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Dedicated to Prof. Hong-Kun Xu on the occasion of his 60^{th} anniversary

Approximating fixed points of enriched nonexpansive mappings in Banach spaces by using a retraction-displacement condition

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ABSTRACT. In this paper, we prove convergence theorems for a fixed point iterative algorithm of Krasnoselskij-Mann type associated to the class of enriched nonexpansive mappings in Banach spaces. The results are direct generalizations of the corresponding ones in [Berinde, V., *Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces*, Carpathian J. Math., **35** (2019), No. 3, 293–304.], from the setting of Hilbert spaces to Banach spaces, and also of some results in [Senter, H. F. and Dotson, Jr., W. G., *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., **44** (1974), No. 2, 375–380.], [Browder, F. E., Petryshyn, W. V., *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20** (1967), 197–228.], by considering enriched nonexpansive mappings instead of nonexpansive mappings. Many other related results in literature can be obtained as particular instances of our results.

1. INTRODUCTION

Let C be a nonempty subset of a real normed linear space X. A map $T : C \to C$ is called *enriched nonexpansive*, see [3], if there exists $b \in [0, \infty)$ such that

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\|x-y\|, \forall x, y \in C. \tag{1.1}$$

To indicate the constant involved in (1.1) we also call T a *b-enriched nonexpansive mapping*.

It is easily seen that nonexpansive mappings are obtained for $b = 0$ in (1.1), hence nonexpansive mappings are included in the class of enriched nonexpansive mappings (and the reverse in not valid, as shown in [3]).

An element $x \in C$ is said to be a *fixed point* of T if $Tx = x$. Denote by $Fix\,(T)$ the set of all fixed points of T. Recall that T is called *quasi-nonexpansive* if $\|Tx-y\| \leq \|x-y\|, \forall x \in C$ and for all $y \in Fix\,(T)$.

It has been shown in [3] that the class of enriched nonexpansive mappings and quasi-nonexpansive mappings are independent to each other. Motivated by the richness of the class of enriched nonexpansive mappings, in [3] we studied the existence of their fixed points in the setting of Hilbert spaces and proved weak and strong convergence theorems for the fixed point iterative scheme defined by $x_0 \in C$ and

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0.$$

The strong convergence result in [3] (Theorem 2.2) is stated as follows.

Theorem 1.1. *Let C be a bounded closed convex subset of a Hilbert space H and $T : C \to C$ be an enriched nonexpansive and demicompact mapping. Then the set $Fix\,(T)$ of fixed points of T is a*

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nonempty convex set and there exists $\lambda \in (0,1)$ *such that, for any given* $x_0 \in C$, *the Krasnoselskij iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0,$$

converges strongly to a fixed point of T.

The main purpose of this paper is to extend the strong convergence results established in [3] from the setting of Hilbert spaces to that of Banach spaces. Our results thus obtained are more general than those from [3], mainly due to the fact that here we use a retraction-displacement condition instead of the demicompactness property of T.

2. The retraction-displacement condition

In order to prove our main results in the next section, we need some definitions and auxiliary results, mainly taken from [20] and [36]. Throughout the rest of this paper, X will be a Banach space and C a nonempty subset of X.

According to [36], a mapping $T: C \to C$ with $Fix\,(T) \neq \emptyset$ is said to satisfy *Condition I*, if there exists a nondecreasing function $f: [0,\infty) \to [0,\infty)$ with the properties $f(0) = 0$ and $f(r) > r$, for $r > 0$, such that

$$\|x - Tx\| \geq f(d(x, Fix\,(T)), \forall x \in C, \tag{2.2}$$

where

$$d(x, Fix\,(T)) = \inf\{\|x - z\| : z \in Fix\,(T)\}$$

is the distance between the point x and the set $Fix\,(T)$.

Remark 2.1. Note that a condition of the form (2.2) is usually referred as *retraction-displacement condition*, see the recent papers [10] and [34], where the authors studied the fixed point equation $x = Tx$ in terms of a retraction-displacement condition and have given various examples, corresponding to Picard, Krasnoselskii, Mann and Halpern iterative algorithms.

To be more explicit, we present Definition 1 in [10], transposed in the case of Banach spaces. Let $r: X \to Fix\,(T)$ be a set retraction.

