CARPATHIAN J. MATH. Online version at https://www.carpathian.cunbm.utcluj.ro
36 (2020), No. 1, 127 - 139 Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Dedicated to Prof. Hong-Kun Xu on the occasion of his 60^{th} anniversary

Modified inertial double Mann type iterative algorithm for a bivariate weakly nonexpansive operator

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ABSTRACT. We introduce a modified inertial double Mann type iterative method to approximate coupled solutions of a bivariate nonexpansive operator $T : C \times C \to C$, where C is a nonempty closed and convex subset of a Hilbert space. The one theorem and complement important old and recent results in coupled fixed point theory. Some appropriate examples to illustrate our results and their generalization are also given.

1. INTRODUCTION

Let X be an arbitrary nonempty set. A fixed point for a nonexpansive mapping $T : X \to X$ is a point $x \in X$ such that $Tx = x$. The fixed point theorems are very important in dealing with problems arising in approximation theory, mathematical economics, theory of differential equations, theory of integral equations, theory of matrix equations, game theory, etc. (see, e.g., [13, 12, 22, 30, 18, 40]).

A significant body of work on iteration methods for fixed points problems has accumulated in literature (for example, see [34, 19, 20, 39]). Specifically, the Mann algorithm [24, 25]:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \tag{1.1}$$

for some suitably chosen scalars $\alpha_n \in [0, 1]$. The iterative sequence $\{x_n\}$ converges weakly to a fixed point of T provided that $\alpha_n \in [0, 1]$ satisfies

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty. \tag{1.2}$$

Mainge [26] introduced the following inertial Mann algorithm by unifying the Mann algorithm and the inertial extrapolation:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = w_n + \lambda_n[T(w_n) - w_n], \end{cases} \tag{1.3}$$

for each $n \geq 1$, $\alpha_n \in [0, \alpha)$ for any $\alpha \in [0, 1)$.

Sakurai and Liduka [37] first proposed an acceleration of the Halpern algorithm to search for a fixed point of a nonexpansive mapping. Inspired by their work, by combining the Mann algorithm (1.1) and conjugate gradient methods [29], the authors [15] proposed

Received: 29.05.2019. In revised form: 30.06.2019. Accepted: 14.07.2019
2010 *Mathematics Subject Classification.* 47H09, 47H10, 49M05.
Key words and phrases. *Inertial Mann method, coupled solutions, bivariate nonexpansive and weekly nonexpansive.*
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the following accelerated Mann algorithm:

$$\begin{cases} d_{n+1} = \frac{1}{\lambda}(T(x_n) - x_n) + \beta_n d_n, \\ z_n = x_n + \lambda d_{n+1}, \\ x_{n+1} = \mu\gamma_n x_n + (1 - \mu\gamma_n) z_n, \end{cases} \tag{1.4}$$

for each $n \geq 0$, where $\mu \in (0, 1]$ and $\lambda > 0$.

Let X be a nonempty set. A pair $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if it is a solution of the system:

$$F(x, y) = x, \quad F(y, x) = y.$$

Berinde et al. [6] introduced a double Krasnoselskij-type algorithm for approximately coupled fixed points of F a bivariate weakly nonexpansive operators :

$$x_{n+1} = \lambda x_n + (1 - \lambda) F(x_n, x_n), \tag{1.5}$$

where $\lambda \in (0, 1)$.

The coupled fixed point results for contractive type mappings have important applications in nonlinear analysis and have been applied successfully for solving various classes of nonlinear functional equations: integral equations and systems of integral equations [1, 10, 17, 38]; nonlinear Hammerstein integral equations [35]; nonlinear matrix and nonlinear quadratic equations [2, 10] initial value problems for ODE [3, 36], etc.

In this paper, motivated and inspired by the work of Dong and Yuan [15], we propose an inertial extrapolation algorithm to approximate coupled solutions of a bivariate nonexpansive operator.

2. Nonexpansive bivariate operators

We define the concept of nonexpansiveness for bivariate mappings as follows.

Definition 2.1. [6] Let X be a normed linear space and C be a subset of X. A mapping $F : C \times C \to X$ is called weakly nonexpansive if

$$\|F(x, y) - F(u, v)\| \leq a\|x - u\| + b\|y - v\|, \tag{2.6}$$

for all $x, y, u, v \in C$, where $a, b \geq 0$ and $a + b \leq 1$.

Definition 2.2. [31] Let X be a normed linear space and C be a subset of X. A mapping $F : C \times C \to X$ is called nonexpansive if

$$\|F(x, y) - F(u, v)\| \leq \frac{1}{2}(\|x - u\| + \|y - v\|), \tag{2.7}$$

for all $x, y, u, v \in C$.

Note that condition (2.6) is more general than (2.7): any nonexpansive mapping F is weakly nonexpansive but the converse is not true, in general, as shown below.

