CARPATHIAN J. MATH. **36** (2020), No. 1, 147 - 157
Online version at https://www.carpathian.cunbm.utcluj.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Dedicated to Prof. Hong-Kun Xu on the occasion of his 60^{th} anniversary

Split equality fixed point problems for asymptotically quasi-pseudocontractive operators

YAQIN WANG[1]†, YANLAI SONG[2], XIAOLI FANG[1] and TAE-HWA KIM[3]*

ABSTRACT. In this paper, we consider a split equality fixed point problem for asymptotically quasi-pseudo contractive operators which includes split feasibility problem, split equality problem, split fixed point problem etc, as special cases. Furthermore we propose a new algorithm for solving the split equality fixed point problem, and prove a weak and strong convergence theorem. The results obtained in this paper generalize and improve the recent ones announced by many others.

1. INTRODUCTION

Throughout this paper, we always assume that H_1 and H_2 are real Hilbert spaces, let $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets, let $A : H_1 \to H_2$ be a bounded linear operator. Recall that the split feasibility problem (SFP) consists of finding a point $x^* \in H_1$ such that

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{1.1}$$

The SFP was first introduced in 1994 by Censor and Elfving [2] in finite-dimensional Hilbert spaces for modeling inverse problems arising from phase retrieval and medical image reconstruction. Recently the SFP has been widely studied by many authors (see, e.g., [1, 16, 15, 14])

Note that if the SFP (1.1) is consistent, it is no hard to see that x^* solves the SFP (1.1) if and only if it solves the fixed point equation

$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*,$$

where P_C and P_Q are the metric projections from H_1 onto C and from H_2 onto Q, respectively, γ is a positive constant and A^* denotes the adjoint of A (see [13, Proposition 3.2] for the details). This implies that the SFP (1.1) can be solved by using fixed point algorithms.

A popular algorithm used in approximating the solution of the SFP (1.1) is the CQ-algorithm of [1]:

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n,$$

for each $n \geq 1$, where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A.

In [3], Censor and Segal consider the following split common fixed-point problem (SCFP):

$$\text{find } x^* \in F(U) \text{ such that } Ax^* \in F(T), \tag{1.2}$$

Received: 29.09.2017. In revised form: 07.07.2018. Accepted: 30.03.2019
2010 *Mathematics Subject Classification.* 47J20, 49J40, 49J52.
Key words and phrases. *Split equality fixed-point problem, asymptotically quasi-pseudocontractive mapping, weak convergence, iterative algorithm.*
Corresponding author: Tae-Hwa Kim; taehwa@pknu.ac.kr

where $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two directed operators with nonempty fixed-point sets. To solve (1.2), Censor and Segal [3] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{n+1} = U(x_n + \gamma A^t(T - I)Ax_k), \;\; k \in N,$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the largest eigenvalue of the matrix $A^tA(A^t$ stands for matrix transposition).

In 2013, Moudafi and Al-Shemas [11] introduced the following split equality fixed point problem (SEFP). Let H_1, H_2, H_3 be real Hilbert spaces, let $A : H_1 \to H_3,\ B : H_2 \to H_3$ be two bounded linear operators, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two firmly quasi-nonexpansive operators. The SEFP in [11] is to

$$\text{find } x^* \in F(U),\ y^* \in F(T) \text{ such that } Ax^* = By^*. \tag{1.3}$$

If $H_2 = H_3$ and $B = I$, the SEFP (1.3) reduces to the SCFP (1.2).

For solving the SEFP (1.3), Moudafi and Al-Shemas [11] introduced the following simultaneous iterative method:

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_k - By_k)) \end{cases}$$

for firmly quasi-nonexpansive operators U and T, where $\gamma_k \in (\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon),\ \lambda_A,\ \lambda_B$ stand for the spectral radiuses of A^*A and B^*B, respectively.

Recently, Che and Li [5] proposed the following iterative algorithm for finding a solution of the SEFP (1.3):

$$\begin{cases} u_n &= x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)Tu_n, \\ v_n &= y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} &= \alpha_n y_n + (1 - \alpha_n)Sv_n \end{cases} \tag{1.4}$$

for quasi-nonexpansive operators T and S. And they obtained the weak convergence of the scheme (1.4).

Very recently, Chang et al. [4] considered the following iterative algorithm for solving the SEFP (1.3):

$$\begin{cases} u_n &= x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)((1 - \xi)I + \xi T((1 - \eta)I + \eta T))u_n, \\ v_n &= y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} &= \alpha_n y_n + (1 - \alpha_n)((1 - \xi)I + \xi S((1 - \eta)I + \eta S))v_n, \end{cases} \tag{1.5}$$

where T and S are quasi-pseudocontractive operators. Furthermore, they established the weak and strong convergence of the scheme (1.5).

Note that the class of quasi-pseudocontractive operators which properly includes the classes of quasi-nonexpansive operators, directed operators and demicontractive operators, is more desirable for example in fixed point methods in image recovery where in many cases, it is possible to map the set of images possessing a certain property to the fixed point set of a nonlinear quasi-nonexpansive operator.

