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Dedicated to Prof. Hong-Kun Xu on the occasion of his 60^{th} anniversary

A class of parabolic evolutionary quasivariational inequalities in contact mechanics

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ABSTRACT. In this paper, we obtain an existence and uniqueness of the solution for a class of parabolic evolutionary quasivariational inequalities in contact mechanics under some mild conditions. We also study an error estimate for the parabolic evolutionary quasivariational inequality by employing the forward Euler difference scheme and the element-free Galerkin spatial approximation.

1. INTRODUCTION

Let V be a Hilbert space endowed with the inner product $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$. For any given $T > 0$, we use $C([0,T];V)$ (resp., $C^1([0,T]);V)$) to denote the space of all V-valued continuous functions (resp., V-valued continuous differentiable functions) on $[0,T]$ with norm $\|u\|_{C([0,T];V)} = \max_{t\in[0,T]} \|u(t)\|_V$ (resp., $\|u\|_{C^1([0,T];V)} = \max_{t\in[0,T]} \|u(t)\|_V + \max_{t\in[0,T]} \|u_t(t)\|_V$), where u_t denotes the derivative of $u(t)$ with respect to the time variable. In the sequel, let $H^1([0,T];V) := W^{1,2}([0,T];V)$.

In 2001, Han and Sofonea [11] showed that a number of quasistatic frictional contact problems for viscoelastic materials can be formulated as the following parabolic evolutionary quasivariational inequality: find a displacement field $u \in C^1([0,T];V)$ such that, for any $t \in [0,T]$,

$$\text{(1.1)}\quad \begin{cases} (Au_t, v-u_t)_V + (Bu, v-u_t)_V + j(u_t, v) - j(u_t,u_t) \geq (f, v-u_t)_V, \quad \forall v \in V, \\ u(x,0) = u_0(x), \end{cases}$$

where $A,\ B : V \to V$ are two operators related to the viscoelastic constitutive law, the functional $j : V \times V \to \mathbb{R}$ is determined by contact conditions and $f : [0,T] \to V$ is a mapping. Various examples have been given in the literature [10, 15, 17, 18, 19] to motivate the study of parabolic evolutionary (quasi)variational inequalities. Some related results concerned with evolutionary (quasi)variational inequalities can be found in [6, 8, 9, 21, 22, 23, 24, 25, 26] and the references therein.

We notice that Han and Sofonea [11] proved the existence and uniqueness for (1.1) under the strong monotonicity and Lipschitz continuity. They also proposed a semi-discrete and fully-discrete scheme and derived error estimates which indicates that the convergence order is 1 with respect to the time. However, the strong monotonicity and Lipschitz continuity are quite strong and may not be satisfied in some practical situations. Thus, it would be important and necessary to relax these conditions. The first purpose of this

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paper is to show the existence and uniqueness of the solution for (1.1) under some mild conditions.

On the other hand, Jonhson [14] proposed a fully discrete scheme for solving (1.1) with $j(u,v) \equiv j(v)$ and used the finite element method to obtain the numerical solutions. Chen et al. [6] proposed a fully discrete for solving (1.1) and showed that the convergence order is 1. Achdou et al. [1] adopt Euler implicit time schemes combined with finite element spatial discretization for the special case of (1.1) and obtained the error estimates. We would like to mention that the finite element method usually needs the mesh in the domain, while the element-free Galerkin method [3] requires only nodal data in the domain. Thus, the element-free Galerkin method provides an efficient numerical tool for solving evolutionary variational inequalities. Recently, Ding et al. [9] provided the error estimate of the element-free Galerkin method for a class of parabolic evolutionary variational inequalities arising from the heat-servo control problem. Very recently, Chen and Xiao [7] extended the main result of Ding et al. [9] and obtained a more efficiency error estimate for the parabolic evolutionary variational inequality. However, we have never seen the study of error estimates of the element-free Galerkin method for solving (1.1). Therefore, it would be important and interesting to employ the element-free Galerkin method for solving (1.1). The second purpose of this paper is to make an attempt to propose an error estimate of the element-free Galerkin method for solving (1.1). Compared with the work due to Ding et al. [9], in this paper, we consider a more general problem called parabolic evolu-tionary quasivariational inequality. Especially, we get rid of the symmetry properties for operators A and B.

The rest of this is organized as follows. The next section presents some necessary preliminaries. A new existence and uniqueness of the solution is obtained in section 3 for (1.1) under some mild conditions. The fully discrete scheme is proposed in section 4 by using both the forward Euler finite difference scheme to approximate the time derivative and the element-free Galerkin method to discretize spatial variable. Finally, the error estimates for the fully discrete scheme is given in section 5.

2. Preliminaries

In order to show a new existence and uniqueness result for the parabolic evolutionary quasivariational inequality (1.1), we employ the following assumptions.

(i) There exist two functions $\varphi_A, \psi_A : [0,+\infty) \to [0,+\infty)$ such that

$$\begin{cases} (a)\ (Au - Av, u - v)_V \geq \varphi_A(\|u - v\|_V)\|u - v\|_V^2, \quad \forall u, v \in V, \\ (b)\ \|Au - Av\|_V \leq \psi_A(\|u - v\|_V)\|u - v\|_V, \quad \forall u, v \in V. \end{cases} \tag{2.2}$$

When $\varphi_A(t) = M$ and $\psi_A(t) = L_A$ for all $t \geq 0$, where $M, L_A > 0$ are two constants, then (2.2) reduces to the following conditions

$$\begin{cases} (a)\ (Au - Av, u - v)_V \geq M\|u - v\|_V^2, \quad \forall u, v \in V, \\ (b)\ \|Au - Av\|_V \leq L_A\|u - v\|_V, \quad \forall u, v \in V. \end{cases} \tag{2.3}$$

which was adopted in Han and Sofonea [11].

