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Dedicated to Prof. Hong-Kun Xu on the occasion of his 60^{th} anniversary

Fréchet vector subdifferential calculus

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ABSTRACT. In this paper, we study Fréchet vector subdifferentials of vector-valued functions in normed spaces which reduce to the known ones of extended-real-valued functions. We establish relations between two kinds of Fréchet vector subdifferentials and between subdifferential and coderivative; some of them improve the existing relations for extended-real-valued functions. Finally, sum and chain rules among others for Fréchet subdifferentials of vector-valued functions are formulated and verified. Many examples are provided.

1. INTRODUCTION

This paper is intended to study Fréchet vector subdifferentials of vector-valued functions and their calculus. It is important to mention that Fréchet subdifferentials of extended-real-valued functions play an important role in necessary optimality conditions. Consider the following minimization problem with a geometric constraint,

$$\text{minimize } \ \varphi(x) \ \text{ subject to } \ x \in \Omega,$$

where $\varphi : X \to I\!R$ is an extended-real-valued function on a normed space X and Ω is a nonempty subset of X. It is equivalently formulated as an unconstrained problem:

$$\text{minimize } \ \varphi(x) + \delta(x;\Omega),$$

where $\delta(\cdot;\Omega) : X \to I\!R \cup \{+\infty\}$ is an indicator function being zero on Ω and $+\infty$, otherwise. Given $\bar{x} \in \operatorname{dom}\varphi$, the sets

$$\widehat{\partial}\varphi(x) := \left\{ x^* \in X^* \ : \ \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0 \right\}, \tag{1.1}$$

$$\widehat{\partial}^+\varphi(x) := \left\{ x^* \in X^* \ : \ \limsup_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0 \right\}, \tag{1.2}$$

are called the *analytic Fréchet lower (resp. upper) subdifferential* of φ at $\bar{x}$. When $\liminf$ in (1.1) and $\limsup$ in (1.2) are replaced by $\lim$ and the inequalities therein by equalities, we have the classical Fréchet derivative/gradient of φ at $\bar{x}$. Assume that $\bar{x} \in \Omega$ is a local solution of the problem. Then, we have

$$0 \in \widehat{\partial}(\varphi + \delta(\cdot;\Omega))(\bar{x})$$

by the generalized Fermat rule. Using the Fréchet subdifferential sum rule in [6], we arrive at

$$-\widehat{\partial}^+\varphi(\bar{x}) \subseteq \widehat{N}(\bar{x};\Omega),$$

where $\widehat{N}(\bar{x};\Omega) := \widehat{\partial}\delta(\bar{x};\Omega)$ is the Fréchet normal cone of Ω at $\bar{x}$.

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In [3], the authors proved that an upper set-less minimizer to a set-valued map (with respect to the image space) is an upper set-less minimal solution to a scalarization of the set-valued map (with respect to the space of real numbers), where the hypergraphical multifunction is involved in the scalarization and vice versa. This means that in order to derive necessary conditions we need to use Fréchet upper subdifferentials. It is the motivation for us to study Fréchet subdifferential objects and calculus rules for them.

The paper is organized as follows. Section 2 provides preliminary material from variational analysis and generalized differentiation needed, recalls Fréchet vector subdifferentials of vector-valued functions, and provide a proof of the Fréchet subdifferential sum rule. In Section 3, we study basic relations among analytic and geometric Fréchet vector subdifferentials and Fréchet coderivatives of vector-valued functions, and several important calculus rules for analytic Fréchet vector subdifferentials.

2. Preliminaries

Our notation is basically standard; see the books by Rockafellar and Wets [12] and by Mordukhovich [9]. All the spaces are assumed to be normed spaces unless otherwise explicitly stated. For a normed space X, we denote its norm by $\|\cdot\|$ and consider the dual space X^* equipped with the weak* topology w^*, where $\langle\cdot,\cdot\rangle$ stands for the canonical pairing between X and X^*. For another normed space Z, $\mathcal{L}(X,Z)$ is the space of all bounded linear operators from X into Z. In this section, we present notions of generalized differentiation for sets, functions and set-valued maps as well as some related results. We follow the book by Mordukhovich [9].

Let $\Omega \subseteq X$ be a subset of a normed space. The *Fréchet normal cone* to Ω at $x \in \Omega$ is defined by

$$\widehat{N}(x;\Omega) := \left\{ x^* \in X^* : \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{\|x-u\|} \le 0 \right\}. \tag{2.3}$$

One can easily observe the following monotonicity property of the Fréchet normal cones with respect to the set order:

$$\widehat{N}(\bar{x};\Omega_1) \subseteq \widehat{N}(\bar{x};\Omega_2) \ \text{ if } \ \bar{x} \in \Omega_2 \subseteq \Omega_1$$

and the representation of normal cones to set products

$$\widehat{N}((\bar{x}_1,\bar{x}_2);\Omega_1\times\Omega_2) = \widehat{N}(\bar{x}_1;\Omega_1)\times\widehat{N}(\bar{x}_2;\Omega_2).$$

Let Y be another normed space. Consider a set-valued map $F\colon X \rightrightarrows Z$ with its graph

$$\operatorname{gph} F := \{(x,z) \in X\times Z : z \in F(x)\}.$$

The *Fréchet coderivative* $\widehat{D}^*F(\bar{x},\bar{z})\colon Z^* \rightrightarrows X^*$ of F at $(\bar{x},\bar{z}) \in \operatorname{gph} F$ is defined by

$$\widehat{D}^*F(\bar{x},\bar{z})(z^*) := \left\{ x^* \in X^* : (x^*,-z^*) \in \widehat{N}((\bar{x},\bar{y});\operatorname{gph} F) \right\}, \tag{2.4}$$

which is a positively homogeneous map of z^*; we omit $\bar{z} = f(\bar{x})$ in (2.4) if $F = f\colon X \to Z$ is single-valued. If f happens to be *Fréchet differentiable* at $\bar{x}$, then

$$\forall z^* \in Z^*, \widehat{D}^*f(\bar{x})(z^*) = \{\nabla f(\bar{x})^* z^*\}.$$

Let $f : X \to Z$ be a vector-valued function between normed spaces, and let Z be partially ordered by a closed, pointed and convex ordering cone K (the pointedness means $K \cap (-K) = \{0\}$). Denote the positive polar cone C^+ of K is defined by

$$K^+ := \{z^* \in Z^* : \forall z \in K, \langle z^*, z\rangle \ge 0\}.$$

The cone K defines a partial order $\leq_K$ in the image space Z: for any $z_1, z_2 \in Z$, $z_1 \leq_K z_2$ if $z_1 \in z_2 - K$. The epigraph and hypograph of f with respect to K are respectively defined by

$$\text{epi}\, f := \{(x, z) \in X \times Z : z \in f(x) + K\},$$
$$\text{hypo}\, f := \{(x, z) \in X \times Z : z \in f(x) - K\}.$$