Example 2.1. Let $X = \mathbb{R}$ (with the usual metric) and $F : X \times X \to X$ be defined by

$$F(x, y) = \frac{x - 3y}{4}, \quad x, y \in X.$$

Then F satisfies condition (2.6) with the constants $a = \frac{1}{4}$ and $b = \frac{3}{4}$ but does not satisfy condition (2.7). Moreover, F possesses a unique coupled fixed point of the form (x, x), i.e., $(0, 0)$, but no coupled fixed point theorem established in [7, 8, 9, 11, 14, 21, 23] (and in other related papers) can be applied to this function F.

Definition 2.3. [33] A mapping $F : C \times C \to H$ is called demicompact if it has the property that whenever $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C with the property that $\{F(u_n, v_n) - u_n\}$ and $\{F(v_n, u_n) - v_n\}$ converge strongly to 0, then there exists a subsequence $\{(u_{n_k}, v_{n_k})\}$ of $\{(u_n, v_n)\}$ such that $u_{n_k} \to u$ and $v_{n_k} \to v$ strongly.

Theorem 2.1. [6] *Let C be a bounded, closed and convex subset of a Hilbert space H and let $F : C \times C \to C$ be a (weakly) nonexpansive operator. Then F has at least one coupled fixed point in C.*

Lemma 2.1. [4] *Let $\{\delta_n\}$ and $\{\delta_n\}$ be nonnegative sequences satisfying $\sum_{n=1}^{\infty} \varphi_n < \infty$ and $\delta_{n+1} \leq \delta_n + \varphi_n$, $n = 1, 2, \ldots$. Then, $\{\delta_n\}$ is a convergent sequence.*

Lemma 2.2. [32] *Let $\{x_n\}$ be a sequence of elements of the Hilbert space H which converges weakly to $x \in H$. Then we have $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - x\|, \quad \forall\, y \in H,\, y \neq x$.*

Lemma 2.3. [5] *Let X be a real inner product space. Then the following statements hold.*

(a) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall\, x, y \in X$;
(b) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \quad \forall\, x, y \in X$;
(c) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2, \quad \forall x, y \in X, \forall \alpha \in \mathbb{R}$.

3. Modified inertial double Mann type iterative

In this section, let C be a bounded, closed and convex subset of a Hilbert space H and let $F : C \times C \to C$ be weakly nonexpansive and demicompact operator.

Algorithm 1. Initialization: *Choose $\mu \in (0, 1]$, $\beta_n \in [0, \infty)$ and $\gamma_n \in (0, 1)$.*
Step 1: *Given $\lambda > 0$ and $x_0 = x_1 \in C$, $y_0 = y_1 \in C$, compute*

$$d_1^x = \frac{F(x_1, y_1) - x_1}{\lambda} \quad \text{and} \quad d_1^y = \frac{F(y_1, x_1) - y_1}{\lambda}.$$

Step 2: *Given $x_{n-1}, x_n, y_{n-1}, y_n \in C$, choose $\tau_n > 0$ and α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, $0 \leq \alpha < 1$, where*

$$\bar{\alpha}_n := \begin{cases} \min\left\{\alpha,\ \max\left\{\frac{\tau_n}{\|x_n - x_{n-1}\|}, \frac{\tau_n}{\|y_n - y_{n-1}\|}\right\}\right\} & \text{if } x_n \neq x_{n-1},\, y_n \neq y_{n-1}, \\ \alpha & \text{otherwise,} \end{cases}$$

Set $n = 1$ and compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}) \quad \text{and} \quad u_n = y_n + \alpha_n(y_n - y_{n-1}).$$

Step 3: *Given $w_n, u_n \in C \times C$ and $d_n^x, d_n^y \in C$, compute $d_{n+1}^x, d_{n+1}^y \in C$ by*

$$d_{n+1}^x = \frac{1}{\lambda}(F(w_n, u_n) - w_n) + \beta_n d_n^x \quad \text{and} \quad d_{n+1}^y = \frac{1}{\lambda}(F(u_n, w_n) - u_n) + \beta_n d_n^y.$$

Compute x_{n+1} as follows:

$$\begin{cases} z_n = w_n + \lambda d_{n+1}^x, \\ x_{n+1} = \mu\gamma_n w_n + (1 - \mu\gamma_n) z_n, \\ v_n = u_n + \lambda d_{n+1}^y, \\ y_{n+1} = \mu\gamma_n u_n + (1 - \mu\gamma_n) v_n. \end{cases}$$

Set $n := n + 1$ and go to Step 2.

Denote by $\mathrm{CFix}(F)$, the set of all coupled fixed points of F with equal components, i.e., $\mathrm{CFix}(F) = \{(x^*, y^*) \in C : F(x^*, y^*) = x^*,\ F(y^*, x^*) = y^*\}$.

Assumption 1. *The sequences $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\tau_n\}$ defined in Algorithm 1 satisfies*

(A1) $\sum_{n=1}^{\infty} \beta_n < \infty$;
(A2) $\sum_{n=1}^{\infty} \tau_n < \infty$;
(A3) $\sum_{n=1}^{\infty} \gamma_n = \infty$.