(ii) There exists a function $\psi_B : [0,+\infty) \to [0,+\infty)$ such that

$$\|B(u_1) - B(u_2)\|_V \leq \psi_B(\|u_1 - u_2\|_V)\|u_1 - u_2\|_V, \quad \forall u_1, u_2 \in V. \tag{2.4}$$

When $\psi_B(t) = L_B$ for all $t \geq 0$, where $L_B > 0$ is a constant, then (2.4) reduces to the following condition There exists a constant $L_B > 0$ such that

$$\|B(u_1) - B(u_2)\|_V \leq L_B\|u_1 - u_2\|_V, \quad \forall u_1, u_2 \in V, \tag{2.5}$$

which was adopted in Han and Sofonea [11].

(iii) $j : V \times V \to \mathbb{R}$ is a functional such that $j(u, \cdot)$ is convex and lower semicontinous on V for all $u \in V$ and there is a function $\psi_j : [0, +\infty) \to [0, +\infty)$ such that

(2.6)
$$j(g_1, v_2)+j(g_2, v_1)-j(g_1, v_1)-j(g_2, v_2) \le \psi_j(\|g_1-g_2\|_V)\|g_1-g_2\|_V\|v_1-v_2\|_V, \forall g_1, g_2, v_1, v_2 \in V.$$

When $\psi_j(t) = L_j$ for all $t \ge 0$, where $L_j > 0$ is a constant, then (2.6) reduces to the following condition

$$j(g_1, v_2)+j(g_2, v_1)-j(g_1, v_1)-j(g_2, v_2) \le L_j\|g_1-g_2\|_V\|v_1-v_2\|_V, \forall g_1, g_2, v_1, v_2 \in V, \tag{2.7}$$

which was adopted in Han and Sofonea [11].

(iv) $f : [0, T] \to V$ and $u_0(x)$ satisfy

$$f \in C((0, T); V), \quad u_0(x) \in V. \tag{2.8}$$

We also need the following fixed point theorem for nonlinear mappings.

Lemma 2.1. ([4]) *Suppose that $(X, \|\cdot\|)$ is a Banach space and that $T : X \to X$ is a mapping such that $\|Tx - Ty\| \le \psi(\|x - y\|)$ for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ satisfies $\psi(t) < t$ for all $t > 0$. Then T has a unique fixed point.*

3. Existence and uniqueness

In this section, we investigate the existence and uniqueness of the solution for (1.1) under the conditions (2.2), (2.4), (2.6) and (2.8).

For any given $u \in V$, it is well-known that the subdifferential $\partial j(u, \cdot)$ of a convex, proper and lower-semicontinous function $j(u, \cdot) : V \to \mathbb{R} \bigcup \{+\infty\}$ is a maximal monotone operator ([5]). Therefore, we can define the following operator

$$J_{j(u)} = (I + \rho\partial j(u, \cdot))^{-1} := (I + \partial j(u))^{-1}, \quad \rho > 0.$$

Clearly, the parabolic evolutionary quasivariational inequality (1.1) is equivalent to the following problem: find a displacement field $u \in C^1([0, T]; V)$ such that, for any $t \in [0, T]$,

$$u_t = J_{j(u_t)}(u_t + \rho f - \rho A u_t - \rho B u), \quad \rho > 0, \quad u(x, 0) = u_0(x). \tag{3.9}$$

Using the definition of $J_{j(u)}$, we have the following lemma.

Lemma 3.2. *If the condition* (2.6) *holds, then*

$$\|J_{j(v)}(u) - J_{j(v)}(w)\|_V \le \|u - w\|_V, \quad \forall u, v, w \in V, \tag{3.10}$$

$$\|J_{j(u)}(v) - J_{j(w)}(v)\|_V \le \rho\psi_j(\|u - w\|_V)\|u - w\|_V, \quad \forall u, v, w \in V. \tag{3.11}$$

Proof. For any given $u, v, w \in V$, let $u_v := J_{j(v)}(u)$ and $w_v := J_{j(v)}(w)$. Then the definition of operator $J_{j(v)}$ yields

$$(u - u_v, l - u_v) \le \rho j(v, l) - \rho j(v, u_v), \quad \forall l \in V \tag{3.12}$$

and

$$(w - w_v, l - w_v) \le \rho j(v, l) - \rho j(v, w_v), \quad \forall l \in V. \tag{3.13}$$

Taking $l = w_v$ and $l = u_v$ in (3.12) and (3.13) respectively, one has

$$(u - u_v, w_v - u_v) \le \rho j(v, w_v) - \rho j(v, u_v) \quad \forall l \in V \tag{3.14}$$

and

$$(w - w_v, u_v - w_v) \le \rho j(v, u_v) - \rho j(v, w_v), \quad \forall l \in V. \tag{3.15}$$

Summing up (3.14) and (3.15), we have $\|u_v - w_v\|_V^2 \le (u - w, u_v - w_v)$ and so $\|J_{j(v)}(u) - J_{j(v)}(w)\|_V \le \|u - w\|_V$. This ends the proof of inequality (3.10).

Similarly, for any given $u, v, w \in V$, we have

$$(v - v_u, l - v_u) \leq \rho j(u, l) - \rho j(u, v_u), \quad \forall l \in V, \tag{3.16}$$

$$(v - v_w, l - v_w) \leq \rho j(w, l) - \rho j(w, v_w), \quad \forall l \in V, \tag{3.17}$$

where $v_u := J_{j(u)}(v)$ and $v_w := J_{j(w)}(v)$. Taking $l = v_w$ and $l = v_u$ in (3.16) and (3.17) respectively and summing them up, it follows from (2.6) that

$$\|v_u - v_w\|_V^2 \leq \rho \|v_u - v_w\|_V \|u - w\|_V \psi_j(\|u - w\|_V)$$

and so

$$\|J_{j(u)}(v) - J_{j(w)}(v)\|_V \leq \rho \psi_j(\|u - w\|_V)\|u - w\|_V.$$

This ends the proof of inequality (3.11). □

Now, we turn to show the existence and uniqueness of the solution for the problem (3.9). The proof can be divided into three steps. In the first step, we consider an auxiliary problem: for any $\eta,\ g \in C([0,T]; V)$, find a unique $u_{\eta g} \in C([0,T]; V)$ such that

$$u_{\eta g} = J_{j(\eta)}(u_{\eta g} + \rho f - \rho A u_{\eta g} - \rho g). \tag{3.18}$$