The purpose of this paper is to extend the above results to the class of operators which are both uniformly Lipschitzian and asymptotically quasi-pseudocontractive. We construct an iterative algorithm for such operators based on the algorithm (1.5) and prove its weak and strong convergence.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H and $F(T)$ denotes the set of the fixed points of an operator T. The notations "$\to$" and "$\rightharpoonup$" stand for strong convergence and weak convergence, respectively. We use $\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}$ to stand for the weak ω-limit set of $\{x_n\}$.

Definition 2.1. An operator $T : C \to C$ is said to be
(i) L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \ \forall x,\ y \in C;$$

(ii) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \ \forall x,\ y \in C,\ n \ge 1.$$

Definition 2.2. An operator $T : C \to C$ is said to be
(i) nonexpansive if $\|Tx - Ty\| \le \|x - y\|, \ \forall\, x,\ y \in C$;
(ii) quasi-nonexpansive if $F(T) \ne \emptyset$ and $\|Tx - q\| \le \|x - q\|, \ \forall\, x \in C,\ q \in F(T)$;
(iii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \ \forall\, x,\ y \in C,$$

or equivalently,

$$\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle, \ \forall\, x,\ y \in C;$$

(iv) directed if $F(T) \ne \emptyset$ and

$$\|Tx - q\|^2 \le \|x - q\|^2 - \|x - Tx\|^2, \ \forall\, x \in C,\ q \in F(T);$$

(v) μ-strictly pseudocontractive if there exists $\mu \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \mu\|x - Tx - (y - Ty)\|^2, \ \forall\, x,\ y \in C;$$

(vi) μ-demicontractive if $F(T) \ne \emptyset$ and the exists a constant $\mu \in (-\infty, 1)$ such that

$$\|Tx - q\|^2 \le \|x - q\|^2 + \mu\|x - Tx\|^2, \ \forall\, x \in H,\ q \in F(T);$$

(vii) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\|^2 \le k_n\|x - y\|^2, \ \forall\, x,\ y \in C.$$

Especially, T is said to be asymptotically quasi-nonexpansive if $F(T) \ne \emptyset$ and

$$\|T^n x - q\|^2 \le k_n\|x - q\|^2, \ \forall\, x \in C, q \in F(T).$$

Definition 2.3. An operator $T : C \to C$ is said to be
(i) pseudocontractive if

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2, \ \forall\, x,\ y \in C.$$

It is well-known that T is pseudocontractive if and only if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|x - Tx - (y - Ty)\|^2, \ \forall\, x,\ y \in C;$$

(ii) quasi-pseudocontractive if $F(T) \ne \emptyset$ and

$$\|Tx - q\|^2 \le \|x - q\|^2 + \|x - Tx\|^2, \ \forall\, x \in C,\ q \in F(T).$$

It is obvious that the class of quasi-pseudocontractive operators includes the class of demicontractive operators as its special case;
(iii) asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y, x - y\rangle \le k_n\|x - y\|^2, \ \forall\, x,\ y \in C.$$

It is easy to see that T is asymptotically pseudocontractive if and only if

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|x - T^n x - (y - T^n y)\|^2, \ \forall\, x,\, y \in C;$$

(iv) asymptotically quasi-pseudocontractive if $F(T) \neq \emptyset$ and if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - q, x - q\rangle \leq k_n \|x - q\|^2, \ \forall\, x \in C,\ q \in F(T). \tag{2.6}$$

It is clear that T is asymptotically quasi-pseudocontractive if and only if

$$\|T^n x - q\|^2 \leq (2k_n - 1)\|x - q\|^2 + \|x - T^n x\|^2, \ \forall\, x \in C,\ q \in F(T).$$

It is worth noting that the class of asymptotically quasi-pseudocontractive operators is more general than the class of asymptotically quasi-nonexpansive operators.

Definition 2.4. (i) An operator $T : H \to H$ is said to be *demiclosed at origin* [8] if, for any sequence $\{x_n\}$ which converges weakly to x, and if the sequence $\{Tx_n\}$ converges strongly to 0, then $Tx = 0$;
(ii) An operator $T : H \to H$ is said to be *semi-compact* if, for any bounded sequence $\{x_n\} \subset H$ with $\|x_n - Tx_n\| \to 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x \in H$.