Consider two set-valued maps $\mathcal{E}_f, \mathcal{H}_f : X \rightrightarrows Z$ by

$$\forall x \in \text{dom}\, f, \mathcal{E}_f(x) := f(x) + K \text{ and } \mathcal{H}_f(x) := f(x) - K.$$

We have epi $f = \text{gph}\, \mathcal{E}_f$ and hypo $f = \text{gph}\, \mathcal{H}_f$.
The *geometric Fréchet vector lower (respectively, upper) subdifferential* $\widehat{\partial}_g f(\bar{x}) : Z^* \rightrightarrows \mathcal{L}(X, Z)$ (respectively, $\partial_g^+ f(\bar{x})$) of f at $\bar{x} \in \text{dom}\, f$ are respectively defined by

$$\widehat{\partial}_g f(\bar{x})(z^*) := \left\{T \in \mathcal{L}(X, Z) : (z^* \circ T, -z^*) \in \widehat{N}((\bar{x}, \bar{z}); \text{epi}\, f)\right\}, \tag{2.5}$$

$$\widehat{\partial}_g^+ f(\bar{x})(z^*) := \left\{T \in \mathcal{L}(X, Z) : (z^* \circ T, -z^*) \in \widehat{N}((\bar{x}, \bar{z}); \text{hypo}\, f)\right\} \quad . \tag{2.6}$$

One can easily observe that

$$\widehat{\partial}_g^+ f(\bar{x})(z^*) = -\widehat{\partial}_g(-f)(\bar{x})(-z^*),$$

and that $z^* \circ \widehat{\partial}_g f(\bar{x})(z^*) = \widehat{D}^* \mathcal{E}_f(\bar{x}, f(\bar{x}))(z^*)$ and $z^* \circ \widehat{\partial}_g^+ f(\bar{x})(z^*) = \widehat{D}^* \mathcal{H}_f(\bar{x}, f(\bar{x}))(z^*)$. Note that the coderivative of f at $\bar{x}$ and z^* is a set in X^* while the subdifferential of f at $\bar{x}$ is a set in $\mathcal{L}(X, Z)$. Fix a dual element $z^* \in Z^*$, $z^* \circ \widehat{\partial}_g f(\bar{x})$ is a set in X^*.

Remark 2.1. 1) Mordukhovich introduced the coderivative for the study of the closed graph set-valued maps in optimal control and used it define nonconvex subdifferentials of extended-real-valued functions. The role of the graph in coderivative is replaced by the epigraph or hypograph of a function in subdifferentials. Then, they were generalized to Mordukhovich vector subdifferentials of vector-valued functions; see a detailed survey in [13]. Subdifferentials of vector-valued functions are natural extensions of subdifferentials of extended-real-valued functions. Precisely, consider an extended-real-valued function $\varphi : X \to \overline{I\!R}$ finite at $\bar{x}$, then Fréchet lower and upper (vector) subdifferentials with respect to $|z^*| = 1$ become the corresponding subdifferentials of φ at $\bar{x}$:

$$\widehat{\partial}_g \varphi(\bar{x}) = \widehat{\partial}_g \varphi(\bar{x})(1) \text{ and } \widehat{\partial}_g^+ \varphi(\bar{x}) = \widehat{\partial}_g^+ \varphi(\bar{x})(-1).$$

They are identical to analytic Fréchet lower and upper subdifferentials defined in (1.1) and (1.2); see [7, Theorem 1.86].

2) Let us recall the notions of subdifferentials of vector-valued functions (indeed, set-valued maps), which were defined by

$$\widehat{\partial} f(\bar{x})(z^*) := \left\{x^* \in X^* : (x^*, -z^*) \in \widehat{N}((\bar{x}, f(\bar{x})); \text{epi}\, f)\right\}.$$

It and its variants were first introduced in [1] and then further developed in [2]; cf. the so-called epi-coderivatives for set-valued maps in [16].

3) In [14, 15], Thibault introduced generalized directional derivatives and subdifferentials of nonconvex vector-valued functions and Lipschitz vector-valued functions via a generalized Clarke-type tangent cone. In [1, 2] and others, authors developed calculus rules for the so-called limiting subdifferentials of vector-valued functions and set-valued maps in the Asplund setting. Recall that a Banach space is *Asplund* if every convex continuous function $\varphi : U \to I\!R$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U. Given $\bar{x} \in \Omega$. Assume that Ω is locally closed around $\bar{x} \in \Omega$,

i.e., there is a neighborhood U of $\bar{x}$ such that $\Omega \cap \operatorname{cl} U$ is a closed set. The (basic, limiting, Mordukhovich) *normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x\to\bar{x}} \widehat{N}(x;\Omega) = \Big\{x^* \in X^* : \exists\, x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in \widehat{N}(x_k;\Omega)\Big\},$$

where $\operatorname{Lim\,sup}$ stands for the sequential Painlevé-Kuratowski outer limit of Fréchet normal cones to Ω at x as x tends to $\bar{x}$.

Proposition 2.1. *Let $f : X \to Z$ be a vector-valued function between normed spaces. The following hold:*

(i) *If $\widehat{\partial}_g f(\bar{x})(z^*) \neq \emptyset$, then $z^* \in K^+$.*
(ii) *If $\widehat{\partial}_g^+ f(\bar{x})(z^*) \neq \emptyset$, then $z^* \in -K^+$.*

Proof. Let us prove (i). Assume that $\widehat{\partial}_g f(\bar{x})(z^*) \neq \emptyset$. Then, there is $x^* \in X^*$ such that $(x^*, -z^*) \in \widehat{N}((\bar{x}, f(\bar{x})); \operatorname{epi} f)$. We have $\bar{x} \times (f(\bar{x}) + K) \subseteq \operatorname{epi} f$. By the monotonicity property of Fréchet normal cones, we have

$$\begin{aligned}(x^*, -z^*) &\in \widehat{N}((\bar{x}, f(\bar{x})); \operatorname{epi} f) \subseteq \widehat{N}((\bar{x}, f(\bar{x})); \bar{x} \times (f(\bar{x}) + K))\\ &= \widehat{N}(\bar{x}; \{\bar{x}\}) \times \widehat{N}(f(\bar{x}); f(\bar{x}) + K\}) = X^* \times -K^+.\end{aligned}$$

obviously verifying that $z^* \in K^+$ and thus the conclusion in (i). Since (ii) is directly derived from (i) due to the fact that $\widehat{\partial}_g^+ f(\bar{x})(z^*) = -\widehat{\partial}_g(-f)(\bar{x})(-z^*)$, the proof is complete. $\square$

Since we use the scalarization approach in establishing new calculus rules for Fréchet vector subdifferentials of vector-valued functions, we need the sum rule for Fréchet subdifferentials of extended-real-valued functions. It was established in [8] by using the smooth variational description of Fréchet subgradients in [9, Theorem 1.88]. Below is a simple proof given in [6]. We also collect conditions under which the sum rule holds as equality.