Lemma 3.3. *Assume that the conditions* (2.2), (2.6) *and* (2.8) *hold. For any $\eta, g \in C([0,T]; V)$, if ψ_A is non-decreasing and φ_A is non-increasing and there exists a real number $\rho > 0$ such that*

$$\rho \psi_A^2(s) < 2\varphi_A(s), \quad \forall s \in [0, \infty), \tag{3.19}$$

then we can find a unique $u_{\eta g} \in C([0,T]; V)$ such that (3.18) *holds.*

Proof. We only need to prove that the mapping $S(u) := J_{j(\eta)}(u + \rho f - \rho A u - g)$ has a unique fixed point in $C([0,T], V)$. It follows from Lemma 3.2 that

$$\begin{aligned}
& \|S(u_1(t)) - S(u_2(t))\|_V^2 \\
\leq\ & \|u_1(t) - u_2(t) + \rho A u_1(t) - \rho A u_2(t)\|_V^2 \\
=\ & \|u_1(t) - u_2(t)\|_V^2 - 2(u_1(t) - u_2(t), \rho A u_1(t) - \rho A u_2(t)) + \rho^2 \|A u_1(t) - A u_2(t)\|_V^2 \\
\leq\ & \|u_1(t) - u_2(t)\|_V^2 - 2\rho \varphi_A(\|u_1(t) - u_2(t)\|_V)\|u_1(t) - u_2(t)\|_V^2 \\
& + \rho^2 \psi_A^2(\|u_1(t) - u_2(t)\|_V)\|u_1(t) - u_2(t)\|_V^2 \\
\leq\ & \left[1 - 2\rho\varphi_A(\|u_1(t) - u_2(t)\|_V) + \rho^2 \psi_A^2(\|u_1(t) - u_2(t)\|_V)\right] \|u_1(t) - u_2(t)\|_V^2
\end{aligned}$$

for all $t \in [0,T]$. Since ψ_A is non-decreasing and φ_A is non-increasing, one has

$$\begin{aligned}
& \|S(u_1) - S(u_2)\|_{C([0,T];V)} \\
\leq\ & \sqrt{\left[1 - 2\rho\varphi_A(\|u_1 - u_2\|_{C([0,T];V)}) + \rho^2 \psi_A^2(\|u_1 - u_2\|_{C([0,T];V)})\right]}\, \|u_1 - u_2\|_{C([0,T];V)}.
\end{aligned} \tag{3.20}$$

Thus, by (3.19), (3.20) and Lemma 2.1, we know that S has a unique fixed point $u_{\eta g} \in C([0,T]; V)$ and so $u_{\eta g}$ is a unique solution of (3.18). □

Example 3.1. Assume that

$$\varphi_A(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}, \quad \psi_A(t) = -\frac{1}{2}e^{-t} + \frac{3}{2}, \quad \forall t \in [0, \infty).$$

Clearly, φ_A is non-increasing and ψ_A is non-decreasing. Moreover, for any $0 < \rho < \frac{2}{9}$, it is easy to check that (3.19) holds.

Lemma 3.4. *Let the conditions* (2.2), (2.6) *and* (2.8) *be satisfied. Assume that* ψ_A *and* ψ_j *are non-decreasing and* φ_A *is non-increasing. For any* $g \in C([0,T];V)$, *if there exists a real number* $\rho > 0$ *such that*

$$\rho\psi_A^2(s) < 2\varphi_A(s), \quad 2\psi_j(s) < 2\varphi_A(s) - \rho\psi_A^2(s) \tag{3.21}$$

for all $s \in [0,\infty)$, *then we can find a unique* $u_g \in C([0,T];V)$ *such that*

$$u_g = J_{j(u_g)}(u_g + \rho f - \rho A u_g - \rho g). \tag{3.22}$$

Proof. We only need to prove that the mapping $\eta \mapsto u_{\eta g}$ has a unique fixed point in $C([0,T],V)$, where $u_{\eta g}$ is defined by (3.18). It follows form Lemmas 3.2 and 3.3 that

$$\begin{aligned}
&\|u_{\eta_1 g}(t) - u_{\eta_2 g}(t)\|_V \\
\le\ & \|J_{j(\eta_1(t))}(u_{\eta_1 g}(t) + \rho f(t) - \rho A u_{\eta_1 g}(t) - \rho g(t)) \\
& - J_{j(\eta_2(t))}(u_{\eta_1 g}(t) + \rho f(t) - \rho A u_{\eta_1 g}(t) - \rho g(t))\| \\
& + \big\|J_{j(\eta_2(t))}(u_{\eta_1 g}(t) + \rho f(t) - \rho A u_{\eta_1 g}(t) - \rho g(t)) \\
& - J_{j(\eta_2)(t)}(u_{\eta_2 g}(t) + \rho f(t) - \rho A u_{\eta_2 g}(t) - \rho g(t))\big\| \\
\le\ & \sqrt{1 - 2\rho\varphi_A(\|u_{\eta_1 g}(t) - u_{\eta_2 g}(t)\|_V) + \rho^2\psi_A^2(\|u_{\eta_1 g}(t) - u_{\eta_2 g}(t)\|_V)}\,\|u_{\eta_1 g}(t) - u_{\eta_2 g}(t)\|_V \\
& + \rho\psi_j(\|\eta_1(t) - \eta_2(t)\|_V)\|\eta_1(t) - \eta_2(t)\|_V.
\end{aligned}$$

