(\Pi_{K^m}x^1, \Pi_{K^n}x_1) &\leq \phi(\Pi_{K^m}x^1, x^1) - \phi(\Pi_{K^n}x^1, x^1) \\ &= \phi(x^m,x^1) - \phi(x^n,x^1) \to 0\ (as\ n \to \infty). \end{aligned}$$

It follows from Lemma 2.3 that $||x^n - x^m|| \to 0$ as $m,n \to \infty$. Hence, $\{x^n\}$ is Cauchy. Thus, there exists $x^* \in X^1$ such that $\lim\limits_{n\to\infty} x^n = x^*$. Following the same argument, we also obtain that $\{y^n\}$ is Cauchy. Hence, there exists $y^* \in X^2$ such that $\lim\limits_{n\to\infty} y^n = y^*$.

Step 3. We show that $\lim\limits_{n\to\infty}||u^n - x^n|| = 0$ and $\lim\limits_{n\to\infty}||v^n - y^n|| = 0$.

From equation (1.7), and for $m \geq n$, $(x^m,y^m) \in K^m \times M^m \subset K^n \times M^n$. Therefore, $\psi(x^m,z^n) \leq \psi(x^m,x^n) \to 0$ as $m,n \to \infty$ and $\psi(y^m,w^n) \leq \psi(y^m,y^n) \to 0$ as $m,n \to \infty$. Hence, by Lemma 2.3, we have that $z^n \to x^*$ as $n \to \infty$ and $w^n \to y^*$ as $n \to \infty$.

From inequality (3.14), set $\eta_n = \big(\psi(p,x^n)-\psi(p,z^n)+\psi(q,y^n)-\psi(q,w^n)\big)$. Then, we have:

$$||Uu^n - Vv^n||^2 \leq \omega^{-1}\eta_n \quad \text{and}$$

$$G(||J_{X^1}x^n - J_{X^1}\mathcal{Y}\theta^n||) + G(||J_{X^2}y^n - J_{X^2}\mathcal{R}\delta^n||) \leq (\beta(1-\beta))^{-1}\eta_n.$$

It follows that:

$$\lim_{n\to\infty}|Uu^n - Vv^n|| = 0; \quad \lim_{n\to\infty} G(||J_{X^1}x^n - J_{X^1}\mathcal{Y}\theta^n||) = 0;$$

$$\lim_{n\to\infty} G(||J_{X^2}y^n - J_{X^2}\mathcal{R}\delta^n||) = 0. \tag{3.16}$$

Applying the property of G and Lemma 2.4, we obtain that:

$$\lim_{n\to\infty}||J_{X^1}x^n - J_{X^1}\mathcal{Y}\theta^n|| = 0 \quad \text{and} \quad \lim_{n\to\infty}||x^n - \mathcal{Y}\theta^n|| = 0, \tag{3.17}$$

$$\lim_{n\to\infty}||J_{X^2}y^n - J_{X^2}\mathcal{R}\delta^n|| = 0 \quad \text{and} \quad \lim_{n\to\infty}||y^n - \mathcal{R}\delta^n|| = 0. \tag{3.18}$$

From equations (1.7), (3.16) and Lemma 2.4, we have that:

$$||u^n - \theta^n|| \leq \frac{\mu}{a}||U||||Uu^n - Vv^n|| \to 0; \quad ||v^n - \delta^n|| \leq \frac{\mu}{a}||V||||Uu^n - Vv^n|| \to 0. \tag{3.19}$$

Also, from equations (1.7), (3.17), (3.18) and Lemma 2.4, we have:

$$\begin{aligned} ||w^n - y^n|| &\leq \frac{(1-\beta)}{a}||J_{X^2}\mathcal{R}\delta^n - J_{X^2}y^n|| \to 0 \text{ as } n\to\infty \\ ||z^n - x^n|| &\leq \frac{(1-\beta)}{a}||J_{X^1}\mathcal{Y}\theta^n - J_{X^1}x^n|| \to 0 \text{ as } n\to\infty. \end{aligned} \tag{3.20}$$

Now, we show that $\lim_{n\to\infty}\psi(u^n,x^n) = 0$ and $\lim_{n\to\infty}\psi(v^n,y^n) = 0$.