Theorem 2.1. (Fréchet subdifferential of a sum). *Let $\varphi_i : X \to \overline{I\!R} := I\!R \cup \{+\infty\}$ for $i = 1, 2$ be finite at $\bar{x}$.*
(i) The lower bound.

$$\widehat{\partial}\varphi_1(\bar{x}) + \widehat{\partial}\varphi_2(\bar{x}) \subseteq \widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x}). \tag{2.7}$$

Inclusion (2.7) holds as equality provided that one of the following conditions is fulfilled:

(a) *Either φ_1 or φ_2 is Fréchet differentiable at $\bar{x}$.*
(b) *X is Asplund, either* $\operatorname{epi}\varphi_1$ *or* $\operatorname{epi}\varphi_2$ *is sequentially normally compact, both φ_1 and φ_2 are epigraphically regular at $(\bar{x}, \varphi_i(\bar{x})), i = 1, 2$, respectively, and that*
$$\widehat{\partial}^\infty\varphi_1(\bar{x}) \cap \big(-\widehat{\partial}^\infty\varphi_2(\bar{x})\big) = \{0\}.$$
(c) *X is finite dimensional, both φ_1 and φ_2 are convex, and* $\operatorname{ri}\operatorname{dom}\varphi_1 \cap \operatorname{ri}\operatorname{dom}\varphi_2 \neq \emptyset$.

(ii) The upper bound. *Assume that $\widehat{\partial}^+\varphi_1(\bar{x}) \neq \emptyset$. Then one has*

$$\widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x}) \subseteq \bigcap_{x_1^* \in \widehat{\partial}^+\varphi_1(\bar{x})} \big[x_1^* + \widehat{\partial}\varphi_2(\bar{x})\big]. \tag{2.8}$$

Inclusion (2.8) becomes an equality provided that one of conditions is satisfied:

(a) *Either φ_1 or φ_2 is Fréchet differentiable at $\bar{x}$.*
(d) *X is Asplund, φ_1 is locally Lipschitz continuous around $\bar{x}$, $\varphi := \varphi_1 + \varphi_2$ and $-\varphi_1$ are epigraphically regular at $(\bar{x}, \varphi(\bar{x}))$ and $(\bar{x}, -\varphi_1(\bar{x})$ respectively.*

Proof. To prove (2.7), we may assume both $\widehat{\partial}\varphi_1(\bar{x})$ and $\widehat{\partial}\varphi_2(\bar{x})$ are nonempty without loss of generality. Otherwise, inclusion (2.7) is trivial. Take any $x_i^* \in \widehat{\partial}\varphi_i(\bar{x})$, $i = 1,2$. Due to (1.1) for any $\varepsilon > 0$ there is $\eta > 0$ such that for all $x \in B(\bar{x},\eta)$ one has

$$\varphi_i(x) - \varphi_i(\bar{x}) - \langle x_i^*, x - \bar{x}\rangle + \frac{\varepsilon}{2}\|x - \bar{x}\| \geq 0,\ i = 1,2.$$

Adding two inequalities together one has

$$\big(\varphi_1 + \varphi_2\big)(x) - \big(\varphi_1 + \varphi_2\big)(\bar{x}) - \langle x_1^* + x_2^*, x - \bar{x}\rangle + \varepsilon\|x - \bar{x}\| \geq 0,$$

which implies

$$\liminf_{x \to \bar{x}} \frac{\big(\varphi_1 + \varphi_2\big)(x) - \big(\varphi_1 + \varphi_2\big)(\bar{x}) - \langle x_1^* + x_2^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0$$

since ε was arbitrary. Hence, $x_1^* + x_2^* \in \widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x})$ and (2.7) follows.

To show that (2.7) holds as equality under condition (a) (assume that φ_1 is differentiable at $\bar{x}$), we apply the lower bound (2.7) to a sum of $(\varphi_1 + \varphi_2)$ and $(-\varphi_1)$ and have

$$\begin{aligned}
\widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x}) + \widehat{\partial}(-\varphi_1)(\bar{x}) &\subseteq \widehat{\partial}\varphi_2(\bar{x})\\
\widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x}) - \nabla\varphi_1(\bar{x}) &\subseteq \widehat{\partial}\varphi_2(\bar{x})\\
\widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x}) &\subseteq \nabla\varphi_1(\bar{x}) + \widehat{\partial}\varphi_2(\bar{x})\\
\widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x}) &\subseteq \widehat{\partial}\varphi_1(\bar{x}) + \widehat{\partial}\varphi_2(\bar{x}).
\end{aligned}$$

The above chain holds thanks to $\widehat{\partial}(-\varphi_1)(\bar{x}) = \{-\nabla\varphi_1(\bar{x})\}$ and $\widehat{\partial}\varphi_1(\bar{x}) = \{\nabla\varphi_1(\bar{x})\}$. The last inclusion and (2.7) justify the equality. In such circumstances one has

$$\widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x}) = \nabla\varphi_1(\bar{x}) + \widehat{\partial}\varphi_2(\bar{x}).$$

Condition (b) comes from the limiting subdiferential sum rule, see [9, Theorem 3.36]. Condition (c) can be found in [11, Theorem 3.36].

To establish (ii), we employ the lower bound (2.7) of the Fréchet subdifferential of a sum between $\big(\varphi_1 + \varphi_2\big)$ and $(-\varphi_1)$, and then rewrite it in the desirable formula. Indeed, we have

$$\begin{gathered}
\widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x}) + \widehat{\partial}(-\varphi_1)(\bar{x}) \subseteq \widehat{\partial}\big(\varphi_1 + \varphi_2 - \varphi_1\big)(\bar{x}) = \widehat{\partial}\varphi_2(\bar{x})\\
\widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x}) - \widehat{\partial}^+\varphi_1(\bar{x}) \subseteq \widehat{\partial}\varphi_2(\bar{x})\\
\widehat{\partial}\big(\varphi_1 + \varphi_2\big)(\bar{x}) \subseteq \textstyle\bigcap_{x_1^* \in \widehat{\partial}^+\varphi_1(\bar{x})} \big[x_1^* + \widehat{\partial}\varphi_2(\bar{x})\big].
\end{gathered}$$