Then, a mapping $T: C \to C$ is said to satisfy a (Ψ, r) *retraction-displacement condition* if

(i) $\Psi: [0,\infty) \to [0,\infty)$ is increasing, continuous at 0 and $\Psi(0) = 0$;

(ii) $\|x - r(x)\| \leq \Psi(\|x - Tx\|), \forall x \in C$.

It is easily seen that, if Ψ is a bijection and $Fix\,(T)$ is a singleton (which happens in most of the cases reported in [10]), then condition (ii) above formally implies (2.2) with $f = \Psi^{-1}$.

To prove our main result in this paper we need the following useful lemmas (Lemma 3 in [20] and Lemma 1 in [36], respectively).

Lemma 2.1. (Lemma 3, [20]) *Suppose* E *is a uniformly convex Banach space. Suppose* $0 < a < b < 1$, *and* $\{t_n\}$ *is a sequence in* $[a,b]$. *Suppose* $\{w_n\}$, $\{y_n\}$ *are sequences in* E *such that* $\{\|w_n\| \leq 1\}$, $\{\|y_n\| \leq 1\}$, *for all* n. *Define* $\{z_n\}$ *in* E *by* $z_n = (1-t_n)w_n + t_n y_n$. *If*

$$\lim_{n\to\infty} \|z_n\| = 1 \text{ then } \lim_{n\to\infty} \|w_n - y_n\| = 0.$$

Lemma 2.2. (Lemma 1, [36]) *Suppose* C *is a closed bounded subset of a Banach space* X *and let* $T: C \to C$ *be a mapping satisfying* $Fix\,(T) \neq \emptyset$. *If* $I - T$ *maps closed bounded subsets of* C *onto closed subsets of* X *then* T *satisfies Condition I.*

A simpler form of Condition I, called Condition II, has been also introduced in [36] and corresponds to the particular case $f(t) = \alpha t$, with $\alpha > 0$ a real number, in Condition I. Condition II is equivalent to the retraction-displacement conditions obtained in [10] for various fixed point algorithms and in connection with some particular contractive conditions.

For example, if T is a contraction with contraction coefficient $c \in (0, 1)$, then we get $f(t) = (1 - c)t$ (in the case of Condition I) and $\alpha = 1 - c$ (in the case of Condition II).

3. Approximating fixed points of enriched nonexpansive mappings in Banach spaces

Now we can state and prove the main result of this paper.

Theorem 3.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a b-enriched nonexpansive mapping. Suppose T satisfies Condition I. Then $Fix\,(T) \neq \emptyset$ and, for any $\lambda \in \left(0, \frac{1}{b+1}\right)$ and for any given $x_0 \in C$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.3}$$

converges strongly to a fixed point of T.

Proof. Since T is b-enriched nonexpansive, there exists a constant b, $b \in [0, \infty)$, such that

$$\|b(x - y) + Tx - Ty\| \leq (b + 1)\|x - y\|, \forall x, y \in C.$$

By putting $b = \dfrac{1}{\mu - 1}$, it follows that $\mu \in (0, 1]$ and the previous inequality is equivalent to

$$\|(1 - \mu)(x - y) + \mu Tx - \mu Ty\| \leq \|x - y\|, \forall x, y \in C. \tag{3.4}$$

Denote $T_\mu x = (1 - \mu)x + \mu Tx$. Then inequality (3.4) expresses the fact that the averaged operator T_μ is nonexpansive, i.e.,

$$\|T_\mu x - T_\mu y\| \leq \|x - y\|, \forall x, y \in C.$$

By means of Browder-Goede-Kirk fixed point theorem (e.g., Theorem 4 in [17]), it follows that T_μ has at least one fixed point.

As a direct consequence, T_μ is also quasi-nonexpansive, i.e., $\|T_\mu x - y\| \leq \|x - y\|, \forall x \in C$ and for all $y \in Fix\,(T_\mu)$.

Now we want to prove that the iterative process $\{u_n\}$ defined by $u_0 \in C$ arbitrary, $\lambda \in (0, 1)$ and

$$u_{n+1} = (1 - \lambda)u_n + \lambda T_\mu u_n,\ n \geq 0, \tag{3.5}$$

converges to a fixed point of T_μ. We split the proof into two steps.

Step 1. We prove that $\lim\limits_{n\to\infty} d(u_n, Fix\,(T_\mu)) = 0$.

Let $u_0 \in C$ be the initial value for the algorithm (3.5). If actually $u_0 \in Fix\,(T_\mu)$, then the conclusion is trivial.