Since ψ_A and ψ_j are non-decreasing and φ_A is non-increasing, one has

$$\begin{aligned}
&\|u_{\eta_1 g} - u_{\eta_2 g}\|_{C([0,T];V)} \\
\le\ & \sqrt{1 - 2\rho\varphi_A(\|u_{\eta_1 g} - u_{\eta_2 g}\|_{C([0,T];V)}) + \rho^2\psi_A^2(\|u_{\eta_1 g} - u_{\eta_2 g}\|_{C([0,T];V)})}\,\|u_{\eta_1 g} - u_{\eta_2 g}\|_{C([0,T];V)} \\
& + \rho\psi_j(\|\eta_1 - \eta_2\|_{C([0,T];V)})\|\eta_1 - \eta_2\|_{C([0,T];V)}.
\end{aligned}$$

Because of the basic inequality $\sqrt{1+x} \le 1 + \frac{x}{2}$ for all $x \in (-1,1)$, we have

$$\begin{aligned}
& \frac{2\varphi_A(\|u_{\eta_1 g} - u_{\eta_2 g}\|_{C([0,T];V)}) - \rho\psi_A^2(\|u_{\eta_1 g} - u_{\eta_2 g}\|_{C([0,T];V)})}{2}\|u_{\eta_1 g} - u_{\eta_2 g}\|_{C([0,T];V)} \\
\le\ & \psi_j(\|\eta_1 - \eta_2\|_{C([0,T];V)})\|\eta_1 - \eta_2\|_{C([0,T];V)}
\end{aligned}$$

and so

$$\begin{aligned}
&\|u_{\eta_1 g} - u_{\eta_2 g}\|_{C([0,T];V)} \\
\le\ & \frac{2\psi_j(\|\eta_1 - \eta_2\|_{C([0,T];V)})}{2\varphi_A(\|\eta_1 - \eta_2\|_{C([0,T];V)}) - \rho\psi_A^2(\|\eta_1 - \eta_2\|_{C([0,T];V)})}\|\eta_1 - \eta_2\|_{C([0,T];V)}.
\end{aligned} \tag{3.23}$$

Thus, it follows from (3.24), (3.23) and Lemma 2.1 that there is a unique $u_g \in C([0,T];V)$ such that (3.22) holds. □

Theorem 3.1. *Let the conditions* (2.2), (2.4), (2.6) *and* (2.8) *be satisfied. Assume that* ψ_A *and* ψ_j *are non-decreasing and* φ_A *is non-increasing. If there exist two constants* $\rho > 0$ *and* $\lambda > 0$ *such that*

$$\rho\psi_A^2(s) < 2\varphi_A(s), \quad 2\psi_j(s) < 2\varphi_A(s) - \rho\psi_A^2(s), \quad \min_{l\in[0,\infty)}(\varphi_A(l) - \psi_j(l)) \ge \lambda\psi_B(s) \tag{3.24}$$

for all $s \in [0,\infty)$, *then we can find a unique function* $u \in C^1([0,T];V)$ *such that*

$$u_t = J_{j(u_t)}(u_t + \rho f - \rho A u_t - Bu), \quad \forall t \in [0,T]. \tag{3.25}$$

Proof. Define a mapping $Q : C([0,T];V) \to C([0,T];V)$ by setting

$$Q(g) = u_0 + \int_0^t u_g(s)ds, \quad \forall g \in C([0,T];V).$$

We only need to prove that the mapping BQ has a unique fixed point in $C([0,T];V)$, where $u_g(t)$ is defined in (3.22). By the proof of Lemma 3.4, we know that

$$\|u_{g_1}(t) - u_{g2}(t)\|_V \le \frac{\|g_1(t) - g_2(t)\|_V}{\varphi_A(\|u_{g_1}(t) - u_{g_2}(t)\|_V) - \psi_j(\|u_{g_1}(t) - u_{g_2}(t)\|_V)}, \quad \forall t \in [0,T].$$

Now we define an equivalent norm $C([0,T];V)$ as $\|v\|^*_{C([0,T];V)} = \max_{t\in[0,T]} e^{-\beta t}\|v(t)\|_V$ with $\beta > \lambda^{-1}$. According to the definition of Q and the conditions (2.4) and (3.24), one has

$$\begin{aligned}
& e^{-\beta t}\|BQ(g_1)(t) - BQ(g_2)(t)\|_V \\
\le\ & e^{-\beta t}\psi_B\left(\left\|\int_0^t (u_{g_1}(s) - u_{g_2}(s))ds\right\|_V\right)\int_0^t \|(u_{g_1}(s) - u_{g_2}(s))\|_V ds \\
\le\ & \psi_B\left(\left\|\int_0^t (u_{g_1}(s) - u_{g_2}(s))ds\right\|_V\right)e^{-\beta t} \\
& \times \int_0^t \frac{\|g_1(s) - g_2(s)\|_V}{\varphi_A(\|u_{g_1}(s) - u_{g_2}(s)\|_V) - \psi_j(\|u_{g_1}(s) - u_{g_2}(s)\|)}ds \\
\le\ & \frac{1}{\lambda\beta}\|g_1 - g_2\|^*_{C([0,T];V)}.
\end{aligned}$$

This shows that

$$\|BQ(g_1) - BQ(g_2)\|^*_{C([0,T];V)} \le \frac{1}{\lambda\beta}\|g_1 - g_2\|^*_{C([0,T];V)}.$$

By the Banach fixed point theorem, we can see that BQ has a unique fixed point in $C([0,T];V)$. □

The following example shows that the condition (3.24) can be satisfied.