Notice that the above calculation works since $\widehat{\partial}\big(\varphi_1 + \varphi_2 - \varphi_1\big)(\bar{x}) = \widehat{\partial}\varphi_2(\bar{x})$ due to the nonemptyness of $\widehat{\partial}^+\varphi_1(\bar{x})$. In details, $\widehat{\partial}^+\varphi_1(\bar{x}) \neq \emptyset$ implies the existence of $x_1^* \in \widehat{\partial}^+\varphi_1(\bar{x})$ such that

$$\varphi_1(x) - \varphi_1(\bar{x}) - \langle x^*, x - \bar{x}\rangle - \varepsilon\|x - \bar{x}\| \leq 0$$

for all x around $\bar{x}$ by (1.2), that is, φ_1 is finite around $\bar{x}$ and $(\varphi_1 + \varphi_2) - \varphi_1 = \varphi_2$ around $\bar{x}$. The proof is complete since the equality of (2.8) under conditions (a) or (d) follows directly from (i) with $\widehat{\partial}^\infty\varphi_1(\bar{x}) = \{0\}$. □

The rest of this section is devoted to illustrate the usefullness of Theorem 2.1. The first example comes from [10]. Let $\varphi_1(x) = |x|\cos^2(x)$, $\varphi_2(x) = -|x|$ and $\varphi(x) = \varphi_1(x) + \varphi_2(x) = -|x|\sin^2(x)$. It is easy to check that $-\varphi_2$ is convex, Lipschitz continuous and φ is differentiable. Hence, $\varphi = (\varphi_1 + \varphi_2)$ is epigraphically regular at $(0,0)$. Moreover, we can compute

$$\widehat{\partial}\varphi_1(0) = [-1,1],\ \widehat{\partial}^+\varphi_2(0) = [-1,1].$$

Then Theorem 2.1(ii)(b′) gives the exact Fréchet subdifferential of φ at 0.

$$\widehat{\partial}\varphi(0) = \bigcap_{x_2^*\in\widehat{\partial}^+\varphi_2(0)} \big[x_2^* + \widehat{\partial}\varphi_1(0)\big] = \bigcap_{x_2^*\in[-1,1]} \big[x_2^* + [-1,1]\big] = \{0\}.$$

Another example is from [9]. It shows that the upper bound (2.8) is "good" enough to estimate the Fréchet subdifferential of functions. Consider $\varphi(x) = |x_1| - |x_2|$ where $x = (x_1, x_2) \in I\!R^2$, then φ can be decomposed as a sum of two functions: $\varphi_1(x_1, x_2) = |x_1|$ and $\varphi_2(x_1, x_2) = -|x_2|$. One can easily check

$$\widehat{\partial}\varphi_1(x) = \begin{cases} (1,0) & \text{if} \quad x_1 > 0,\\ \big\{(x_1^*,0) \;\big|\; x_1^* \in [-1,1]\big\} & \text{if} \quad x_1 = 0,\\ (-1,0) & \text{if} \quad x_1 < 0, \end{cases} \quad \widehat{\partial}\varphi_2(x) = \begin{cases} (0,-1) & \text{if} \quad x_2 > 0,\\ \emptyset & \text{if} \quad x_2 = 0,\\ (0,1) & \text{if} \quad x_2 < 0. \end{cases}$$

Applying Theorem 2.1(ii)(a) one has

$$\widehat{\partial}\varphi(x) = \begin{cases} (1,-1) & \text{if} \quad x_1 > 0, x_2 > 0,\\ (-1,-1) & \text{if} \quad x_1 < 0, x_2 > 0,\\ (-1,1) & \text{if} \quad x_1 < 0, x_2 < 0,\\ (1,1) & \text{if} \quad x_1 > 0, x_2 > 0,\\ \big\{(x_1^*,1) \;\big|\; x_1^* \in [-1,1]\big\} & \text{if} \quad x_1 = 0, x_2 < 0,\\ \big\{(x_1^*,-1) \;\big|\; x_1^* \in [-1,1]\big\} & \text{if} \quad x_1 = 0, x_2 > 0, \end{cases}$$

due to the Fréchet differentiability of φ_1 and φ_2 everywhere except $(0,0)$. At $\bar{x} = (\bar{x}_1, \bar{x}_2) = (0,0)$ one also have

$$\widehat{\partial}\varphi(0) \subseteq \bigcap_{(0,x_2^*)\in\widehat{\partial}\varphi_2(0)} \big[\widehat{\partial}\varphi_1(0) - (0,x_2^*)\big] = \bigcap_{x_2^*\in[-1,1]} \big[[-1,1]\times\{0\} - (0,x_2^*)\big] = \emptyset$$

by employing again Theorem 2.1(ii). The emptiness of the right hand side implies that $\widehat{\partial}\varphi(0) = \emptyset$.

The following example shows that inclusion (2.8) may hold as equality without the epigraphical regularity assumption of the function $\varphi = \varphi_1 + \varphi_2$. Take $\varphi_1(x) = |x|$ and $\varphi_2(x) = -\sqrt{|x|}$. One has $\widehat{\partial}\varphi_1(0) = [-1,1]$, $\widehat{\partial}^+\varphi_2(0) = I\!R$ and

$$\widehat{\partial}\varphi(0) \subseteq \bigcap_{x_2^*\in\widehat{\partial}^+\varphi_2(\bar{x})} \big[\widehat{\partial}\varphi_1(\bar{x}) - x^*\big] = \bigcap_{x_2^*\in I\!R} \big[[-1,1] - x^*\big] = \emptyset.$$

Again, the emptiness of the right hand side implies that $\widehat{\partial}\varphi(0) = \emptyset$, which justifies that inclusion (2.8) holds as equality.

The last example indicates that inclusion (2.8), in general, is strict. Consider $\varphi = \varphi_1 + \varphi_2$ where φ_1, φ_2 are given by

$$\varphi_1(x) = \begin{cases} -\max\{0, x\sin\big(\frac{1}{x}\big)\} & \text{if} \quad x \neq 0,\\ 0 & \text{if} \quad x = 0, \end{cases} \quad \varphi_2(x) = \begin{cases} 0 & \text{if} \quad x \in [0,+\infty),\\ +\infty & \text{if} \quad x \in (-\infty,0). \end{cases}$$

It can be easily checked that $\widehat{\partial}\big(\varphi_1 + \varphi_2\big)(0) = \emptyset$ but one has

$$\bigcap_{x^*\in\widehat{\partial}^+\varphi_1(0)} \big[x^* + \widehat{\partial}\varphi_2(0)\big] = (-\infty,0]$$

since $\widehat{\partial}^+\varphi_1(0) = \{0\}$ and $\widehat{\partial}\varphi_2(0) = N\big(0;[0,+\infty)\big) = (-\infty,0]$.

Another observation is that authors [8] stated that it could not be computed the Fréchet subdifferential of $(\varphi_1 - \varphi_2)$ with $\varphi_1(x) = |x|$ and $\varphi_2(x) = |x|$ due to the emptyness of the

Fréchet upper subdifferential of φ_2. However, by using Theorem 2.1, one has

$$\widehat{\partial}\varphi_1(0) = [-1,1],\ \widehat{\partial}^+(-\varphi_2)(0) = [-1,1],$$
$$\widehat{\partial}\big(\varphi_1 + (-\varphi_2)\big)(0) = \bigcap_{x_2^* \in \widehat{\partial}(-\varphi_2)(0)} \big[x_2^* + \widehat{\partial}\varphi_1(0)\big] = \{0\},$$

where the equality holds due to condition (b′) of Theorem 2.1 (ii).