Definition 2.5. Let S be a nonempty closed convex subset of a real Hilbert space H. We say that a sequence $\{x_n\} \subset H$ is a *quasi-Fejér* of Type II relative to the target set S (later called *S-quasi-Fejérian*) [6] if

$$\|x_{n+1} - a\|^2 \leq \|x_n - a\|^2 + \epsilon_n,\ a \in S,\ n \geq 1,$$

where $\{\epsilon_n\}$ is a sequence in $(0, \infty)$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

In any Hilbert space, the following conclusion holds:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2. \tag{2.7}$$

Lemma 2.1. ([7]) *Let X and Y be Banach spaces, A be a continuous linear operator from X to Y. Then A is weakly continuous.*

Lemma 2.2. ([6]) *Let S be a nonempty closed convex subset of a real Hilbert space H. Suppose that $\{x_n\} \subset H$ is S-quasi-Fejérian. Then $x_n \rightharpoonup x \in S$ if and only if $\omega_w(x_n) \subset S$.*

Lemma 2.3. ([12]) *Let $\{s_n\}$ and $\{\delta_n\}$ be positive real sequences. Assume that $\sum_{n=1}^{\infty} \delta_n < \infty$. If either $s_{n+1} \leq (1 + \delta_n)s_n$ or $s_{n+1} \leq s_n + \delta_n$ for all n, then the limit of the sequence $\{s_n\}$ exists.*

Lemma 2.4. *Let C be a nonempty closed convex subset of H and $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically quasi-pseudocontractive operator. Then $F(T)$ is a closed convex subset of C.*

Proof. Since T is L-Lipschitzian, then we deduce $F(T)$ is closed. Next we only need to prove that $F(T)$ is convex. To this aim, let $p_1, p_2 \in F(T)$ and write $p = tp_1 + (1-t)p_2$ for any $t \in (0, 1)$. we plan to show $p = Tp$, i.e., $p \in F(T)$. Take $\alpha \in (0, \frac{1}{1+L})$ and define $y_{\alpha,n} = (1-\alpha)p + \alpha T^n p$. Since T is L-Lipschitzian, we have

$$\begin{aligned} & \langle p - y_{\alpha,n}, (p - T^n p) - (y_{\alpha,n} - T^n y_{\alpha,n})\rangle \\ = \ & \langle p - y_{\alpha,n}, (p - y_{\alpha,n}) - (T^n p - T^n y_{\alpha,n})\rangle \\ \leq \ & \|p - y_{\alpha,n}\|^2 + \|p - y_{\alpha,n}\|\|T^n p - T^n y_{\alpha,n}\| \\ \leq \ & (1+L)\|p - y_{\alpha,n}\|^2 \\ = \ & (1+L)\alpha^2\|p - T^n p\|^2. \end{aligned} \tag{2.8}$$

Hence, for any $q \in F(T)$, from (2.6) and (2.8) we obtain

$$\begin{aligned}
& \|p - T^n p\|^2 = \langle p - T^n p, p - T^n p\rangle = \frac{1}{\alpha}\langle p - y_{\alpha,n}, p - T^n p\rangle \\
= \; & \frac{1}{\alpha}\langle p - y_{\alpha,n}, (p - T^n p) - (y_{\alpha,n} - T^n y_{\alpha,n})\rangle + \frac{1}{\alpha}\langle p - y_{\alpha,n}, y_{\alpha,n} - T^n y_{\alpha,n}\rangle \\
= \; & \frac{1}{\alpha}\langle p - y_{\alpha,n}, (p - T^n p) - (y_{\alpha,n} - T^n y_{\alpha,n})\rangle + \frac{1}{\alpha}\langle p - q, y_{\alpha,n} - T^n y_{\alpha,n}\rangle \\
& + \frac{1}{\alpha}\langle q - y_{\alpha,n}, y_{\alpha,n} - q\rangle + \frac{1}{\alpha}\langle q - y_{\alpha,n}, q - T^n y_{\alpha,n}\rangle \\
\leq \; & \alpha(1+L)\|p - T^n p\|^2 + \frac{1}{\alpha}\langle p - q, y_{\alpha,n} - T^n y_{\alpha,n}\rangle + \frac{1}{\alpha}(k_n - 1)\|q - y_{\alpha,n}\|^2,
\end{aligned}$$

which implies that

$$\alpha(1 - \alpha(1+L))\|p - T^n p\|^2 \leq \langle p - q, y_{\alpha,n} - T^n y_{\alpha,n}\rangle + (k_n - 1)\|q - y_{\alpha,n}\|^2. \tag{2.9}$$

Since T is L-Lipschitzian, we get

$$\begin{aligned}
\|q - y_{\alpha,n}\|^2 &= \|(1-\alpha)(p - q) + \alpha(T^n p - q)\|^2 \\
&\leq (1-\alpha)\|p - q\|^2 + \alpha\|T^n p - q\|^2 \\
&\leq (1 - \alpha + \alpha L^2)\|p - q\|^2 = M.
\end{aligned} \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\alpha(1 - \alpha(1+L))\|p - T^n p\|^2 \leq \langle p - q, y_{\alpha,n} - T^n y_{\alpha,n}\rangle + (k_n - 1)M. \tag{2.11}$$

Taking $q = p_1$ and $q = p_2$ in (2.11), multiplying t and $1-t$ on both sides of (2.11), respectively, and adding up yield

$$\alpha(1 - \alpha(1+L))\|p - T^n p\|^2 \leq (k_n - 1)M \to 0,$$

which implies that $T^n p \to p$, then we have $T^{n+1}p \to p$. Noting that

$$\begin{aligned}
\|p - Tp\| &\leq \|p - T^{n+1}p\| + \|T^{n+1}p - Tp\| \\
&\leq \|p - T^{n+1}p\| + L\|T^n p - p\| \to 0,
\end{aligned}$$

we have $p = Tp$, completing the proof. □

Remark 2.1. Comparing with Lemma 1.3 in [9], we do not require that the subset C of H is bounded in Lemma 2.4.