Moreover, sequence $\{(w_n, u_n)\}$ *and* $\{(u_n, w_n)\}$ *are satisfied*

(A4) $\{F(w_n, u_n) - w_n\}$ *is bounded*;
(A5) $\{F(u_n, w_n) - u_n\}$ *is bounded*;
(A6) $\{F(w_n, u_n) - x^*\}$ *is bounded for any* $x^* \in \mathrm{Fix}(F)$;
(A7) $\{F(u_n, w_n) - y^*\}$ *is bounded for any* $y^* \in \mathrm{Fix}(F)$.

Theorem 3.2. *Suppose that* $F : C \times C \to C$ *be weakly nonexpansive and demicompact operator with* $\mathrm{CFix}(F) \neq \emptyset$ *and suppose Assumption 1 holds. Then the sequence* $\{(x_n, y_n)\}$ *converges weakly to a coupled fixed point of* F.

Proof. It follows from (A1) that $\lim_{n\to\infty} \beta_n = 0$. This implies that there exists $n_0 \in \mathbb{N}$ such that $\beta_n \leq \frac{1}{2}$ for all $n \geq n_0$. Define a number

$$M_1^x := \max\left\{ \max_{1\leq k\leq n_0} \|d_k^x\|, \frac{2}{\lambda} \sup_{n\in\mathbb{N}} \|F(w_n, u_n) - w_n\| \right\}.$$

Then (A4) implies that $M_1^x < \infty$. Assume that $\|d_n^x\| \leq M_1^x$ for some $n \geq n_0$. From $d_{n+1}^x = \frac{1}{\lambda}(F(w_n, u_n) - w_n) + \beta_n d_n^x$, we have

$$\|d_{n+1}^x\| = \|\frac{1}{\lambda}(F(w_n, u_n) - w_n) + \beta_n d_n^x\| \leq \|\frac{1}{\lambda}(F(w_n, u_n) - w_n)\| + \beta_n \|d_n^x\| \leq M_1^x.$$

Hence, $\|d_{n+1}^x\| \leq M_1^x$ for all $n \geq n_0$. So, $\{d_n^x\}$ is bounded.
Similarly, we have $\{d_n^y\}$ is bounded.
Note that $\alpha_n\|x_n - x_{n-1}\| \leq \tau_n$ and $\alpha_n\|y_n - y_{n-1}\| \leq \tau_n$ this implies that

$$\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\| \leq \sum_{n=1}^{\infty} \tau_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n \|y_n - y_{n-1}\| \leq \sum_{n=1}^{\infty} \tau_n < \infty. \tag{3.8}$$

Denote $w_n := x_n + \alpha_n(x_n - x_{n-1})$ and $u_n := y_n + \alpha_n(y_n - y_{n-1})$ for each $n \geq 1$. Then

$$\begin{aligned} x_{n+1} &= \mu\gamma_n w_n + (1 - \mu\gamma_n)(F(w_n, u_n) + \lambda\beta_n d_n^x) \\ &= w_n + (1 - \mu\gamma_n)(F(w_n, u_n) - w_n + \lambda\beta_n d_n^x)) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} y_{n+1} &= \mu\gamma_n u_n + (1 - \mu\gamma_n)(F(u_n, w_n) + \lambda\beta_n d_n^y) \\ &= u_n + (1 - \mu\gamma_n)(F(u_n, w_n) - u_n + \lambda\beta_n d_n^y). \end{aligned} \tag{3.10}$$

Let $(x^*, y^*) \in \mathrm{CFix}(F)$ and (3.9) and Lemma 2.3(a), we have

(3.11)

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|w_n - x^* + (1-\mu\gamma_n)(F(w_n,u_n) - w_n + \lambda\beta_n d_n^x)\|^2 \\
&= \|w_n - x^* + (1-\mu\gamma_n)(F(w_n,u_n) - w_n) + (1-\mu\gamma_n)\lambda\beta_n d_n^x\|^2 \\
&\le \|w_n - x^* + (1-\mu\gamma_n)(F(w_n,u_n) - w_n)\|^2 + 2(1-\mu\gamma_n)\langle \lambda\beta_n d_n^x, x_{n+1} - x^*\rangle \\
&= \|w_n - x^*\|^2 + (1-\mu\gamma_n)^2\|F(w_n,u_n) - w_n\|^2 \\
&\quad + 2(1-\mu\gamma_n)\langle w_n - x^*, F(w_n,u_n) - w_n\rangle \\
&\quad + 2(1-\mu\gamma_n)\langle \lambda\beta_n d_n^x, x_{n+1} - x^*\rangle \\
&\le \|w_n - x^*\|^2 + (1-\mu\gamma_n)\|F(w_n,u_n) - w_n\|^2 \\
&\quad + 2(1-\mu\gamma_n)\langle w_n - F(w_n,u_n), F(w_n,u_n) - w_n\rangle \\
&\quad + 2(1-\mu\gamma_n)\langle F(w_n,u_n) - x^*, F(w_n,u_n) - w_n\rangle \\
&\quad + 2(1-\mu\gamma_n)\langle \lambda\beta_n d_n^x, x_{n+1} - x^*\rangle \\
&\le \|w_n - x^*\|^2 - (1-\mu\gamma_n)\|F(w_n,u_n) - w_n\|^2 \\
&\quad + 2(1-\mu\gamma_n)\left[\|F(w_n,u_n) - x^*\|\|F(w_n,u_n) - w_n\| + \lambda\beta_n\|d_n^x\|\|x_{n+1} - x^*\|\right] \\
&= \|w_n - x^*\|^2 - (1-\mu\gamma_n)\|F(w_n,u_n) - w_n\|^2 + \phi_n^x,
\end{aligned}$$