3. Fréchet vector subdifferential sum rule for vector-valued functions

This section is devoted to study (1) relations between analytic and geometric Fréchet vector subdifferentials and between analytic Fréchet vector subdifferentials and coderivatives of vector-valued functions and (2) sum and chain rules for Fréchet vector subdifferentials.

It is known from [9, Theorem 1.96] that $\widehat{\partial}\varphi(\bar{x}) = \widehat{\partial}_g\varphi(\bar{x})$. Given a vector-valued function $f : X \to Z$ between normed spaces, we scalarize f by a linear operator $z^* \in Z^*$

$$\forall x \in X, \langle z^*, f\rangle(x) = \langle z^*, f(x)\rangle.$$

By [9, Theorem 1.90] $D^*f(\bar{x})(z^*) = \widehat{\partial}\langle z^*, f\rangle(\bar{x})$ provided that f is Lipschitz continuous around $\bar{x}$. We consider another type of subdifferentials of vector-valued functions. Using this idea, we define the *analytic* Fréchet subdifferential of vector-valued functions.

Definition 3.1. Given a vector-valued function $f : X \to Z$ between normed spaces. The *analytic* Fréchet subdifferential of f at $\bar{x}$ with respect to $z^* \in Z^*$ is defined by

$$\widehat{\partial}_a f(\bar{x})(z^*) = \Big\{T \in \mathcal{L}(X,Z) : z^* \circ T \in \widehat{\partial}\langle z^*, f\rangle(\bar{x})\Big\}. \tag{3.9}$$

Obviously, we have

$$\forall z^* \in Z^*, z^* \circ \widehat{\partial}_a f(\bar{x})(z^*) = \widehat{\partial}\langle z^*, f\rangle(\bar{x}).$$

Before formulating and justifying calculus rules for analytic Fréchet vector subdifferentials, we study relations between analytic and geometric Fréchet vector subdifferentials of f at $\bar{x}$ with respect to z^*.

Theorem 3.2. (relationships between analytic and geometric Fréchet subdifferentials). *Let $f : X \to Y$ be a vector-valued function between normed spaces. Then, we have:*

$$\forall z^* \in K^+, \widehat{\partial}_a f(\bar{x})(z^*) \subseteq \widehat{\partial}_g f(\bar{x})(z^*). \tag{3.10}$$

This inclusion holds as equality provided that one of the following conditions is satisfied:

$$\exists\, \ell > 0,\ \big|\langle z^*, z - \bar{z}\rangle\big| \geq \ell\|z - \bar{z}\|,\ \forall\, z \in f(x) + K, \tag{3.11}$$

$$f \ \text{ is locally Lipchitz continuous around } \ \bar{x}. \tag{3.12}$$

Proof. Prove (3.10). Fix an arbitrary element $z^* \in K^+$ and an arbitrary operator $T \in \widehat{\partial}_a f(\bar{x})(z^*)$. Set $x^* := z^* \circ T$ and $\varphi(x) := \langle z^*, f(x)\rangle$. Then, we have $x^* \in \widehat{\partial}\varphi(\bar{x})$, i.e.,

$$\liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0.$$

For every number $\varepsilon > 0$, we find a neighborhood U of $\bar{x}$ such that

$$\forall x \in \operatorname{dom} f \cap U, \langle z^*, f(x)\rangle - \langle z^*, f(\bar{x})\rangle - \langle x^*, x - \bar{x}\rangle \geq -\varepsilon\|x - \bar{x}\|. \tag{3.13}$$

For any pair $(x,z) \in \operatorname{epi} f$, i.e., $z - f(x) \in K$, we have $\langle z^*, z - f(x)\rangle \geq 0$. Rearranging terms in (3.13) while taking into account the last inequality, we have

$$\langle (x^*, -z^*), (x,z) - (\bar{x}, f(\bar{x}))\rangle \leq \varepsilon\|x - \bar{x}\| \leq \varepsilon\|(x,z) - (\bar{x}, f(\bar{x}))\|.$$

clearly verifying $(x^*, -z^*) \in \widehat{N}((\bar{x}, f(\bar{x}); \mathrm{epi}\, f)$ and thus $T \in \widehat{\partial}_g f(\bar{x})(z^*)$. Since z^* was arbitrary in K^+, (3.10) is proved.

Next, we prove that

$$\widehat{\partial}_g f(\bar{x})(z^*) \subseteq \widehat{\partial}_a f(\bar{x})(z^*), \tag{3.14}$$

under one of additional assumptions (3.11) or (3.12).

Case 1: Assume that (3.11) holds. Arguing by contradiction, assume that there exists $T \in \widehat{\partial}_g f(\bar{x})(z^*)$ but $T \notin \widehat{\partial}_a f(\bar{x})(z^*) = \widehat{\partial}\langle z^*, f\rangle(\bar{x})$. Again, set $x^* := z^* \circ T$. By the definition of analytic Fréchet subdifferential, we can find $\varepsilon > 0$ and a sequence $x_k \to \bar{x}$ as $k \to \infty$ such that

$$\eta_k := \big\langle z^*, f(x_k)\big\rangle - \big\langle z^*, f(\bar{x})\big\rangle - \langle x^*, x_k - \bar{x}\rangle + \varepsilon\|x_k - \bar{x}\| < 0. \tag{3.15}$$

Since K is a cone and $z^* \in K^+ \setminus \{0\}$, we find $\theta_k \in K$ with $\langle z^*, \theta_k\rangle = -\eta_k$. By choosing $z_k = f(x_k) + \theta_k$, the sequence $(x_k, z_k) \in \mathrm{epi}\, f$ and

$$\langle (x^*, -z^*), (x_k, z_k) - (\bar{x}, \bar{z})\rangle = \varepsilon\|x_k - \bar{x}\|.$$

Taking into account (3.11), one can estimate

$$\ell\|z_k - \bar{z}\| \le \big|\langle z^*, z_k - \bar{z}\rangle\big| = \big|\varepsilon\|x_k - \bar{x}\| - \langle x^*, x_k - \bar{x}\rangle\big| \le (\varepsilon + \|x^*\|)\|x_k - \bar{x}\|.$$