Example 3.2. Let

$$\varphi_A(t) = e^{-t} + 1, \quad \psi_A(t) = 3 + \frac{2\arctan(t)}{\pi}, \quad \psi_B(t) = \psi_j(t) = e^{-t}, \quad \forall t \in [0,\infty).$$

Then φ_A is non-increasing, ψ_A, $\psi_B(t)$ and $\psi_j(t)$ are non-decreasing. Moreover, for $0 < \rho < \frac{1}{16}$ and $\lambda = 1$, it is easy to see that

$$\psi_A(t) > \varphi_A(t), \quad \min_{l\in[0,\infty)}(\varphi_A(l) - \phi_j(l)) > \psi_B(t), \quad \forall t \in [0,\infty)$$

and so the condition (3.24) holds.

Remark 3.1. We would like to mention that Theorem 3.1 is a generalization of Theorem 2.1 of [11]. In fact, under the assumptions (2.3), (2.5), (2.7) and (2.8), it is easy to check that all the conditions of Theorem 3.1 are satisfied.

4. Fully discrete scheme

In this section, we introduce the fully discrete scheme to discretize the parabolic evolutionary quasivariational inequality (1.1). To this end, we utilize the finite dimensional moving least-squares approximation method [16] to approximate the function u of V as follows.

For convenience, we omit the subscript V in the symbol of inner product. Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a convex subset. Given $h > 0$ and $X_h = \{\xi_1, \xi_2, \cdots, \xi_n\}$, which is a set of points in Ω, let $u_j = u(\xi_j)$ with $1 \leq j \leq n$ and $0 \leq w_j(x) \leq 1$ be a weighted function such that

$$\text{supp}(w_j) \subset \overline{B_h(\xi_j)} = \{z \in \mathbb{R}^{\mathbb{N}} : \|z - \xi_j\| \leq h\},$$

where h is the support radius. Let $\{p_0, \cdots, p_m\}$ be a basis of the polynomial space P_m (for example, $p_0 = 1, p_1 = x, \cdots, p_m = x^m$ in $\mathbb{R}^1$) with $m \ll n$. For each $x \in \Omega$, the approximation $u^h(x)$ of a function $u(x)$ has the following form: $u^h(x) = P^*(x,x) = \sum_{k=0}^{m} p_k(x)\alpha_k(x)$, where $\alpha_k(x)$ is chosen such that

$$J_x(\alpha) = \sum_{j=1}^{n} w_j(x) \left(u_j - \sum_{k=0}^{m} p_k(\xi_j)\alpha_k(x) \right)^2$$

is minimized. The minimization leads $u^h(x) \in V^h$ with $V^h = span\{\Phi_k : 1 \leq k \leq M\}$, where $\Phi_k(x)$ is the moving least-squares shapes function (see also in [9]), which can be written as

$$\Phi_k(x) = [P(x)^T A(x)^{-1} B(x)]_k, \quad A(x) = W(x)P(x)(P(x))^T, \quad B(x) = W(x)P(x) \tag{4.26}$$

with

$$W(x) = diag(w_1(x), ..., w_n(x)), \quad P(x) = [p(x_1), ..., p(x_n)]^T. \tag{4.27}$$

For more details on the moving least-squares approximation, we refer the reader to [2, 27, 28].

Next we turn to discretize the time. The time interval $[0, T]$ is divided into N equal parts $\{t_n\}_{n=1}^{N}$ and let $t_n - t_{n-1} = k = T/N$ for $n = 1, 2, \cdots, N$. Similar to [11], for a continuous function $w(t)$, we use the following notations

$$w_n = w(t_n), \quad \Delta w_n = w_n - w_{n-1}, \quad \delta w_n = \Delta w_n / k.$$

Thus, the fully discrete scheme of the parabolic evolutionary quasivariational inequality (1.1) can be described as follows:

Find $\{\delta u_n^{hk}\}_{n=1}^{N} \subset V^h$ such that, for $n = 1, \cdots, N$ and any $v^h \in V^h$,

$$\begin{cases} (A\delta u_n^{hk}, v^h - \delta u_n^{hk}) + (Bu_n^{hk}, v^h - \delta u_n^{hk}) + j(\delta u_n^{hk}, v^h) - j(\delta u_n^{hk}, \delta u_n^{hk}) \geq (f_n, v^h - \delta u_n^{hk}), \\ u_0^{hk} = u_0^h, \end{cases} \tag{4.28}$$

where u_0^h is the moving least-squares approximation of u_0. To simplify the notation, we use w_n^{hk} as the substitute of δu_n^{hk} for $n = 1, \cdots, N$. It follows that

$$u_n^{hk} = \sum_{j=1}^{n} w_j^{hk} + u_0^h \tag{4.29}$$

and so we can rewrite (4.28) as follows: find $w_n^{hk} \in V^h$ such that, for all $v^h \in V^h$,

$$\begin{aligned} &\left(Aw_n^{hk} + B\left(\sum\nolimits_{1 \leq j \leq n} w_j^{hk} + u_0^h\right), v^h - w_n^{hk}\right) + j(w_n^{hk}, v^h) - j(w_n^{hk}, w_n^{hk}) \\ \geq\ & (f_n, v^h - w_n^{hk}). \end{aligned} \tag{4.30}$$

Let $F_n = f_n - Bu_{n-1}^{hk}$. If B is linear, then it is easy to see that (4.30) is equivalent to the following problem:

Problem P^{hk}: find $w_n^{hk} \in V^h$ such that, for any $v^h \in V^h$,

$$((A + B)w_n^{hk}, v^h - w_n^{hk}) + j(w_n^{hk}, v^h) - j(w_n^{hk}, w_n^{hk}) \geq (F_n, v^h - w_n^{hk}). \tag{4.31}$$

Employing the auxiliary principle ([11, 13]), we prove the existence and uniqueness of solutions for Problem P^{hk}.