Since $(p,q)\in\mathcal{F}$, $u_n = Q_r x_n$ and $v_n = Q_r y_n$, by a result of Zhang [21], we have that:

$$\psi(p,u^n) \leq \psi(p,x^n) - \psi(u^n,x^n) \quad \text{and} \quad \psi(q,v^n) \leq \psi(q,y^n) - \psi(v^n,y^n). \tag{3.21}$$

From inequalities (3.7), (3.8) and (3.21), we have that:

$$\begin{aligned} \psi(p,z^n) &\leq \beta\psi(p,x^n) + (1-\beta)\big[\psi(p,u^n) - 2\mu\langle U(\theta^n - p), e^n\rangle\big] \\ &\leq \psi(p,x^n) - (1-\beta)\psi(u^n,x^n) - 2\mu(1-\beta)\langle U(\theta^n - p), e^n\rangle. \end{aligned} \tag{3.22}$$

Similarly, we obtain that:

$$\psi(q,w^n) \leq \psi(q,y^n) - (1-\beta)\psi(v^n,y^n) + 2\mu(1-\beta)\langle V(\delta^n - q), e^n\rangle. \tag{3.23}$$

Utilizing inequalities (3.22), (3.23), (3.12), (3.13) and Step 2, we obtain that:

$$\psi(u^n,x^n)+\psi(v^n,y^n) \leq \frac{1}{(1-\beta)}\big(\psi(p,x^n)-\psi(p,z^n)+\psi(q,y^n)-\psi(q,w^n)\big) \to 0 \text{ as } n\to\infty.$$

By Lemma 2.3, we obtain that:

$$\lim_{n\to\infty}||x_n - u_n|| = 0 \text{ and } \lim_{n\to\infty}||y_n - v_n|| = 0. \tag{3.24}$$

Step 4. We show that $(x^*, y^*)\in\mathcal{F}$ and $Ux^* = Vy^*$. From equations (3.16) and (3.24), we obtain that:

$$Ux^* = Vy^*. \tag{3.25}$$

Using equations (3.19), (3.24) and (3.17), we have that:

$$||\theta^n - \mathcal{Y}\theta^n|| \leq ||\theta^n - u^n|| + ||u^n - x^n|| + ||x^n - \mathcal{Y}\theta^n|| \to 0 \text{ as } n\to\infty. \tag{3.26}$$

Similarly, using equations (3.19), (3.24) and (3.18), we have that:

$$||\delta^n - \mathcal{R}\delta^n|| \leq ||\delta^n - v^n|| + ||v^n - y^n|| + ||y^n - \mathcal{R}\delta^n|| \to 0 \text{ as } n\to\infty. \tag{3.27}$$

Since, $\mathcal{Y}$, $\mathcal{R}$ are closed and the fact that $\lim_{n\to\infty} \theta^n = x^*$, $\lim_{n\to\infty} \delta^n = y^*$, we conclude that $(x^*, y^*) \in F(\mathcal{Y})\times F(\mathcal{R})$. This together with equation (3.25) implies that $(x^*, y^*) \in SEFPP$. Furthermore, from equation (1.7), $u_n = Q_r x_n$ and $v_n = Q_r y_n$. By a result of Zhang [21], we have that:

$$F(u^n, w) + \frac{1}{r}\langle w - u^n, Ju^n - Jx^n\rangle \geq 0,\ \forall\, w \in X^1, \tag{3.28}$$

where $F(u^n, w) = f(u^n, w) + \chi(w) - \chi(u^n) + \langle \mathcal{A}u^n, w - u^n\rangle$. By condition (A_2), we have that $\frac{1}{r}\langle w - u^n, Ju^n - Jx^n\rangle \geq F(w, u^n)$. Since $w \mapsto F(u, w)$ is convex and lower semi-continuous, applying equation (3.24), we obtain from the above inequality that $0 \geq F(w, x^*),\ \forall\, w \in X^1$. For $\lambda \in (0, 1]$, $w \in X^1$, let $w^\lambda = \lambda w + (1-\lambda)x^* \in X^1$. Hence, $0 \geq F(w^\lambda, x^*),\ \forall\, w \in X^1$. By condition (A_1), we have that

$$0 \;=\; F(w^\lambda, w^\lambda) \leq \lambda F(w^\lambda, w) + (1-\lambda)F(w^\lambda, x^*) \leq F(x^* + \lambda(w - x^*), w)$$

Letting $\lambda \downarrow 0$, by condition (A_3), we obtain that:

$$F(w, x^*) \geq 0,\ \forall\, w \in H_1. \tag{3.29}$$

This implies that $x^* \in GMEP(f, \mathcal{A}, \chi)$. Similarly, we also have that $y^* \in GMEP(g, \mathcal{B}, \varphi)$. These together with equation (3.25) imply that $(x^*, y^*) \in SGMEEP$. Hence, we conclude that $(x^*, y^*) \in \mathcal{F}$. This completes the proof. □

3.1. **Conclusion.** In Theorem 3.1, it is proved that the sequence of Algorithm (1.7) converges strongly to a solution of $(SEFPP) \cap (SGMEEP)$ in 2-uniformly convex and uniformly smooth real Banach spaces. Moreover, no compactness-type condition is imposed on any of the operators involved. The theorem proved is an improvement on the result of Chidume *et al.* [13] in the sense that $(SGMEEP)$ was not studied. The result of Zhaoli *et al.* [17] is a special case of Theorem 3.1 in which X is a real Hilbert space. Furthermore, the compactness-type condition imposed on the operators and the fact that $\{r_n\} \subset [0, \infty)$ is such that $\lim |r_{n+1} - r_n| = 0$ in the theorem of Zhaoli *et al.* [18] were dispensed with in Theorem 3.1. The class of $(SMEEP)$ studied in the theorem of Zhaoli *et al.* [18] is contained in the class of $(SGMEEP)$ considered in Theorem 3.1. Finally, Algorithm (1.7) studied is slightly different from the algorithm considered by Zhaoli *et al.* [18].