Lemma 2.5. ([17])Let H be a real Hilbert space. Let $T : H \to H$ be a uniformly L-Lipschitzian asymptotically pseudocontractive operator with coefficient k_n. If $0 < \zeta < \eta < \frac{1}{\sqrt{k_n^2 + L^2} + k_n}$ for all $n \geq 1$, then

$$\begin{aligned}
& \|(1-\zeta)x + \zeta T^n((1-\eta)I + \eta T^n)x - x^*\|^2 \\
\leq \; & [1 + 2(k_n - 1)\zeta + 2(k_n - 1)(2k_n - 1)\eta\zeta]\|x - x^*\|^2, \quad \forall x \in H, \; x^* \in F(T).
\end{aligned}$$

Remark 2.2. From the proof of Proposition 3.2 in [17], we know that if the asymptotically pseudocontractive operator T is an asymptotically quasi-pseudocontractive operator with coefficient k_n, the conclusion of Lemma 2.5 still holds.

3. Main results

In this section, we always assume that H_1, H_2, H_3 are real Hilbert spaces. Let an operator $T : H_1 \to H_1$ be both uniformly L_1-Lipschitzian and asymptotically quasi-pseudocontractive with coefficient $k_n^{(1)}$ and $S : H_2 \to H_2$ be both uniformly L_2-Lipschitzian and asymptotically quasi-pseudocontractive with coefficient $k_n^{(2)}$, $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with their adjoints A^* and B^*, respectively.

Put $H^* = H_1 \times H_2$. Define the inner product of H^* as follows:

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \ \forall\, (x_1, y_1), (x_2, y_2) \in H^*.$$

It is easy to see that H^* is also a real Hilbert space and

$$\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}, \ \forall\, (x, y) \in H^*.$$

We use Γ to stand for the solution set of the SEFP (1.3), i.e.,

$$\Gamma = \{(x, y) \in H^* |\, x \in F(T),\ y \in F(S) \text{ such that } Ax = By\}.$$

Now we present our algorithm for solving the SEFP (1.3).

Algorithm 3.1. *Choose $\{\alpha_n\} \subset (0, 1)$. Take arbitrary $x_0 \in H_1, y_0 \in H_2$. Assume that the nth iterate $x_n \in H_1,\ y_n \in H_2$ has been constructed, then we calculate $(n + 1)$th iterate $(x_{n+1},\ y_{n+1})$ via the formula*

$$\left\{ \begin{array}{lcl} u_n & = & x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} & = & \alpha_n x_n + (1-\alpha_n)((1-\xi_n)I + \xi_n T^n((1-\eta_n)I + \eta_n T^n))u_n, \\ v_n & = & y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} & = & \alpha_n y_n + (1-\alpha_n)((1-\xi_n)I + \xi_n S^n((1-\eta_n)I + \eta_n S^n))v_n. \end{array} \right. \tag{3.12}$$

Put $k_n = \max\{k_n^{(1)}, k_n^{(2)}\}$, $L = \max\{L_1, L_2\}$. Based on the assumption on the operators T and S, we can readily see that S and T are both uniformly L-Lipschitzian and asymptotically quasi-pseudocontractive with coefficient k_n.

Theorem 3.2. *Let $H_1, H_2, H_3, A, B, S, T$ and Γ be the same as above. If $I - T$ and $I - S$ are demiclosed at 0 and the following conditions are satisfied:*
(a) $\gamma_n \in (\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon), \ \ \forall n \geq 1$;
(b) $0 < a^* < \xi_n < \eta_n < b^* < \frac{1}{\sqrt{k_n^2 + L^2} + k_n}$;
(c) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and* $\sum_{n=1}^{\infty} (k_n - 1) < \infty$,
where λ_A, λ_B stand for the spectral radiuses of A^*A and B^*B, respectively and $\varepsilon > 0$ is small enough. Then the following conclusions hold:
(I) the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges weakly to a solution of the SEFP (1.3);
(II) In addition, if S, T are also semi-compact, then $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to a solution of the SEFP (1.3).