Theorem 4.2. *Let* (2.3), (2.5) *and* (2.7) *hold. If B is linear, $M > L_j$ and $L_B < M - L_j$, then there exists a unique solution $u^h \in V^h$ to Problem P^{hk}.*

Proof. Let $\eta \in V^h$ and $g \in V^h$ be given. Firstly, we consider the following variational inequality of finding $u_{\eta g} \in V^h$ such that

$$(Au_{\eta g} + \eta, v^h - u_{\eta g}) + j(g, v^h) - j(g, u_{\eta g}) \geq (F_n, v^h - u_{\eta g}), \quad \forall v^h \in V^h. \tag{4.32}$$

For any given $\eta,\ g \in V$, we note that (4.32) is the well-known Hartman-Stampacchia variational inequality and so (4.32) has a unique solution $u_{\eta g}$ by the classical result of [12]. Consider the operator $\Gamma_\eta : V^h \to V^h$ defined by $\Gamma_\eta g = u_{\eta g}$. Let $g_1, g_2 \in V$ and $\eta \in V$. One has

$$\begin{cases} (A\Gamma_\eta g_2 + \eta, v^h - \Gamma_\eta g_1) + j(g_1, v^h) - j(g_1, \Gamma_\eta g_1) \geq (F_n, v^h - \Gamma_\eta g_1), \\ (A\Gamma_\eta g_2 + \eta, v^h - \Gamma_\eta g_2) + j(g_2, v^h) - j(g_2, \Gamma_\eta g_2) \geq (F_n, v^h - \Gamma_\eta g_2). \end{cases} \tag{4.33}$$

Taking $v^h = g_2$ and $v^h = g_1$ in the first and second inequality of (4.33), respectively. We can derive the relation from (4.33) by adding them together that

$$j(g_1, \Gamma_\eta g_2) + j(g_2, \Gamma_\eta g_1) - j(g_1, \Gamma_\eta g_1) - j(g_2, \Gamma_\eta g_2) \geq (A(\Gamma_\eta g_1 - \Gamma_\eta g_2), \Gamma_\eta g_1 - \Gamma_\eta g_2).$$

Form the conditions (2.3)(a) and (2.7)(b), we have

$$L_j \|g_1 - g_2\| \|\Gamma_\eta g_1 - \Gamma_\eta g_2\| \geq M \|\Gamma_\eta g_1 - \Gamma_\eta g_2)\|^2.$$

If $M > L_j$, then the operator Γ_η has a unique fixed point $g_\eta \in V^h$ from the Banach fixed point theorem. Let $u_\eta \in V^h$ be defined by $u_\eta = u_{\eta g_\eta}$. Then $u_\eta = g_\eta$. Taking $g = g_\eta$ in (4.32), we have

$$(Au_\eta + \eta, v^h - u_\eta) + j(u_\eta, v^h) - j(u_\eta, u_\eta) \geq (F_n, v^h - u_\eta), \quad \forall v^h \in V^h. \tag{4.34}$$

Similarly, we have

$$\begin{aligned} & j(u_{\eta_1}, u_{\eta_2}) + j(u_{\eta_2}, u_{\eta_1}) - j(u_{\eta_1}, u_{\eta_1}) - j(u_{\eta_1}, u_{\eta_1}) + (\eta_1 - \eta_2, u_{\eta_2} - u_{\eta_1}) \\ \geq \; & (A(u_{\eta_1} - u_{\eta_2}), u_{\eta_1} - u_{\eta_2}). \end{aligned}$$

Thus, form the conditions (2.3)(a) and (2.7)(b), we have

$$L_j \|u_{\eta_1} - u_{\eta_1}\|^2 + \|\eta_1 - \eta_2\| \|u_{\eta_1} - u_{\eta_2}\| \geq M \|u_{\eta_1} - u_{\eta_1}\|^2$$

and so $\frac{1}{M - L_j} \|\eta_1 - \eta_2\| \geq \|u_{\eta_1} - u_{\eta_1}\|$. Taking $\Lambda\eta = Bu_\eta$, we know that

$$\|\Lambda\eta_1 - \Lambda\eta_1\| \leq L_B \|u_{\eta_1} - u_{\eta_1}\| \leq \frac{L_B}{M - L_j} \|\eta_1 - \eta_2\|$$

Therefore, if $L_B < M - L_j$, the operator Λ has a unique fixed point η^*. Apparently, u_{η^*} is the unique solution of (4.35), which means that

$$(Au_{\eta^*} + Bu_{\eta^*}, v^h - u_{\eta^*}) + j(u_{\eta^*}, v^h) - j(u_{\eta^*}, u_{\eta^*}) \geq (F_n, v^h - u_{\eta^*}), \quad \forall v^h \in V^h. \tag{4.35}$$

That ends our proof. □

Remark 4.2. If we replace Bu_n^{hk} by Bu_{n-1}^{hk} in (4.28), then Problem P^{hk} considered in this paper reduces to Problem P^{hk} of [11].

5. Error estimates for the fully discrete scheme

In this section, we discus the error estimates for the fully discrete scheme. To this end, we set

$$h_I = \max_{x\in\partial B_{h_I}(x_I)}\{\|x-x_I\|\}, \quad \overline{\Omega}\subset \cup_{I=1}^{N} B_{h_I}(x_I),$$

where $B_{h_{\rm I}}(x_{\rm I})$ is the open ball with the radius $h_{\rm I}$ and the center $x_{\rm I}$. Let $x_{h_I}(x_I)=\{x_1,x_2,..,x_l\}$ denote a set of nodes in ball $B_{h_I}(x_I)$ and $card\{x_{h_I}(x_I)\}$ represents the number of nodes in $B_{h_I}(x_I)$.

In order to get the error estimates, we assume the nodes distribution in the element-free Galerkin method satisfies the following conditions (see [2, 9]):

H1: For a given node $x\in\Omega$, there exists at least $m+1$ nodes which satisfies $x_j\in x_{h(x)}\cap B_{h/2}(x)$.
H2: There exists a constant c_0 such that the weight function $\omega(x)\geq c_0$.
H3: There exists a constant $c_*>0$ such that $card\{x_{h(x)}\cap B_{2h}(x)\}\leq c_*$ for all $x\in\Omega$.
H4: Weight function $\omega\subseteq C^m(B_h(0))\cap W^{m,\infty}(\mathbb{R})$.
H5: There exists a constant c_p such that $\frac{h}{\sigma}\leq c_p$, where $\sigma=\min_{i\neq k}\|x_i-x_k\|$, x_i and x_k are the ones of the $m+1$ nodes in $x_{h(x)}\cap B_{h/2}(x)$ mentioned in condition H1.