Thus, $\|z_k - \bar{z}\| \le \bar{\varepsilon}\|x_k - \bar{x}\|$ with $\bar{\varepsilon} = \ell^{-1}(\varepsilon + \|x^*\|)$. We now have

$$\begin{aligned} \frac{\big\langle (x^*, -z^*), (x_k, z_k) - (\bar{x}, \bar{z})\big\rangle}{\|(x_k, z_k) - (\bar{x}, \bar{z})\|} &= \frac{\varepsilon\|x_k - \bar{x}\|}{\|x_k - \bar{x}\| + \|z_k - \bar{z}\|} \\ &\ge \frac{\varepsilon\|x_k - \bar{x}\|}{\|x_k - \bar{x}\| + \bar{\varepsilon}\|x_k - \bar{x}\|} \ge \frac{\varepsilon}{1 + \bar{\varepsilon}} \end{aligned}$$

for all $k \in \mathbb{N}$. Passing to the limit as $k \to \infty$, we have

$$\limsup_{(x,z) \xrightarrow{\mathrm{epi}\ f} (\bar{x}, \bar{z})} \frac{\big\langle (x^*, -z^*), (x, z) - (\bar{x}, \bar{z})\big\rangle}{\|(x, z) - (\bar{x}, \bar{z})\|} \ge \frac{\varepsilon}{1 + \bar{\varepsilon}} > 0,$$

which implies that $T \notin \widehat{\partial}_g f(\bar{x})(z^*)$. This contradiction establishes (3.14) under condition (3.11).

Case 2: Assume that (3.12) holds. Fix an arbitrary element $z^* \in K^+$ and an arbitrary operator $T \in \widehat{\partial}_g f(\bar{x})(z^*)$, and set $x^* := z^* \circ T$ and $\varphi(x) = \langle z^*, f(x)\rangle$. Then, we have

$$(x^*, -z^*) \in \widehat{N}((\bar{x}, f(\bar{x})); \mathrm{epi}\, f) \subseteq \widehat{N}((\bar{x}, f(\bar{x})); \mathrm{gph}\, f).$$

By the definition of Fréchet coderivative, we have

$$\limsup_{\substack{(x,z) \to (\bar{x}, f(\bar{x})) \\ (x,z) \in \mathrm{gph} f}} \frac{\langle (x^*, -z^*), (x, z) - (\bar{x}, f(\bar{x}))\rangle}{\|(x, z) - (\bar{x}, f(\bar{x}))\|} \le 0.$$

For every number $\varepsilon > 0$, we find a neighborhood U of $\bar{x}$ such that

$$\begin{aligned} & \forall x \in \mathrm{dom}\, f \cap U, \langle x^*, x - \bar{x}\rangle - \langle z^*, f(x)\rangle + \langle z^*, f(\bar{x})\rangle \le \varepsilon\|(x, z) - (\bar{x}, f(\bar{x}))\| \\ \Leftrightarrow\ & \forall x \in \mathrm{dom}\, f \cap U, \langle z^*, f(x)\rangle - \langle z^*, f(\bar{x})\rangle - \langle x^*, x - \bar{x}\rangle \\ & \qquad \ge -\varepsilon\|(x, z) - (\bar{x}, f(\bar{x}))\| \ge -\varepsilon(1 + \ell)\|x - \bar{x}\| \\ \Leftrightarrow\ & \liminf_{x \to \bar{x}} \frac{\langle z^*, f(x)\rangle - \langle z^*, f(\bar{x})\rangle - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \ge -\varepsilon(1 + \ell). \end{aligned}$$

Since ε was arbitrary, we have $x^* \in \widehat{\partial}\langle z^*, f(\bar{x})\rangle$ and thus $T \in \widehat{\partial}_a f(\bar{x})(z^*)$. The proof is complete.

□

Remark 3.2. When $Z = I\!R$, condition (3.11) automatically holds. Therefore, Theorem 3.2 confirms that the analytic and geometric subdifferentials of extended-real-valued functions are identical.

The next result provides a relation between Fréchet subdifferentials and Fréchet coderivatives of vector-valued functions.

Theorem 3.3. (relationships between Fréchet subdifferentials and Fréchet coderivatives). *Let $f : X \to Z$ be a vector-valued function between normed spaces. Then, we have*

$$\forall z^* \in Z^*, z^* \circ \widehat{\partial}_g f(\bar{x})(z^*) \subseteq \widehat{D}^* f(\bar{x})(z^*). \tag{3.16}$$

Assume in addition that f enjoys the order-lowersemicontinuity at $\bar{x}$ in the sense that for every sequence $\{(x_k, z_k)\} \subseteq$ epi f converging to $(\bar{x}, f(\bar{x}))$, a subsequence of $\{f(x_k)\}$ converges to $f(\bar{x})$; which is automatic if f is continuous at $\bar{x}$. Then, inclusion (3.16) holds as equality.

Proof. Fix an arbitrary element $z^* \in Z^*$. By the monotonicity of Fréchet normal cones, we get from gph $f \subseteq$ epi f that

$$\widehat{N}((\bar{x}, f(\bar{x})); \text{epi } f) \subseteq \widehat{N}((\bar{x}, f(\bar{x})); \text{gph } f)$$

and thus we have $z^* \circ \widehat{\partial}_g f(\bar{x})(z^*) \subseteq \widehat{D}^* f(\bar{x})(z^*)$. Since z^* was arbitrary, (3.16) is satisfied.

Next, we will show the inverse inclusion $\widehat{D}^* f(\bar{x})(z^*) \subseteq z^* \circ \widehat{\partial}_g f(\bar{x})(z^*)$ under the additional assumption. Fix an arbitrary element $z^* \in Z^*$. Assume that there exists $x^* \in \widehat{D}^* f(\bar{x})(z^*)$; otherwise, the inclusion is trivial. By the definition of coderivative, we have

$$\limsup_{\substack{(x,z)\to(\bar{x},f(\bar{x}))\\ (x,z)\in \text{gph} f}} \frac{\langle (x^*, -z^*), (x,z) - (\bar{x}, f(\bar{x}))\rangle}{\|(x,z) - (\bar{x}, f(\bar{x}))\|} \le 0.$$

To justify that $x^* \in z^* \circ \widehat{\partial}_g f(\bar{x})(z^*)$, we choose a sequence $\{(x_k, z_k)\} \subseteq$ epi f converging to $(\bar{x}, f(\bar{x}))$ such that

$$\limsup_{\substack{(x,z)\to(\bar{x},f(\bar{x}))\\ (x,z)\in \text{epi} f}} \frac{\langle (x^*, -z^*), (x,z) - (\bar{x}, f(\bar{x}))\rangle}{\|(x,z) - (\bar{x}, f(\bar{x}))\|} = \lim_{k\to\infty} \frac{\langle (x^*, -z^*), (x_k,z_k) - (\bar{x}, f(\bar{x}))\rangle}{\|(x_k,z_k) - (\bar{x}, f(\bar{x}))\|}.$$