Proof. By Lemma 2.4 we have $F(T)$ and $F(S)$ are both closed convex sets. Since A and B are both linear, it is easy to see that Γ is a closed convex subset in H^*. Given any $(p, q) \in \Gamma$, then $p \in F(T),\ q \in F(S)$ such that $Ap = Bq$. By (3.12) and the definitions of λ_A and λ_B, we have

$$\begin{aligned}
& \|u_n - p\|^2 \\
= & \|x_n - \gamma_n A^*(Ax_n - By_n) - p\|^2 \\
= & \|x_n - p\|^2 - 2\gamma_n \langle x_n - p, A^*(Ax_n - By_n)\rangle + \gamma_n^2 \|A^*(Ax_n - By_n)\|^2 \\
= & \|x_n - p\|^2 - 2\gamma_n \langle Ax_n - Ap, Ax_n - By_n\rangle \\
& + \gamma_n^2 \langle A^*(Ax_n - By_n), A^*(Ax_n - By_n)\rangle \\
= & \|x_n - p\|^2 - 2\gamma_n \langle Ax_n - Ap, Ax_n - By_n\rangle \\
& + \gamma_n^2 \langle Ax_n - By_n, AA^*(Ax_n - By_n)\rangle \\
\leq & \|x_n - p\|^2 - 2\gamma_n \langle Ax_n - Ap, Ax_n - By_n\rangle + \gamma_n^2 \lambda_A \|Ax_n - By_n\|^2,
\end{aligned}$$

and

$$\begin{aligned}
& \|v_n - q\|^2 \\
= & \|y_n + \gamma_n B^*(Ax_n - By_n) - q\|^2 \\
= & \|y_n - q\|^2 + 2\gamma_n \langle By_n - Bq, Ax_n - By_n\rangle + \gamma_n^2 \|B^*(Ax_n - By_n)\|^2 \\
= & \|y_n - q\|^2 + 2\gamma_n \langle By_n - Bq, Ax_n - By_n\rangle \\
& + \gamma_n^2 \langle Ax_n - By_n, BB^*(Ax_n - By_n)\rangle \\
\leq & \|y_n - q\|^2 + 2\gamma_n \langle By_n - Bq, Ax_n - By_n\rangle + \gamma_n^2 \lambda_B \|Ax_n - By_n\|^2.
\end{aligned}$$

Adding the above inequalities and noticing $Ap = Bq$, we have

$$\begin{aligned}
& \|u_n - p\|^2 + \|v_n - q\|^2 \\
\leq & \|x_n - p\|^2 + \|y_n - q\|^2 - \gamma_n[2 - (\lambda_A + \lambda_B)\gamma_n]\|Ax_n - By_n\|^2.
\end{aligned} \tag{3.13}$$

Put

$$K_n := (1 - \xi_n)I + \xi_n T^n((1 - \eta_n)I + \eta_n T^n),$$

$$G_n := (1 - \xi_n)I + \xi_n S^n((1 - \eta_n)I + \eta_n S^n).$$

It follows from Algorithm 3.1, Lemma 2.5 and (2.6) that

$$\begin{aligned}
& \|x_{n+1} - p\|^2 \\
= & \|\alpha_n(x_n - p) + (1 - \alpha_n)(K_n u_n - p)\|^2 \\
= & \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|K_n u_n - p\|^2 - \alpha_n(1 - \alpha_n)\|K_n u_n - x_n\|^2 \\
\leq & \alpha_n \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\|K_n u_n - x_n\|^2 \\
& + (1 - \alpha_n)[1 + 2(k_n - 1)\xi_n + 2(k_n - 1)(2k_n - 1)\eta_n \xi_n]\|u_n - p\|^2,
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
& \|y_{n+1} - q\|^2 \\
= & \|\alpha_n(y_n - q) + (1 - \alpha_n)(G_n v_n - q)\|^2 \\
= & \alpha_n \|y_n - q\|^2 + (1 - \alpha_n)\|G_n v_n - q\|^2 - \alpha_n(1 - \alpha_n)\|G_n v_n - y_n\|^2 \\
\leq & \alpha_n \|y_n - q\|^2 - \alpha_n(1 - \alpha_n)\|G_n v_n - y_n\|^2 \\
& + (1 - \alpha_n)[1 + 2(k_n - 1)\xi_n + 2(k_n - 1)(2k_n - 1)\eta_n \xi_n]\|v_n - q\|^2.
\end{aligned} \tag{3.15}$$