Now define an index set

$$ST(x):=\{j:\omega_j(x)\neq 0\}, \quad \forall x\in\overline{\Omega}.$$

We note that, for any node $x\in\overline{\Omega}$ satisfying $ST(x)=\{j_1,j_2,\cdots,j_k\}$, one has

$$h=h(ST(x)):=\max\{h_{j1},h_{j2},\cdots,h_{jk}\}.$$

Thus, for any $u\in H^{m+1}(\bar{(\Omega)})$, the semi-norm $|u|_{m+1}$ is denoted by

$$|u|_{m+1}=\left(\sum\nolimits_{m+1}\|D^\mu u(x)\|^2_{L^2(\Omega)}\right)^{1/2}, \quad x\in\Omega. \tag{5.36}$$

Lemma 5.5. ([2, 27, 28]) *Assume the conditions H1 - H5 are satisfied and u_h is the moving least-squares approximation of u. For $u\in H^{m+1}(\overline{\Omega})$ with $0\leq|\mu|\leq m$, where $m\geq 1$ is the number of the basis functions used to approximate u in the moving least-squares approximation, there exists a constant C, which is independent on h, such that*

$$\|D^\mu u-D^\mu u_h\|_{L^\infty}(\Omega)\leq Ch^{m+1-|\mu|}|u|_{m+1}, \tag{5.37}$$

$$\|D^\mu u-D^\mu u_h\|_{L^2}(\Omega)\leq Ch^{m+1-|\mu|}|u|_{m+1}, \tag{5.38}$$

$$\|D^\mu u-D^\mu u_h\|_{H^1}(\Omega)\leq Ch^{m-|\mu|}|u|_{m+1}. \tag{5.39}$$

We also need the following lemma.

Lemma 5.6. [10, Lemma 7.25] *Assume that $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of non-negative numbers satisfying $e_n\leq cg_n+c\sum_{j=1}^n k_je_j$. Then $e_n\leq c\left(g_n+\sum_{j=1}^n k_je_j\right)$ and $\max_{1\leq n\leq N}e_n\leq c\max_{1\leq n\leq N}g_n$.*

Now we can give an error analysis of the element-free Galerkin method to (1.1). To complete the proof of Theorem 5.3, we also need the following assumption

$$\exists L_j^*>0 \text{ such that } |j(g,v_1)-j(g,v_2)|\leq L_j^*\|g\|\|v_1-v_2\|, \quad \forall g,v_1,v_2\in V. \tag{5.40}$$

Theorem 5.3. *Assume u and u_n^{hk} are the solutions of* (1.1) *and* (4.30)*, respectively, and $u_{tt}\in L^2([0,T],V)$. If B is linear, then the following error estimate holds:*

$$\max_{1\leq n\leq N}\|w_n-w_n^{hk}\|\leq C\left(h^{m+1}+k+h^{\frac{m+1}{2}}\right), \quad \max_{1\leq n\leq N}\|u_n-u_n^{hk}\|\leq \tilde{C}\left(h^{m+1}+k+h^{\frac{m+1}{2}}\right),$$

where C and $\tilde{C}$ are constants depending on $\|u_{tt}\|_{L^2([0,T],V)}$ and M, L_A, L_B, L_j and L_j^*. Here M, L_A, L_B, L_j and L_j^* are defined by (2.3), (2.5), (2.7) and (5.40), *respectively.*

Proof. Because $w_n^{hk} \in V^h \subset V$, we can take $v = w_n^{hk}$ in (1.1) at $t = t_n$, where $w_n^{hk} = \delta u_n^{hk}$ and δu_n^{hk} is defined in (4.28). To simplify the equation, we denote $u(t_n) = u_n, u_t(t_n) = w_n$ and $f(t_n) = f_n$ at $t = t_n$. Therefore, (1.1) can be transformed as follows:

$$(Aw_n, w_n^{hk} - w_n) + (Bu_n, w_n^{hk} - w_n) + j(w_n, w_n^{hk}) - j(w_n, w_n) \geq (f_n, w_n^{hk} - w_n) \tag{5.41}$$

Adding (4.30) and (5.41), we have

$$\begin{aligned}
&(Aw_n - Aw_n^{hk}, w_n - w_n^{hk}) \\
\leq\ & (Aw_n, v^h - w_n) + (Bu_n^{hk}, v^h - w_n^{hk}) + (Bu_n, w_n^{hk} - w_n) + j(w_n, w_n^{hk}) \\
& + j(w_n^{hk}, v^h) - j(w_n^{hk}, w_n^{hk}) - j(w_n, w_n) - (f_n, v^h - w_n) \\
=\ & (Aw_n^{hk} - Aw_n, v^h - w_n) + (Bu_n^{hk} - Bu_n, v^h - w_n^{hk}) \\
& + j(w_n, w_n^{hk}) + j(w_n^{hk}, v^h) - j(w_n^{hk}, w_n^{hk}) - j(w_n, v^h) - (f_n, v^h - w_n) \\
& + (Aw_n, v^h - w_n) + (Bu_n, v^h - w_n) + j(w_n, v^h) - j(w_n, w_n) \\
=\ & (Aw_n^{hk} - Aw_n, v^h - w_n) + (Bu_n^{hk} - Bu_n, w_n - w_n^{hk}) \\
& + j(w_n, w_n^{hk}) + j(w_n^{hk}, w_n) - j(w_n^{hk}, w_n^{hk}) - j(w_n, w_n) - (f_n, v^h - w_n) \\
& + (Aw_n, v^h - w_n) + (Bu_n, v^h - w_n) + j(w_n, v^h) - j(w_n, w_n) \\
& + (Bu_n^{hk} - Bu_n, v^h - w_n).
\end{aligned} \tag{5.42}$$