Since $z_k - f(x_k) \in K$ and $z_k - f(x_k) \to 0$ as $k \to \infty$, we have

$$\begin{aligned}\lim_{k\to\infty} \frac{\langle (x^*, -z^*), (x_k,z_k) - (\bar{x}, f(\bar{x}))\rangle}{\|(x_k,z_k) - (\bar{x}, f(\bar{x}))\|} &= \lim_{k\to\infty} \frac{\langle (x^*, -z^*), (x_k,f(x_k)) - (\bar{x}, f(\bar{x}))\rangle}{\|(x_k,f(x_k)) - (\bar{x}, f(\bar{x}))\|} \\ &\le \limsup_{\substack{(x,z)\to(\bar{x},f(\bar{x}))\\ (x,z)\in \text{gph} f}} \frac{\langle (x^*, -z^*), (x,z) - (\bar{x}, f(\bar{x}))\rangle}{\|(x,z) - (\bar{x}, f(\bar{x}))\|}.\end{aligned}$$

Therefore, we have

$$\limsup_{\substack{(x,z)\to(\bar{x},f(\bar{x}))\\ (x,z)\in \text{epi} f}} \frac{\langle (x^*, -z^*), (x,z) - (\bar{x}, f(\bar{x}))\rangle}{\|(x,z) - (\bar{x}, f(\bar{x}))\|} \le 0$$

clearly justifying that $x^* \in z^* \circ \widehat{\partial}_g f(\bar{x})(z^*)$. □

Proposition 3.2. *If $\varphi : X \to I\!R \cup \{+\infty\}$ is lower semicontinuous at $\bar{x}$, then φ enjoys the order-semicontinuity property at $\bar{x}$.*

Proof. For every sequence $\{(x_k, \alpha_k)\} \subseteq \operatorname{epi}\varphi$ converging to $(\bar{x}, \varphi(\bar{x}))$. Since φ is lower semicontinuous at $\bar{x}$, we have

$$\varphi(\bar{x}) = \lim_{k\to\infty} z_k \geq \liminf \varphi(x_k) \geq \varphi(\bar{x})$$

which implies that there is a subsequence of $\{\varphi(x_k)\}$ converging to $\varphi(\bar{x})$. The proof is complete. $\square$

Remark 3.3. 1) [9, Theorem 1.80] proved a similar result for limiting subdifferential objects for continuous real-valued functions.

2) Let us show by example that the order-semicontinuous condition is essential. Consider $\varphi : I\!R \to I\!R$ with $\varphi(x) = x$ for $x \leq 0$ and -1 otherwise. Then, φ is not order-semicontinuous at 0; indeed, the sequence $\{1/k, 0\} \subseteq \operatorname{epi}\varphi$ converges to $(0,0)$, but the sequence $\varphi(1/k) = -1$ does not converge to $\varphi(0)$. Obviously, we have $\widehat{\partial}_g\varphi(0) = \emptyset$, but $\widehat{D}\varphi(0)(1) = [1, +\infty)$.

Now, we establish a Fréchet vector subdifferential sum rule for vector-valued functions.

Theorem 3.4. (Fréchet vector subdifferential sum rule). *Let $f_i : X \to Z$ between normed spaces with $\bar{z}_i = f_i(\bar{x})$, $i = 1, 2$ and $\bar{z} = \bar{z}_1 + \bar{z}_2$. Assume that $\widehat{\partial}^+_a f_1(\bar{x})(-z^*) \neq \emptyset$. Then one has*

$$\widehat{\partial}_a(f_1 + f_2)(\bar{x})(z^*) \subseteq \bigcap_{T_1 \in -\widehat{\partial}^+_a f_1(\bar{x})(-z^*)} \big[T_1 + \widehat{\partial}_a f_2(\bar{x})(z^*)\big]. \tag{3.17}$$

Inclusion (3.17) holds as equality provided that either f_1 or f_2 is Fréchet differentiable at $\bar{x}$.

Proof. Set $\varphi_i(x) := \langle z^*, f_i\rangle(x)$ for $i = 1, 2$. Fix an arbitrary element $T \in \widehat{\partial}_a\big(f_1 + f_2\big)(\bar{x})(z^*)$; if not, inclusion (3.17) is trivial. By the definition of analytic Fréchet vector subdifferentials, $x^* = z^* \circ T \in \widehat{\partial}\langle z^*, f_1 + f_2\rangle(\bar{x})$. Since $\widehat{\partial}^+_a f_1(\bar{x})(-z^*) \neq \emptyset$, we have $\widehat{\partial}^+\langle z^*, f_1\rangle(\bar{x}) \neq \emptyset$ and thus we could apply the Fréchet sum rule to obtain

$$x^* \in \widehat{\partial}\langle z^*, f_1 + f_2\rangle(\bar{x}) \subseteq \bigcap_{x_1^* \in \widehat{\partial}^+\langle z^*, f_1\rangle(\bar{x})} \Big[x_1^* + \widehat{\partial}\langle z^*, f_2\rangle(\bar{x})\Big].$$

For an arbitrary element $x_1^* \in \widehat{\partial}^+\langle z^*, f_1\rangle(\bar{x})$, we find $T_1 \in \widehat{\partial}^+_a f_1(\bar{x})(-z^*)$ such that $x_1^* = (-z^*) \circ T_1$. Hence, we have

$$x^* = z^* \circ T \in (-z^*) \circ T_1 + \widehat{\partial}\langle z^*, f_2\rangle(\bar{x}),$$

which is equivalent to $z^* \circ (T + T_1) \in \widehat{\partial}\langle z^*, f_2\rangle(\bar{x})$ or $T_2 = T + T_1 \in \widehat{\partial}_a f_2(\bar{x})(z^*)$. This justifies inclusion (3.17). $\square$

Theorem 3.4 reduces to the known Fréchet sum rule by taking $Z = \mathbb{R}$, $K = \mathbb{R}_+$, $z^* = 1 \in K^+ = [0, +\infty)$ since in this special case, we have $\widehat{\partial}_a f(\bar{x})(z^*) = \widehat{\partial}\langle z^*, f\rangle(\bar{x}) = \widehat{\partial}_g f(\bar{x})$.

In the rest of this section we are going to formulate the analytic Fréchet subdifferentials of generalized compositions given by

$$\big(f \circ g\big)(x) := f\big(x, g(x)\big)$$

where $f : X \times Y \to Z$ and $g : X \to Y$ are vector-valued functions between normed spaces via the corresponding analytic Fréchet lower/upper vector subdifferentials of f and g.