where $\phi_n^x := 2(1-\mu\gamma_n)\left[\|F(w_n,u_n) - x^*\|\|F(w_n,u_n) - w_n\| + \lambda\beta_n\|d_n^x\|\|x_{n+1} - x^*\|\right]$. Using (A1), (A4) and (A5), it follow that $\{\phi_n^x\}$ is bounded. Thus there exists $M_2^x > 0$ such that $\phi_n^x \le M_2^x$ for all $n \ge 1$.

Following similar process as in (3.11), we get

(3.12)

$$\begin{aligned}
\|y_{n+1} - y^*\|^2 &= \|u_n - y^* + (1-\mu\gamma_n)(F(u_n,w_n) - u_n + \lambda\beta_n d_n^y)\|^2 \\
&\le \|u_n - y^*\|^2 - (1-\mu\gamma_n)\|F(u_n,w_n) - u_n\|^2 \\
&\quad + 2(1-\mu\gamma_n)\left[\|F(u_n,w_n) - y^*\|\|F(u_n,w_n) - u_n\| + \lambda\beta_n\|d_n^y\|\|y_{n+1} - y^*\|\right] \\
&= \|u_n - y^*\|^2 - (1-\mu\gamma_n)\|F(u_n,w_n) - u_n\|^2 + \phi_n^y,
\end{aligned}$$

where $\phi_n^y := 2(1-\mu\gamma_n)\left[\|F(u_n,w_n) - y^*\|\|F(u_n,w_n) - u_n\| + \lambda\beta_n\|d_n^y\|\|y_{n+1} - y^*\|\right]$. Using (A2), (A5) and (A6), it follow sthat $\{\phi_n^y\}$ is bounded. Thus there exists $M_2^y > 0$ such that $\phi_n^y \le M_2^y$ for all $n \ge 1$.

Using Lemma 2.3(c), we get

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - x^*\|^2 \\
&= \|(1+\alpha_n)(x_n - x^*) - \alpha_n(x_{n-1} - x^*)\|^2 \qquad (3.13)\\
&= (1+\alpha_n)\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\|u_n - y^*\|^2 &= \|y_n + \alpha_n(y_n - y_{n-1}) - y^*\|^2 \\
&= \|(1+\alpha_n)(y_n - y^*) - \alpha_n(y_{n-1} - y^*)\|^2 \qquad (3.14)\\
&= (1+\alpha_n)\|y_n - y^*\|^2 - \alpha_n\|y_{n-1} - y^*\|^2 + \alpha_n(1+\alpha_n)\|y_n - y_{n-1}\|^2.
\end{aligned}$$

Using (3.13) in (3.11), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 - (1+\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x_{n-1} - x^*\|^2 \qquad (3.15)\\
&\le -(1-\mu\gamma_n)\|F(w_n,u_n) - w_n\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 + \phi_n^x.
\end{aligned}$$

Using (3.14) in (3.12), we obtain

$$\begin{aligned} &\|y_{n+1}-y^*\|^2-(1+\alpha_n)\|y_n-y^*\|^2+\alpha_n\|y_{n-1}-y^*\|^2 \\ &\le -(1-\mu\gamma_n)\|F(u_n,w_n)-u_n\|^2+\alpha_n(1+\alpha_n)\|y_n-y_{n-1}\|^2+\phi_n^y. \end{aligned} \tag{3.16}$$

Observe that

$$\begin{aligned} \alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2 &= \alpha_n\|x_n-x_{n-1}\|\Big((1+\alpha_n)\|x_n-x_{n-1}\|\Big) \\ &\le \alpha_n\|x_n-x_{n-1}\|M_3^x, \end{aligned} \tag{3.17}$$

where $M_3^x := \sup_{n\ge 1}\Big((1+\alpha_n)\|x_n-x_{n-1}\|\Big)$ and

$$\begin{aligned} \alpha_n(1+\alpha_n)\|y_n-y_{n-1}\|^2 &= \alpha_n\|y_n-y_{n-1}\|\Big((1+\alpha_n)\|y_n-y_{n-1}\|\Big) \\ &\le \alpha_n\|y_n-y_{n-1}\|M_3^y, \end{aligned} \tag{3.18}$$

where $M_3^y := \sup_{n\ge 1}\Big((1+\alpha_n)\|y_n-y_{n-1}\|\Big)$.