So, assume $u_0 \in C \setminus Fix\,(T_\mu)$. Then, for any $z \in Fix\,(T_\mu)$ we have

$$\|T_\mu u_n - z\| \leq \|x_n - z\|, \forall n \geq 0$$

and hence

$$\|u_{n+1} - z\| \leq (1 - \lambda)\|u_n - z\| + \lambda\|T_\mu u_n - z\| \leq \|u_n - z\|.$$

This shows that the sequence of nonnegative real numbers $\{d(u_n, Fix\,(T_\mu))\}$ is non increasing, hence $\lim\limits_{n\to\infty} d(u_n, Fix\,(T_\mu))$ exists. Denote

$$\lim_{n\to\infty} d(u_n, Fix\,(T_\mu)) = d \geq 0.$$

To prove that $d = 0$, let us admit that $d > 0$. Then for any $p \in Fix\,(T_\mu)$, we have

$$\lim_{n\to\infty} \|u_n - p\| = d_1 \geq d > 0, \tag{3.6}$$

and, since $\{\|u_n - p\|\}$ is decreasing, one can choose $N > 0$ such that

$$\|u_n - p\| \leq 2d_1, \forall n \geq N.$$

If we consider now the sequences $\{w_n\}$ and $\{y_n\}$ defined for $n \geq 0$ by

$$y_n = \frac{1}{\|u_n - p\|}\,(T_\mu u_n - p) \text{ and } w_n = \frac{1}{\|u_n - p\|}\,(u_n - p)$$

then it is easily seen that, by the nonexpansivity of T_μ, we get

$$\|y_n\| \leq 1 \text{ and } \|w_n\| = 1, \forall n \geq 0.$$

Now, by Condition I and in view of (3.6), for $n \geq N$ we have

$$\|w_n - y_n\| = \frac{\|Tu_n - x_n\|}{\|u_n - p\|} \geq \frac{d(u_n, Fix\,(T_\mu))}{\|u_n - p\|} \geq \frac{f(d)}{2d_1} > 0,$$

which shows that

$$\lim_{n\to\infty} \|w_n - y_n\| \neq 0. \tag{3.7}$$

On the other hand, we also have

$$\lim_{n\to\infty} \|(1-\lambda)w_n + \lambda y_n\| = \lim \frac{\|u_{n+1} - p\|}{\|u_n - p\|} = \frac{d_1}{d_1} = 1. \tag{3.8}$$

Therefore, in view of (3.7) and (3.8), by Lemma 2.1 (with $t_n = \lambda \in (0,1)$) we would obtain

$$\lim \|(1-\lambda)w_n + \lambda y_n\| \neq 1,$$

a contradiction. This proves that $d = 0$, that is,

$$\lim_{n\to\infty} d(u_n, Fix\,(T_\mu)) = 0. \tag{3.9}$$

Step 2. We prove that $\{u_n\}$ converges to $q \in Fix\,(T_\mu)$.

By (3.9) we deduce that, for any $\varepsilon > 0$, there exists a rank N_ε and $z_\varepsilon \in Fix\,(T_\mu)$ such that

$$\|x_{N_\varepsilon} - z_\varepsilon\| < \varepsilon,$$

which implies

$$\|u_n - z_\varepsilon\| < \varepsilon, \forall n \geq N_\varepsilon,$$

in view of the fact that the sequence $\{\|u_n - z_\varepsilon\|\}$ is nonincreasing. Now, if we let $\varepsilon_k = \dfrac{1}{2^k}$, for $k \in \mathbb{N}$, then, corresponding to each ε_k, there is a rank $N_k > 0$ and a fixed point $z_k \in Fix\,(T_\mu)$ such that

$$\|u_n - z_k\| < \frac{\varepsilon}{4}, \forall n \geq N_k. \tag{3.10}$$

We can construct the sequence $\{N_k\}$ such that $N_{k+1} \geq N_k$, again due to the fact that the sequence $\{\|u_n - z_\varepsilon\|\}$ is nonincreasing and, by the same reason, for $k \in \mathbb{N}$ we have

$$\|z_{k+1} - z_k\| = \|z_{k+1} - u_{N_{k+1}} + u_{N_{k+1}} - z_k\|$$

$$\leq \|z_{k+1} - u_{N_{k+1}}\| + \|u_{N_{k+1}} - z_k\| \leq \|u_{N_k} - z_k\| + \|u_{N_{k+1}} - z_{k+1}\|$$

$$< \frac{\varepsilon_k}{4} + \frac{\varepsilon_{k+1}}{4} = \frac{3}{4}\varepsilon_{k+1}.$$