From (3.13), (3.14) and (3.15) we have

$$\begin{aligned}
& \|x_{n+1}-p\|^2+\|y_{n+1}-q\|^2 \\
\leq\ & \alpha_n(\|x_n-p\|^2+\|y_n-q\|^2) \\
& -\alpha_n(1-\alpha_n)(\|K_nu_n-x_n\|^2+\|G_nv_n-y_n\|^2)+(1-\alpha_n) \\
& \times[1+2(k_n-1)\xi_n+2(k_n-1)(2k_n-1)\eta_n\xi_n](\|u_n-p\|^2+\|v_n-q\|^2) \\
\leq\ & \alpha_n(\|x_n-p\|^2+\|y_n-q\|^2) \\
& -\alpha_n(1-\alpha_n)(\|K_nu_n-x_n\|^2+\|G_nv_n-y_n\|^2)+(1-\alpha_n) \\
& \times[1+2(k_n-1)\xi_n+2(k_n-1)(2k_n-1)\eta_n\xi_n](\|x_n-p\|^2+\|y_n-q\|^2) \\
& -(1-\alpha_n)[1+2(k_n-1)\xi_n+2(k_n-1)(2k_n-1)\eta_n\xi_n]\gamma_n \\
& \times[2-(\lambda_A+\lambda_B)\gamma_n]\|Ax_n-By_n\|^2 \\
=\ & \{1+(1-\alpha_n)[2(k_n-1)\xi_n+2(k_n-1)(2k_n-1)\eta_n\xi_n]\}(\|x_n-p\|^2+\|y_n-q\|^2) \\
& -\alpha_n(1-\alpha_n)(\|K_nu_n-x_n\|^2+\|G_nv_n-y_n\|^2) \\
& -(1-\alpha_n)[1+2(k_n-1)\xi_n+2(k_n-1)(2k_n-1)\eta_n\xi_n]\gamma_n \\
& \times[2-(\lambda_A+\lambda_B)\gamma_n]\|Ax_n-By_n\|^2.
\end{aligned}$$

Setting $s_n=\|x_n-p\|^2+\|y_n-q\|^2$, we have

$$\begin{aligned}
s_{n+1} \leq\ & \{1+(k_n-1)[2\xi_n+2(2k_n-1)\eta_n\xi_n]\}s_n \\
& -\alpha_n(1-\alpha_n)(\|K_nu_n-x_n\|^2+\|G_nv_n-y_n\|^2) \\
& -(1-\alpha_n)[1+2(k_n-1)\xi_n+2(k_n-1)(2k_n-1)\eta_n\xi_n]\gamma_n \\
& \times[2-(\lambda_A+\lambda_B)\gamma_k]\|Ax_n-By_n\|^2
\end{aligned} \tag{3.16}$$

for all n. Since $\sum_{n=1}^{\infty}(k_n-1)<\infty$, now use Lemma 2.3 to get that $\lim_{n\to\infty} s_n$ exists. Subsequently, the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ are all bounded. The following simplicity of (3.16) implies that the sequence $\{(x_n,y_n)\}\subset H^*$ is also Γ-quasi-Fejérian:

$$\begin{aligned}
& \|(x_{n+1},y_{n+1})-(p,q)\|^2=s_{n+1} \\
\leq\ & \|(x_n,y_n)-(p,q)\|^2+\epsilon_n, \quad (p,q)\in\Gamma,
\end{aligned}$$

where $\epsilon_n=(k_n-1)M$ with a positive constant M such that $[2\xi_n+2(2k_n-1)\eta_n\xi_n]\|s_n\|\leq M$ for all n. On one hand, from (3.16) we obtain

$$\begin{aligned}
& \alpha_n(1-\alpha_n)(\|K_nu_n-x_n\|^2+\|G_nv_n-y_n\|^2) \\
& +(1-\alpha_n)[1+2(k_n-1)\xi_n+2(k_n-1)(2k_n-1)\eta_n\xi_n]\gamma_n \\
& \times[2-(\lambda_A+\lambda_B)\gamma_n]\|Ax_n-By_n\|^2 \\
& \leq\{1+(k_n-1)[2\xi_n+2(2k_n-1)\eta_n\xi_n]\}s_n-s_{n+1}\to 0,
\end{aligned} \tag{3.17}$$

since $\lim_{n\to\infty} s_n$ exists and $k_n\to 1$. Hence it follows from (3.17), the conditions (a)-(c) and $\lim_{n\to\infty}k_n=1$ that

$$\lim_{n\to\infty}\|K_nu_n-x_n\|=\lim_{n\to\infty}\|G_nv_n-y_n\|=0, \tag{3.18}$$

$$\lim_{n\to\infty}\|Ax_n-By_n\|=0. \tag{3.19}$$

From (3.12), (3.18) and (3.19) we get

$$
\begin{cases}
\lim_{n\to\infty} \|u_n - x_n\| & = & \lim_{n\to\infty} \gamma_n \|A^*(Ax_n - By_n)\| & = & 0, \\
\lim_{n\to\infty} \|v_n - y_n\| & = & \lim_{n\to\infty} \gamma_n \|B^*(Ax_n - By_n)\| & = & 0, \\
\lim_{n\to\infty} \|x_{n+1} - x_n\| & = & \lim_{n\to\infty} (1-\alpha_n)\|K_n u_n - x_n\| & = & 0, \\
\lim_{n\to\infty} \|y_{n+1} - y_n\| & = & \lim_{n\to\infty} (1-\alpha_n)\|G_n v_n - y_n\| & = & 0.
\end{cases} \tag{3.20}
$$