Observe that M_3^x exists and M_3^y exists. Since $\{x_n\}$ and $\{y_n\}$ are bounded. Thus,

$$\begin{aligned} \sum_{n=1}^{\infty}\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2 &\le \sum_{n=1}^{\infty}\alpha_n\|x_n-x_{n-1}\|M_3^x<\infty, \\ \sum_{n=1}^{\infty}\alpha_n(1+\alpha_n)\|y_n-y_{n-1}\|^2 &\le \sum_{n=1}^{\infty}\alpha_n\|y_n-y_{n-1}\|M_3^y<\infty. \end{aligned} \tag{3.19}$$

Hence,

$$\lim_{n\to\infty}\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|=0 \quad \text{and} \quad \lim_{n\to\infty}\alpha_n(1+\alpha_n)\|y_n-y_{n-1}\|=0. \tag{3.20}$$

From (3.15), we get

$$\begin{aligned} &(1-\mu\gamma_n)\|F(w_n,u_n)-w_n\|^2 \\ &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n(\|x_n-x^*\|^2-\|x_{n-1}-x^*\|^2) \\ &\quad +\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2+\phi_n^x. \end{aligned} \tag{3.21}$$

From (3.16), we get

$$\begin{aligned} &(1-\mu\gamma_n)\|F(u_n,w_n)-u_n\|^2 \\ &\le \|y_n-y^*\|^2-\|y_{n+1}-y^*\|^2+\alpha_n(\|y_n-y^*\|^2-\|y_{n-1}-y^*\|^2) \\ &\quad +\alpha_n(1+\alpha_n)\|y_n-y_{n-1}\|^2+\phi_n^y. \end{aligned} \tag{3.22}$$

Observe that $\lim_{n\to\infty}\phi_n^x=0$, $\lim_{n\to\infty}\phi_n^y=0$ in view of $\sum_{n=1}^{\infty}\beta_n<\infty$, $\{x_n\}$ and $\{y_n\}$ are bounded. Hence, we get from (3.21) and (3.22) that

$$\sum_{n=1}^{\infty}(1-\mu\gamma_n)\|F(w_n,u_n)-w_n\|^2<\infty \quad \text{and} \quad \sum_{n=1}^{\infty}(1-\mu\gamma_n)\|F(u_n,w_n)-u_n\|^2<\infty.$$

Thus,

$$\liminf_{n\to\infty}\|F(w_n,u_n)-w_n\|=0 \quad \text{and} \quad \liminf_{n\to\infty}\|F(u_n,w_n)-u_n\|=0.$$

Furthermore,

$$\begin{aligned} \|w_n-x_n\| &= \alpha_n\|x_n-x_{n-1}\|\to 0 \text{ as } n\to\infty, \\ \|u_n-y_n\| &= \alpha_n\|y_n-y_{n-1}\|\to 0 \text{ as } n\to\infty, \end{aligned} \tag{3.23}$$

since $\sum_{n=1}^{\infty}\alpha_n\|x_n-x_{n-1}\|<\infty$ and $\sum_{n=1}^{\infty}\alpha_n\|y_n-y_{n-1}\|<\infty$.

We show that $\lim_{n\to\infty}\|F(w_n,u_n)-w_n\|=0$. Observe from (3.9), we have

$$\begin{aligned}\|F(w_n,u_n)-w_n\| &= \left\|\frac{x_{n+1}-w_n}{1-\mu\gamma_n}-\lambda\beta_n d_n^x\right\|\\ &= \left\|\frac{x_{n+1}-x_n-\alpha(x_n-x_{n-1})}{1-\mu\gamma_n}-\lambda\beta_n d_n^x\right\|\\ &\leq \frac{\|x_{n+1}-x_n\|+\alpha\|x_n-x_{n-1}\|}{1-\mu\gamma_n}+\lambda\beta_n\|d_n^x\|.\end{aligned} \tag{3.24}$$

Thus, by (A1), $\{d_n^x\}$ is bounded and (3.8), we obtain

$$\lim_{n\to\infty}\|F(w_n,u_n)-w_n\|=0. \tag{3.25}$$

Next, we show that $\lim_{n\to\infty}\|F(u_n,w_n)-u_n\|=0$. Observe from (3.9), we have

$$\begin{aligned}\|F(u_n,w_n)-u_n\| &= \left\|\frac{y_{n+1}-y_n}{1-\mu\gamma_n}-\lambda\beta_n d_n^y\right\|\\ &= \left\|\frac{y_{n+1}-y_n-\alpha(y_n-y_{n-1})}{1-\mu\gamma_n}-\lambda\beta_n d_n^y\right\|\\ &\leq \frac{\|y_{n+1}-y_n\|+\alpha\|y_n-y_{n-1}\|}{1-\mu\gamma_n}+\lambda\beta_n\|d_n^y\|.\end{aligned} \tag{3.26}$$