3.2. **Competing interest.** The author declares that he has no conflict of interest.

Acknowledgements. The author thank the anonymous referees for their very useful remarks which helped to improve the final version of this paper.

References

[1] Alber, Y., *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Applications of Nonlininear Operators of Accretive and Monotone Type (A. G. Kartsatos, Ed.), Marcel Dekker, New York (1996), 15–50

[2] Alber, Y. and Ryazantseva, I., *Nonlinear Ill Posed problemss of Monotone Type*, Springer, London, UK, 2006

[3] Attouch, H., Bolte, J. and Redont, P., et al., *Alternating proximal algorithms for weakly coupled minimization problems, applications to dynamical games and PDE's*, J. Convex Anal., **15** (2008), No. 3, 485–506

[4] Blum, E. and Oettli, W., *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994), No. 1-4, 123–145

[5] Bnouhachem, A., *A modified projection method for a common solution of a system of variational inequalities, a split equilibrium problem and a hierarchical fxed-point problem*, Fixed Point Theory Appl., 2014, 2014:22, 25

[6] Byrne, C., *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl., **20** (2004), No. 1, 103–120

[7] Byrne, C., *Iterative oblique projection onto convex subsets and the split feasibility problem*, Inverse Probl., **18** (2002), 441–453

[8] Censor, Y., Bortfeld, T. and Marti, B., et al., *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol., **51** (2006), 2353–2365

[9] Censor, Y. and Segal, A., *The split common fixed point problem for directed operators*, J. Convex Anal., **16** (2009), 587–600

[10] Chang, S. S., Wang, L. and Zhao, Y., *Strongly convergent iterative methods for split equality variational inclusion problems in Banach spaces*, Acta Math. Sci. Ser. B (Engl. Ed.), **36** (2016), No. 6, 1641–1650

[11] Chang, S. S., Lee H. W. and Chan C. K., *A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization*, Nonlinear Anal., **70** (2009), 3307–3319

[12] Chidume, C. E. and Nnakwe, M. O., *A new Halpern-type algorithm for a generalized mixed equilibrium problem and a countable family of generalized-J-nonexpansive maps, with applications*, Carpathian J. Math., **34** (2018), No. 2, 191–198

[13] Chidume, C. E., Romanus, O. M. and Nnyaba, U. V., *An iterative algorithm for solving split equilibrium problems and split equality variational inclusions for a class of nonexpansive-type maps*, Optimization, DOI: 10.1080/02331934.2018.1503270

[14] Chidume, C. E., Ndambomve, P. and Bello, A. U., *The split equality fixed point problem for demi-contractive mappings*, J. Nonlinear Anal. Optim., **6** (2015), No. 1, 61–69

[15] Giang, D. M., Strodiot, J. J. and Nguyen, V. H., *Strong convergence of an iterative method for solving the multiple-set split equality fixed point problem in a real Hilbert space*, RACSAM. 2017, doi:10.1007/s13398-016-0338-7

[16] Kamimura S. and Takahashi W., *Strong convergence of a proximal-type algorithm in a Banach space*, SIAMJ. Optim., **13** (2002), No. 3, 938–945

[17] Ma, Z., Wang, L., Chang, S.S. and Duan, W., *Convergence theorems for split equality mixed equilibrium problems with applications*, Fixed Point Theory Appl., 2015, 2015:31, 18 pp.

[18] Ma, Z., Wang, L. and Cho, Y. J., *Some results for split equality equilibrium problems in Banach Spaces*, Symmetry, **11** (2019), No. 2, 194

[19] Moudafi, A., *Alternating CQ-algorithm for convex feasibility and split fixed-point problems*, J. Nonlinear Convex Anal., **15** (2014), No. 4, 809–818

[20] Xu, H. K., *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16** (1991), No. 12, 1127–1138

[21] Zhang, S., *Generalized mixed equilibrium problems in Banach spaces* , Appl. Math. Mech. -Engl. Ed., **30** (2009), No. 9,1105-1112 DOI: 10.1007/s10483-009-0904-6

[22] Zhao, J., *Solving split equality fixed-point problem of quasi-nonexpansive mappings without prior knowledge of operators norms*, Optim. 2014. doi:10.1080/02331934.2014.883515.

INSTITUTE OF MATHEMATICS
AFRICAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
ABUJA, KM 10 AIRPORT ROAD, FCT, GALADIMAWA, NIGERIA
E-mail address: mondaynnakwe@gmail.com