Hence

$$\lim_{n\to\infty} \|K_n u_n - u_n\| = \lim_{n\to\infty} \|G_n v_n - v_n\| = 0, \tag{3.21}$$

$$
\begin{aligned}
&\|u_{n+1} - u_n\| \le \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - u_n\| \to 0, \\
&\|v_{n+1} - v_n\| \le \|v_{n+1} - y_{n+1}\| + \|y_{n+1} - y_n\| + \|y_n - v_n\| \to 0.
\end{aligned} \tag{3.22}
$$

From the condition (b) we have

$$0 < a^* < \xi_n < \eta_n < b^* < \frac{1}{\sqrt{k_n^2 + L^2} + k_n} < \frac{1}{L}. \tag{3.23}$$

Since T is uniformly L-Lipschitzian, we can derive

$$
\begin{aligned}
\|u_n - T^n u_n\| &\le \|u_n - T^n((1-\eta_n)I + \eta_n T^n)u_n\| \\
&\quad + \|T^n((1-\eta_n)I + \eta_n T^n)u_n - T^n u_n\| \\
&\le \frac{1}{\xi_n}\|u_n - (1-\xi_n)u_n - \xi_n T^n((1-\eta_n)I + \eta_n T^n)u_n\| \\
&\quad + L\|(1-\eta_n)u_n + \eta_n T^n u_n - u_n\| \\
&= \frac{1}{\xi_n}\|u_n - K_n u_n\| + L\eta_n \|u_n - T^n u_n\|,
\end{aligned}
$$

which together with (3.21) and (3.23) implies that

$$\|u_n - T^n u_n\| \le \frac{1}{\xi_n(1 - L\eta_n)}\|u_n - K_n u_n\| \to 0. \tag{3.24}$$

Since T is uniformly L-Lipschitzian, from (3.22) and (3.24) we can obtain

$$
\begin{aligned}
&\|u_{n+1} - Tu_{n+1}\| \\
\le\ &\|u_{n+1} - T^{n+1}u_{n+1}\| + \|T^{n+1}u_{n+1} - T^{n+1}u_n\| + \|T^{n+1}u_n - Tu_{n+1}\| \\
\le\ &\|u_{n+1} - T^{n+1}u_{n+1}\| + L\|u_{n+1} - u_n\| + L\|T^n u_n - u_{n+1}\| \\
\le\ &\|u_{n+1} - T^{n+1}u_{n+1}\| + 2L\|u_{n+1} - u_n\| + L\|T^n u_n - u_n\| \to 0.
\end{aligned}
$$

This combined with (3.22) and uniform L-Lipschitz of T again yields

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0. \tag{3.25}$$

Similarly, we can get

$$\lim_{n\to\infty} \|v_n - Sv_n\| = 0. \tag{3.26}$$

Next, we prove that $\omega_w(x_n, y_n) \subset \Gamma$. Indeed, taking $(\widetilde{x}, \widetilde{y}) \in \omega_w(x_n, y_n)$, from (3.20) we have $(\widetilde{x}, \widetilde{y}) \in \omega_w(u_n, v_n)$. Then $\widetilde{x} \in \omega_w(u_n)$ and $\widetilde{y} \in \omega_w(v_n)$. Since $I - T$ and $I - S$ are demiclosed at 0, it follows from (3.25) and (3.26) that $\widetilde{x} \in F(T)$ and $\widetilde{y} \in F(S)$. On the other hand, by Lemma 2.1 we have $A\widetilde{x} - B\widetilde{y} \in \omega_w(Ax_n - By_n)$, which together with weakly lower semicontinuity of the norm implies that

$$\|A\widetilde{x} - B\widetilde{y}\| \le \liminf_{n\to\infty} \|Ax_n - By_n\| = 0.$$

Therefore, $(\widetilde{x}, \widetilde{y}) \in \Gamma$. So $\omega_w(x_n, y_n) \subset \Gamma$.

Since Γ is closed convex set and we have shown that $\{(x_n, y_n)\}$ is Γ-quasi-Fejérian and $\omega_w(x_n, y_n) \subset \Gamma$, Lemma 2.2 ensures that the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges weakly to a point of Γ. This completes the proof of the conclusion (I).

Now we prove that the conclusion (II) holds. In fact, since $\{u_n\}$,$\{v_n\}$ are bounded and S, T are semi-compact, from (3.25) and (3.26), there exist subsequences $\{u_{n_i}\} \subset \{u_n\}$ and $\{v_{n_i}\} \subset \{v_n\}$ such that $u_{n_i} \to x^*$ and $v_{n_i} \to y^*$. Then $(x^*, y^*) \in \omega_w(u_n, v_n)$, furthermore $(x^*, y^*) \in \omega_w(x_n, y_n)$. Being similar to the proof of $\omega_w(x_n, y_n) \subset \Gamma$, we have $(x^*, y^*) \in \Gamma$. Also from (3.20) we have $x_{n_i} \to x^*$ and $y_{n_i} \to y^*$. Repeating the previous proof with $s_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$, we also arrive at the existence of $\lim_{n\to\infty} s_n$. Combined with the fact $s_{n_i} \to 0$, it results $s_n \to 0$; hence

$$\lim_{n\to\infty} \|x_n - x^*\| = 0 \ \text{ and } \ \lim_{n\to\infty} \|y_n - y^*\| = 0.$$