Consider the ball centered at z and of radius ε, that is, $S(z,\varepsilon) := \{x \in X : \|x-z\| \le \varepsilon\}$. If $x \in S(z_{k+1}, \varepsilon_{k+1})$ arbitrary, we have

$$\|z_k - x\| = \|z_k - z_{k+1} + z_{k+1} - x\| < \frac{3}{4}\varepsilon_{k+1} + \|z_{k+1} - x\|$$
$$< \frac{3}{4}\varepsilon_{k+1} + \varepsilon_{k+1} < 2\varepsilon_{k+1} = \varepsilon_k.$$

This proves that $S(z_{k+1}, \varepsilon_{k+1}) \subset S(z_k, \varepsilon_k)$, for all $k \in \mathbb{N}$ and therefore, $\{S(z_{k+1}, \varepsilon_{k+1})\}_{k \in \mathbb{N}}$ is a non increasing sequence of nonempty closed balls with radii $\varepsilon_k \to 0$, as $k \to \infty$.

By the Cantor intersection theorem,

$$\bigcap_{k \in \mathbb{N}} S(z_k, \varepsilon_k) = \{q\} \text{ (say)}.$$

Now, since $Fix\,(T_\mu)$ is closed (by a result in [21]) and $\{z_k\} \subset Fix\,(T_\mu)$ converges to q, it follows that $q \in Fix\,(T_\mu)$.

Hence, by (3.10) and the inequality $\|u_n - q\| \le \|u_n - z_k\| + \|z_k - q\|$, it follows that

$$\lim_{n \to \infty} u_n = q.$$

But $Fix\,(T_\mu) = Fix\,(T)$ and

$$(1-\lambda)x + \lambda T_\mu x = (1-\lambda\mu)x + \lambda\mu T x.$$

Since $0 < \lambda < 1$ and $\mu = \dfrac{1}{b+1}$, it follows that $\lambda\mu \in \left(0, \dfrac{1}{b+1}\right)$. This proves that, for any $\lambda \in \left(0, \dfrac{1}{b+1}\right)$, the sequence $\{x_n\}$ given by (3.11) converges strongly to $q \in Fix\,(T)$. □

Example 3.1. (Example 2.1, [3])

Let $X = \left[\dfrac{1}{2}, 2\right]$ be endowed with the usual norm and let $T : X \to X$ be defined by $Tx = \dfrac{1}{x}$, for all $x \in \left[\dfrac{1}{2}, 2\right]$. Then

(i) T is Lipschitzian continuous with Lipschitz constant $L = 4$ and T is not nonexpansive;

(ii) T is a $3/2$-enriched nonexpansive mapping

(iii) $Fix\,(T) = \{1\}$;

(iv) T is not quasi-nonexpansive.

Remark 3.2. As shown by the previous example, Theorem 3.2 is an extension of Theorem 1 of Senter and Dotson [36], by considering the larger class of enriched nonexpansive mappings instead of the class of nonexpansive mappings used in [36].

We note also that the proof of Theorem 3.2 is patterned after the proofs of Theorems 1 and 2 in [36].

As a consequence of Theorem 3.2 and Lemma 2.2 we obtain the following result.

Corollary 3.1. *Let C be a closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be a b-enriched nonexpansive mapping. If $Fix\,(T) \ne \emptyset$ and if $I - T$ maps closed bounded subsets of C onto closed subsets of X, then for any $\lambda \in \left(0, \dfrac{1}{b+1}\right)$ and any given $x_0 \in C$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \ge 0,$$

converges strongly to a fixed point of T.

Remark 3.3. Note that in Corollary 3.1, the set C is not supposed to be bounded. Therefore, it was necessary to compensate this lack by assuming $Fix\,(T) \neq \emptyset$. If C is also bounded, then we can remove this assumption to obtain the following result.

Corollary 3.2. *Let C be a bounded closed convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a b-enriched nonexpansive mapping. If $I - T$ maps closed bounded subsets of C onto closed subsets of X, then for any $\lambda \in \left(0, \dfrac{1}{b+1}\right)$ and any given $x_0 \in C$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \geq 0,$$

converges strongly to a fixed point of T.