Thus, by (A1), $\{d_n^y\}$ is bounded and (3.8), we obtain

$$\lim_{n\to\infty}\|F(u_n,w_n)-u_n\|=0. \tag{3.27}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$, $\{y_{n_k}\}$ of $\{y_n\}$ and $(x,y)\in C\times C$ such that $x_{n_k}\rightharpoonup x$ and $y_{n_k}\rightharpoonup y$. By (3.23), we get $w_{n_k}\rightharpoonup x$ and $u_{n_k}\rightharpoonup y$. Applying Lemma 2.3(b), we obtain

$$\|F(x,y)-x_n\|^2=\|F(x,y)-x\|^2+\|x-x_n\|^2+2\langle F(x,y)-x,x-x_n\rangle.$$

Since, $\{x_n\}$ convergent weakly to x, we obtain

$$\lim_{n\to\infty}\langle F(x,y)-x,x-x_n\rangle=0.$$

Hence,

$$\limsup_{n\to\infty}\|F(x,y)-x_n\|^2=\|F(x,y)-x\|^2+\limsup_{n\to\infty}\|x-x_n\|^2.$$

Also , using the condition of coupled weakly nonexpansive , we get

$$\lim_{n\to\infty}\|F(x,y)-x_n\|\leq\|F(x,y)-F(x_n,y_n)\|+\lim_{n\to\infty}\|F(x_n,y_n)-x_n\|^2\leq\|x-x_n\|.$$

Thus, we have

$$\|F(x,y)-x\|^2+\limsup_{n\to\infty}\|x-x_n\|^2\leq\limsup_{n\to\infty}\|x-x_n\|^2.$$

Therefore,

$$\|F(x,y)-x\|^2=0.$$

Therefore, we get $F(x,y)=x$ and similarly we can prove that $F(y,x)=y$. Thus, $(x,y)\in \mathrm{CFix}(F)$.

Finally, we claim that $x_n\rightharpoonup x$ and $y_n\rightharpoonup y$. On the contrary, assume that there are subsequence $\{x_{m_k}\}$ of $\{x_n\}$ such that $x_{m_k}\rightharpoonup\overline{x}$ as $k\to\infty$ and subsequence $\{y_{l_k}\}$ of $\{y_n\}$ such that $y_{l_k}\rightharpoonup\overline{y}$ as $k\to\infty$, as well as $(x,y)\neq(\overline{x},\overline{y})$.

By the Lemma 2.2 (Opial lemma), we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n - x\|^2 &= \lim_{k\to\infty}\|x_{n_k} - x\|^2 = \liminf_{k\to\infty}\|x_{n_k} - x\|^2\\ &< \liminf_{k\to\infty}\|x_{n_k} - \overline{x}\|^2\\ &= \lim_{k\to\infty}\|x_{n_k} - \overline{x}\|^2 = \lim_{n\to\infty}\|x_n - \overline{x}\|^2\\ &= \lim_{k\to\infty}\|x_{m_k} - \overline{x}\|^2 = \liminf_{k\to\infty}\|x_{m_k} - \overline{x}\|^2\\ &< \liminf_{k\to\infty}\|x_{m_k} - x\|^2\\ &= \lim_{k\to\infty}\|x_{m_k} - x\|^2 = \lim_{n\to\infty}\|x_n - x\|^2,\end{aligned}$$

which is a contradiction. Thus, $x_n \rightharpoonup x$. By the same method, we can prove $y_n \rightharpoonup y$. Hence $(x_n, y_n) \rightharpoonup (x, y) \in \mathrm{CFix}(F)$. □

4. Numerical experiments

Example 4.2. *Let $H = \mathbb{R}$, we consider $C = [-1, 1]$. Define a mapping $F : C \times C \to C$ by $F(x, y) = \frac{x^2-5y}{7}$ for all $x, y \in C$.*

Since $|x^2 - 5y - (u^2 - 5v)| \leq |x^2 - u^2| + 5|y - v|$ holds for all $x, y, u, v \in C$. Therefore, we have

$$\begin{aligned}|F(x,y) - F(u,v)| &= \left|\frac{x^2 - 5y}{7} - \frac{u^2 - 5v}{7}\right|\\ &\leq \frac{1}{7}(|x^2 - u^2| + 5|y - v|).\end{aligned}$$

Thus, F satisfies all the hypotheses of Theorem 3.2. Then there exists a coupled fixed point of F, i.e., $(0, 0)$. In Algorithm 1, we set $\alpha_n = 0.718$, $\lambda = 0.5$, $\mu = 1$, $\tau_n = 100n^{-\frac{3}{2}}, \gamma_n = 0.9$ and $\beta_n = \frac{1}{(n+1)^2}$. Then, we have results in Table 1, Table 2, Figure 1 and Figure 2.