Therefore $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \Gamma$ which is a solution of the SEFP (1.3), completing the proof. □

Remark 3.3. Theorem 3.2 extends and improves Theorem 3.2 in [4] from quasi-pseudocontractive operators to asymptotically quasi-pseudocontractive operators, and modifies the conditions on $\{\gamma_n\}$ and $\{\alpha_n\}$. Meanwhile, our proof is different from that of Theorem 3.2 in [4]. Also, Theorem 3.2 is still remained in a special case which the operators T and S are asymptotically quasi-nonexpansive, under the same parameter conditions (a)-(c)

For giving an example of an operator which satisfies all hypotheses of our main theorem, we could revisit the example in [10] which is not asymptotically quasi-nonexpanive for $k = 3/2$.

Acknowledgments.† Supported by the Natural Science Foundation of China (no. 11401388, 11671365), Zhejiang Provincial Natural Science Foundation of China (no. LQ13A010007, LY14A010006).

References

[1] Byrne, C., *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems, **18** (2002), 441–453

[2] Censor, Y. and Elfving, T., *A multiprojection algorithm using Bergman projections in a product space*, Numer. Algorithms, **8** (1994), No. 2, 221–239

[3] Censor, Y. and Segal, A., *The split common fixed point problem for directed operators*, J. Convex Anal., **16** (2009), 587–600

[4] Chang, S. S., Wang, L. and Qin, L. J., *Split equality fixed point problem for quasi-pseudo-contractive mappings with applications*, Fixed Point Theory Appl., **2015** (2015), 12 pp.

[5] Che, H. and Li, M., *A simultaneous iterative method for split equality problems of two finite families of strictly pseudononspreading mappings without prior knowledge of operator norms*, Fixed Point Theory Appl., **2015** (2015), 14 pp.

[6] Combettes, P. L., *Quasi-Fejérian analysis of some optimization algorithms*, in Inherently Parallel Algorithms in Feasibility and Optimization anf Their Applications, New York: Elsevier (2001), pp. 115–152

[7] Joshi, M. C. and Bose, R. K., *Some topics in nonlinear functional analysis*, John Wiley and Sons, Inc., New York, 1985

[8] Goebel, K. and Kirk, W. A., Topics in Metric Fixed Point Theory, vol. 28 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, UK, 1990

[9] Kang, S. M., Cho, S. Y. and Qin, X., *Hybrid projection algorithms for approximating fixed points of asymptotically quasi-pseudocontractive mappings*, J. Nonlinear Sci. Appl., **5** (2012), 466–474

[10] Kim, T. H. and Park, K. M., *Some results on metric fixed point theory and open problems*, Comm. Korean Matj. Soc., **11** (1996), No. 3, 725–742

[11] Moudafi, A. and Al-Shemas, E., *Simultaneous iterative methods for split equality problem*, Trans. Math. Program. Appl., **1** (2013), No. 2, 1–11

[12] Tan, K. K. and Xu, H. K., *Approximating fixed points of nonexpansive operators by the Ishikawa iteration process*, J. Math. Anal. Appl., **178** (1993), 301–308

[13] Xu, H. K., *Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces*, Inverse Problems, **26** (2010), 105018, 17 pp.
[14] Yao, Y. H., and Wu, J. and Liou, Y. C., *Regularized methods for the split feasibility problem*, Abstr. Appl. Anal., **2012** (2012), 13 pp.
[15] Yao, Y. H., Postolache, M. and Liou, Y. C., *Strong convergence of a self-adaptive method for the split feasibility problem*, Fixed Point Theory Appl., **2013** (2013), 12 pp.
[16] Yao, Y. H., Agarwal, R. P., Postolache, M. and Liou, Y. C., *Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem*, Fixed Point Theory Appl., **2014** (2014), 14 pp.
[17] Yao, Y. H., Leng, L. M., Postolache, M. H. and Zheng, X. X., *A unified framework for the two-sets common fixed point problem in Hilbert spaces*, J. Nonlinear Sci. Appl., **9** (2016), 6113–6125

[1]DEPARTMENT OF MATHEMATICS
SHAOXING UNIVERSITY
SHAOXING 312000, CHINA
E-mail address: wangyaqin0579@126.com
E-mail address: fxl0418@126.com

[2]ZHONGYUAN UNIVERSITY OF TECHNOLOGY
COLLEGE OF SCIENCE
ZHENGZHOU 450007, CHINA
E-mail address: songyanlai2009@163.com

[3]DEPARTMENT OF APPLIED MATHEMATICS
PUKYONG NATIONAL UNIVERSITY
BUSAN 48513, KOREA
E-mail address: taehwa@pknu.ac.kr