Since any nonexpansive mappings is a 0-enriched nonexpansive mapping, see [3], by Corollary 3.1 we obtain in particular the following result of Browder and Petryshyn [17].

Corollary 3.3. (Browder and Petryshyn, [17]) *Let C be a closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be a nonexpansive mapping. If $Fix\,(T) \neq \emptyset$ and if $I - T$ maps closed bounded subsets of C onto closed subsets of X, then $\lambda \in (0, 1)$ and any given $x_0 \in C$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$,*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \geq 0,$$

converges strongly to a fixed point of T.

Remind that a mapping $T : C \to X$ of a subset C of a Banach space X is said to be *demicompact* [32], provided whenever $\{u_n\}$ is a bounded sequence in X and $\{Tu_n - u_n\}$ is strongly convergent, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which is strongly convergent.

If a mapping T is continuous as well as demicompact then, according to Opial ([30], p. 41), the mapping $I - T$ maps closed bounded subsets of C onto closed subsets of X.

In particular, if $T : C \to C$ is nonexpansive and demicompact and has a fixed point in C, it follows from Opial's result and Lemma 2.2 that T also satisfies Condition I.

By using the previous results and remarks, we can extend directly Theorem 1.1 (i.e, Theorem 2.2 in [3]) from Hilbert spaces to the setting of uniformly convex Banach spaces.

Theorem 3.3. *Let C be a bounded closed convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a b-enriched nonexpansive and demicompact mapping. Then the set $Fix\,(T)$ of fixed points of T is a nonempty and there exists $\lambda \in (0, 1)$ such that, for any given $x_0 \in C$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{3.11}$$

converges strongly to a fixed point of T.

Remark 3.4. The class of demicompact operators contains, among many classes of operators (see [32]), the compact operators and, in particular, the completely continuous and strongly continuous operators.

Hence, from Theorem 3.3 one obtains the pioneering result of Krasnoselskij from 1955 ([27], [28])

$$\frac{1}{2}(x_n + Tx_n) \to q \in Fix\,(T)\,(\text{ as } n \to \infty)$$

and also the result of Schaefer [35] for the general Krasnoselskij scheme, i.e.,

$$(1 - \lambda)x_n + \lambda T x_n \to q \in Fix\,(T),\,(\text{ as } n \to \infty,\ 0 < \lambda < 1).$$

The above results for compact operators have been extended to strictly convex Banach spaces by Edelstein in [22].

The enriched nonexpansive mapping T in Example 3.1 is neither nonexpansive nor quasi-nonexpansive. However, there exist quasi-nonexpansive mappings which satisfy Condition II (and hence Condition I), see [26], [33] and [36]. Indeed, If T is a weak Kannan mapping, that is, a mapping satisfying

$$\|Tx - Ty\| \leq \frac{1}{2}\left[\|x - Tx\| + \|y - Ty\|\right], x, y \in X,$$

and T has a (unique) fixed point, then Condition II is satisfied with $\alpha = 2$. On the other hand, in [6] we have studied the class of (strict) enriched Kannan contractions, that corresponds to the strict Kannan contraction condition [26]

$$\|Tx - Ty\| \leq k[\|x - Tx\| + \|y - Ty\|], x, y \in X,$$

where $0 \leq k < \frac{1}{2}$. Therefore, it will be an interesting problem to consider the class of enriched weakly Kannan mappings and study its properties.

4. Conclusions

In this paper we studied in the setting of a uniformly Banach space the class of *enriched nonexpansive mappings*, introduced and studied in [3] for the case of Hilbert spaces. We have shown that any enriched nonexpansive mapping defined on a bounded, closed and convex subset C of a Banach space X has a fixed point in C. In order to approximate a fixed point of an enriched nonexpansive mapping, we used the Krasnoselskij iteration for which we have proven a strong convergence result (Theorem 1.1) as well as some other important results (Corrolaries 3.1, 3.2 and 3.3, Theorem 3.3).

We illustrated the richness of the new class of mappings by means of an example which shows that, alongside all nonexpansive mappings, some Lipschitzian but not all quasi-nonexpansive mappings, are also included in the class of *enriched nonexpansive mappings*.

Our results extend some classical strong convergence theorems in [17] and [36] from nonexpansive mappings to enriched nonexpansive mappings and thus include many other important related results from literature as particular cases, see [16], [19], [27], [32], [35] etc.

For other very recent related developments, see also [2], [6], [7], [8] and [9].

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