Theorem 3.5. (Fréchet subdifferentials of generalized compositions). *Consider the generalized composition $\big(f \circ g\big)$ above. Assume that $\widehat{\partial}^+ f(\bar{x},\bar{y})(-z^*) \neq \emptyset$ with $z^* \in K^+$. Then, one has*

$$z^* \circ \widehat{\partial}_a\big(f \circ g\big)(\bar{x})(z^*) \subseteq \bigcap_{(x^*,y^*)\in(-z^*)\circ\widehat{\partial}_a^+ f(\bar{x},\bar{y})(-z^*)} \big[x^* + \widehat{D}^* g(\bar{x})(y^*)\big]. \tag{3.18}$$

If, in addition, either g satisfies condition (3.11) with respect to y^* or g is locally Lipschitz continuous around $\bar{x}$, the coderivative $\widehat{D}^* g(\bar{x})(y^*)$ in (3.18) is replaced by $\widehat{\partial}_a g(\bar{x})(y^*)$, i.e.,

$$\widehat{\partial}_a\big(f \circ g\big)(\bar{x})(z^*) \subseteq \bigcap_{(T_x,T_y)\in\widehat{\partial}_a^+ f(\bar{x},\bar{y})(-z^*)} \big[T_x + \widehat{\partial}_a g(\bar{x})(z^* \circ T_y)\big].$$

Inclusion (3.18) holds as equality if f is Fréchet differentiable at $(\bar{x},\bar{y})$.

Proof. Take $x^* \in z^* \circ \widehat{\partial}_a\big(f \circ g\big)(\bar{x})(z^*)$ and consider the special sum function $\Phi : X \times Y \to Z$ given in the form

$$\Phi(x,y) := f(x,y) + \Delta\big((x,y);\operatorname{gph} g\big),$$

where $\Delta(\cdot;\operatorname{gph} g) : X \times Y \to Z$ is the indicator function with $\Delta((x,y);\operatorname{gph} g) = 0$ for all $(x,y) \in \operatorname{dom}\Delta(\cdot;\operatorname{gph} g) := \operatorname{gph} g$. We observe that

$$\begin{aligned}
&\liminf_{\substack{(x,y)\to(\bar{x},\bar{y})\\ z=\langle z^*,\Phi(x,y)\rangle}} \frac{\big\langle(-x^*,0,z^*),(x,y,z)-(\bar{x},\bar{y},\bar{z})\big\rangle}{\|(x,y)-(\bar{x},\bar{y})\|}\\
= &\liminf_{\substack{(x,y)\to(\bar{x},\bar{y})\\ y=g(x),z=\langle z^*,f(x,y)\rangle}} \frac{\big\langle(-x^*,z^*),(x,z)-(\bar{x},\bar{z})\big\rangle}{\|(x,y)-(\bar{x},\bar{y})\|}\\
\geq &\liminf_{\substack{x\to\bar{x}\\ z=\langle z^*,(f\circ g)(x)\rangle}} \min\Big\{0, \frac{\big\langle(-x^*,z^*),(x,z)-(\bar{x},\bar{z})\big\rangle}{\|x-\bar{x}\|}\Big\} \geq 0
\end{aligned}$$

where the last inequality (≥ 0) holds since $x^* \in z^* \circ \widehat{\partial}_a\big(f \circ g\big)(\bar{x})(z^*)$.

Applying Theorem 3.4 with $f_1 = f$ and $f_2 = \Delta(\cdot;\operatorname{gph} g)$ together with $\widehat{D}^*\Delta(\cdot;\operatorname{gph} g)(\bar{x},\bar{y})(z^*) = \widehat{N}\big((x,y);\operatorname{gph} g\big) = \widehat{\partial}_a\Delta(\cdot;\operatorname{gph} g)(\bar{x},\bar{y})(z^*)$ one has

$$\widehat{\partial}_a\big(f \circ g\big)(\bar{x})(z^*) \subseteq \bigcap_{(x^*,y^*)\in(-z^*)\circ\widehat{\partial}_a^+ f(\bar{x},\bar{y})(-z^*)} \big[x^* + \widehat{D}^* g(\bar{x})(y^*)\big].$$

Other assertions follows from Theorem 3.4 and Theorem 3.3. □

It is worth mentioning that Theorem 3.7 [8] holds without the Lipschitzian continuity of f by Theorem 3.5, and that if we take $f(x,y) = f(y)$, we has the usual composition. Since

$$y^* \in (-z^*) \circ \widehat{\partial}_a^+ f(\bar{y})(-z^*) \quad \text{implies} \quad (0,y^*) \in (-z^*) \circ \widehat{\partial}_a^+ f(\bar{x},\bar{y})(-z^*),$$

we obtain the following corollary.

Corollary 3.1. (Fréchet vector subdifferentials of usual compositions). *Let $g : X \to Y$ and $f : Y \to Z$ be vector-valued functions between normed spaces, and let $\bar{y} = g(\bar{x})$ and $\bar{z} = \big(f \circ g\big)(\bar{x})$. Assume that $\widehat{\partial}_a^+ f(\bar{y}) \neq \emptyset$ for some dual element $z^* \in K^+$. Then one has*

$$z^* \circ \widehat{\partial}_a\big(f \circ g\big)(\bar{x})(z^*) \subseteq \bigcap_{y^*\in(-z^*)\circ\widehat{\partial}_a^+ f(\bar{y})(-z^*)} \widehat{D}^* g(\bar{x})(y^*) \subseteq \widehat{D}^* g(\bar{x}) \circ \widehat{\partial}_a^+ f(\bar{y})(-z^*).$$

As consequences of Theorem 3.5 and Corollary 3.1 with $f = \varphi : X \times Y \to I\!R$, we arrive at the results in [8] without the Lipschitzian continuity assumption imposed on the function g.

Corollary 3.2. (Fréchet subdifferentials of real-valued generalized compositions). *Let X, Y be normed spaces, $g : X \to Y$ be a vector-valued function, $\varphi : X \times Y \to I\!R$ be an extended-real-valued function. Assume that $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) \neq \emptyset$ and $\bar{y} = g(\bar{x})$. Then, one has*

$$\widehat{\partial}\big(\varphi \circ g\big)(\bar{x}) \subseteq \bigcap_{(x^*,y^*)\in\widehat{\partial}^+\varphi(\bar{x},\bar{y})} \big[x^* + \widehat{D}^* g(\bar{x})(y^*)\big].$$

The above inclusion holds as equality if φ is Fréchet differentiable at $(\bar{x}, \bar{y})$.

Corollary 3.3. (Fréchet subdifferentials of real-valued usual compositions). *Let X, Y be normed spaces, $g : X \to Y$ be a vector-valued function, and $\varphi : Y \to I\!R$ be a real-valued function. Assume that $\widehat{\partial}^+\varphi(\bar{y}) \neq \emptyset$ and $\bar{y} = g(\bar{x})$. Then one has*

$$\widehat{\partial}\big(\varphi \circ g\big)(\bar{x}) \subseteq \bigcap_{y^*\in\widehat{\partial}^+\varphi(\bar{y})} \widehat{D}^* g(\bar{x})(y^*) \subseteq \widehat{D}^* g(\bar{x}) \circ \widehat{\partial}^+ f(\bar{y}).$$

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