TABLE 1. Result of Example 4.2 with initial point $(x_0, y_0) = (0.9, -0.9)$.

Algorithm 1			
Time	Iter	Approximation	$\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\|$
0.039870	1	(0.882321, -0.853393)	0.064286
	2	(0.850071, -0.785270)	0.100373
	3	(0.805367, -0.711875)	0.118099
	4	(0.750706, -0.641306)	0.125229
	5	(0.689107, -0.576913)	0.125993
	10	(0.377858, -0.340034)	0.094650
	20	(0.086018, -0.084588)	0.028891
	30	(0.015962, -0.015988)	0.006140
	35	(0.006466, -0.006466)	0.002617
	40	(0.002509, -0.002502)	0.001068
	45	(0.000925, -0.000926)	0.000417
	50	(0.000323, -0.000323)	0.000156
	55	(0.000104, -0.000104)	0.000055
	60	(0.000029, -0.000029)	0.000018
	65	(0.000006, -0.000006)	0.000005
	66	(0.000004, -0.000004)	0.000004
	67	(0.000003, -0.000003)	0.000003
	68	(0.000002, -0.000002)	0.000002
	69	(0.000001, -0.000001)	0.000002
	70	(0.000000, -0.000000)	0.000001
	71	(-0.000000, 0.000000)	0.000001

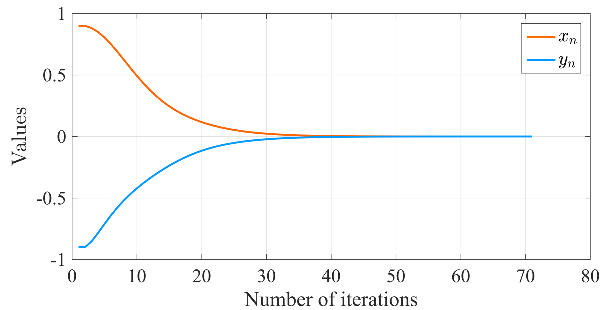

FIGURE 1. Result of Example 4.2 with initial point $(x_0, y_0) = (0.9, -0.9)$.

TABLE 2. Result of Example 4.2 with initial point $(x_0, y_0) = (-0.5, 0.7)$.

Algorithm 1			
Time	Iter	Approximation	$\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\|$
0.036877	1	(-0.495536, 0.665893)	0.038571
	2	(-0.484953, 0.614517)	0.061958
	3	(-0.467162, 0.557050)	0.075258
	4	(-0.442434, 0.499643)	0.082135
	5	(-0.412048, 0.445470)	0.084559
	10	(-0.237505, 0.242598)	0.065258
	20	(-0.055294, 0.055903)	0.019071
	30	(-0.010219, 0.010270)	0.003974
	35	(-0.004111, 0.004119)	0.001681
	40	(-0.001584, 0.001581)	0.000681
	45	(-0.000579, 0.000579)	0.000264
	50	(-0.000199, 0.000200)	0.000098
	55	(-0.000063, 0.000063)	0.000034
	60	(-0.000017, 0.000017)	0.000011
	65	(-0.000003, 0.000003)	0.000003
	66	(-0.000002, 0.000002)	0.000002
	67	(-0.000001, 0.000001)	0.000002
	68	(-0.000001, 0.000001)	0.000001
	69	(-0.000000, 0.000000)	0.000001

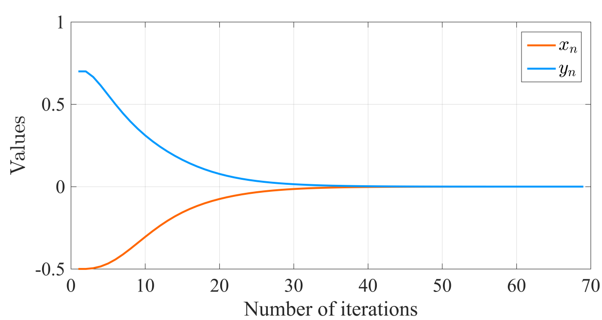

FIGURE 2. Result of Example 4.2 with initial point $(x_0, y_0) = (-0.5, 0.7)$.

Example 4.3. Let $H = \mathbb{R}$, we consider $C = [0,1]$. Define a mapping $F : C \times C \to C$ by $F(x,y) = \frac{x^2+3y^2}{8}$ for all $x, y \in C$.

Since $|x^2+3y^2-(u^2+3v^2)| \leq |x^2+u^2|+3|y^2-v^2|$ holds for all $x, y, u, v \in C$. Therefore, we have

$$|F(x,y)-F(u,v)| = \Big|\frac{x^2+3y^2}{8} - \frac{u^2+3v^2}{8}\Big| \leq \frac{1}{8}(|x^2-u^2|+3|y^2-v^2|).$$

Thus, F satisfies all the hypotheses of Theorem 3.2. Then there exists a coupled fixed point of F, i.e., $(0,0)$. In Algorithm 1, we set $\alpha_n = 0.53$, $\lambda = 0.5$, $\mu = 1$, $\tau_n = 100n^{-\frac{3}{2}}, \gamma_n = 0.9$ and $\beta_n = \frac{1}{(n+1)^2}$. Then, we have the results in Table 3, Table 4, Figure 3 and Figure 4.

TABLE 3. Result of Example 4.3 with initial point $(x_0, y_0) = (0.8,\ 0.9)$.

Algorithm 1			
Time	Iter	Approximation	$\|x_{n+1}-x_n\| + \|y_{n+1}-y_n\|$
0.048907	1	(0.747969, 0.830156)	0.121875
	2	(0.672649, 0.733389)	0.172087
	3	(0.588924, 0.630987)	0.186127
	4	(0.504522, 0.532275)	0.183114
	5	(0.423979, 0.441535)	0.171283
	10	(0.139360, 0.140016)	0.080673
	20	(0.006394, 0.006302)	0.005244
	30	(0.000141, 0.000137)	0.000142
	31	(0.000092, 0.000090)	0.000096
	32	(0.000060, 0.000058)	0.000064
	33	(0.000038, 0.000037)	0.000042
	34	(0.000024, 0.000024)	0.000028
	35	(0.000015, 0.000015)	0.000018
	36	(0.000009, 0.000009)	0.000012
	37	(0.000006, 0.000005)	0.000007
	38	(0.000003, 0.000003)	0.000005
	39	(0.000002, 0.000002)	0.000003
	40	(0.000001, 0.000001)	0.000002
	41	(0.000000, 0.000000)	0.000001

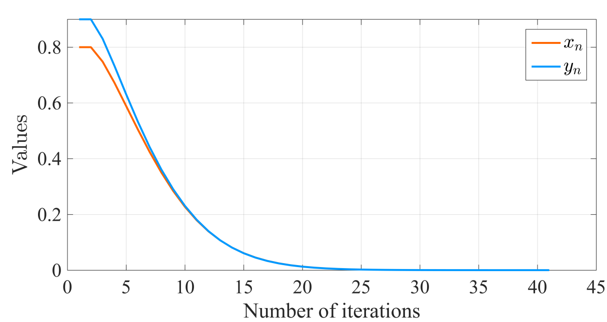

FIGURE 3. Result of Example 4.3 with initial point $(x_0, y_0) = (0.9,\ 0.8)$.

TABLE 4. Result of Example 4.3 with initial point $(x_0, y_0) = (0.6,\ 0.4)$.

Algorithm 1			
Time	Iter	Approximation	$\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\|$
0.030499	1	(0.538125, 0.369375)	0.092500
	2	(0.455792, 0.325561)	0.126147
	3	(0.373395, 0.277895)	0.130063
	4	(0.298954, 0.231266)	0.121070
	5	(0.235137, 0.188362)	0.106721
	10	(0.058413, 0.052795)	0.037620
	20	(0.001889, 0.001920)	0.001701
	30	(0.000029, 0.000034)	0.000036
	31	(0.000018, 0.000022)	0.000024
	32	(0.000011, 0.000013)	0.000015
	33	(0.000007, 0.000008)	0.000010
	34	(0.000004, 0.000005)	0.000006
	35	(0.000002, 0.000003)	0.000004
	36	(0.000001, 0.000002)	0.000002
	37	(0.000000, 0.000001)	0.000001
	38	(0.000000, 0.000000)	0.000001

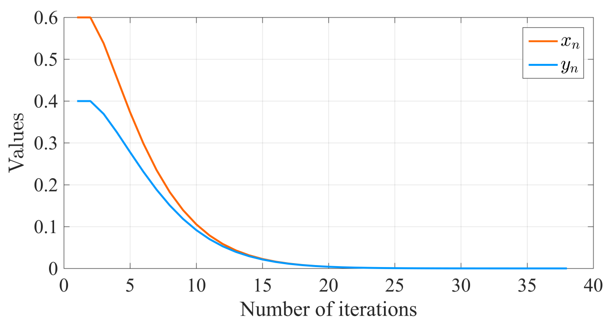

FIGURE 4. Result of Example 4.3 with initial point $(x_0, y_0) = (0.6,\ 0.4)$.

Acknowledgments. The first author thanks for the support of Rambhai Barni Rajabhat University. The authors thank you very much Prof. Vasile Berinde for his suggestions and comments. Finally, Kamonrat Sombut was financial supported by RMUTT annual government statement of expendture in 2019 and RMUTT Research Grant for New Scholar for fiscal year of 2019 (Grant no.NSF62D0603) was gratefully acknowledged.

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