According to the conditions (2.3), (2.5) and (2.7) and the Cauthy-Schwartz inequality, it follows from (5.42) that

$$\begin{aligned}
& M\|w_n - w_n^{hk}\|^2 \\
\leq\ & L_A\|w_n - w_n^{hk}\|\|v^h - w_n\| + L_B\|u_n^{hk} - u_n\|\|v^h - w_n\| \\
& + L_j\|w_n^{hk} - w_n\|\|w_n - v^h\| + \|f_n\|\|v^h - w_n\| + L_A\|w_n\|\|v^h - w_n\| \\
& + L_B\|u_n\|\|v^h - w_n\| + L_j^*\|w_n\|\|w_n - v^h\| + L_B\|u_n^{hk} - u_n\|\|w_n - w_n^{hk}\|.
\end{aligned} \tag{5.43}$$

Using the basic inequality

$$ab \leq \frac{a^2}{2\varepsilon^2} + \frac{b^2\varepsilon^2}{2}, \quad \forall a, b \geq 0 \quad \forall \varepsilon \neq 0$$

to (5.43), one has

$$\begin{aligned}
& (M - L_j)\|w_n - w_n^{hk}\|^2 \\
\leq\ & \frac{\varepsilon_1^2 L_A}{2}\|w_n - w_n^{hk}\|^2 + \frac{L_A}{2\varepsilon_1^2}\|v^h - w_n\|^2 + \frac{\varepsilon_2^2 L_B}{2}\|u_n - u_n^{hk}\|^2 + \frac{L_B}{2\varepsilon_2^2}\|v^h - w_n\|^2 \\
& + \frac{\varepsilon_3^2 L_B}{2}\|u_n - u_n^{hk}\|^2 + \frac{L_B}{2\varepsilon_3^2}\|w_n - w_n^{hk}\|^2 + \frac{\varepsilon_4^2 L_B}{2}\|w_n - w_n^{hk}\|^2 + \frac{L_B}{2\varepsilon_4^2}\|w_n - v^h\|^2 \\
& + (\|f_n\| + L_A\|w_n\| + L_B\|u_n\|)\|v^h - w_n\| + L_j^*\|w_n\|\|w_n - v^h\|.
\end{aligned}$$

This shows that

$$\begin{aligned}
& \left(M - L_j - \frac{\varepsilon_1^2 L_A}{2} - \frac{L_B}{2\varepsilon_3^2} - \frac{\varepsilon_4^2 L_B}{2}\right)\|w_n - w_n^{hk}\|^2 \\
\leq\ & \left(\frac{L_A}{2\varepsilon_1^2} + \frac{L_B}{2\varepsilon_2^2}\right)\|v^h - w_n\|^2 + \left(\frac{\varepsilon_2^2 L_B}{2} + \frac{\varepsilon_3^2 L_B +}{2} + \frac{L_B}{2\varepsilon_4^2}\right)\|u_n - u_n^{hk}\|^2 \\
& + (\|f_n\| + L_A\|w_n\| + L_j^*\|w_n\| + L_B\|u_n\|)\|v^h - w_n\|.
\end{aligned} \tag{5.44}$$

Choose ε_1, ε_3 and ε_4 such that $\left(M - L_j - \frac{\varepsilon_1^2 L_A}{2} - \frac{L_B}{\varepsilon_3^2 2}\right) > 0$. Since $f \in C([0,T];V)$ and $u_{tt} \in L^2([0,T],V)$, it follows from (5.44) that there exists a constant $c' > 0$ such that

$$\|w_n - w_n^{hk}\|^2 \le c' \left(\|v^h - w_n\|^2 + \|u_n - u_n^{hk}\|^2 + \|v^h - w_n\|\right)$$

and so

$$\|w_n - w_n^{hk}\| \le c\left(\|w_n - v^h\| + \|u_n^{hk} - u_n\| + \sqrt{\|v^h - w_n\|}\right), \tag{5.45}$$

where the constant c depends on M, L_A, L_B, L_j, L_j^* and $\|u_{tt}\|_{L^2([0,T],V)}$.

Now we turn to estimate the term $\|u_n^{hk} - u_n\|$. From (4.29) and the fact $u_n - u_0 = \int_0^{t_n} w(s)ds$, we have

$$u_n^{hk} - u_n = \sum\nolimits_{j=1}^{n} (w_j^{hk} - w_j)k + u_0^h - u_0 + \sum\nolimits_{j=1}^{n}\left(\int_{t_{j-1}}^{t_j} w_j - w(s)ds\right)$$

and so

$$\|u_n^{hk} - u_n\| \le \sum_{j=1}^{n}\|w_j^{hk} - w_j\|k + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|w_j - w(s)\|ds + \|u_0^{hk} - u_0\|. \tag{5.46}$$

Thus, by (5.45) and (5.46), we have

$$\begin{aligned} &\|w_n - w_n^{hk}\| \\ \le\;& c\left(\|w_n - v^h\| + \sum\nolimits_{j=1}^{n}\|w_j^{hk} - w_j\|k + \sum\nolimits_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|w_j - w(s)\|ds\right. \\ &\left. + \|u_0^{hk} - u_0\| + \sqrt{\|v^h - w_n\|}\right). \end{aligned} \tag{5.47}$$

Since $w_j - w(s) = \int_s^{t_j} u_{tt}dt$, one has

$$\|w_j - w(s)\| \le \int_s^{t_j}\|u_{tt}\|dt \le c^*(s - t_j) \le c^*(t_j - t_{j-1}) \le c^*k,$$

where c^* is a constant depending on $\|u_{tt}\|_{L^2([0,T],V)}$. Thus, it follows from (5.47) that

$$\begin{aligned} &\|w_n - w_n^{hk}\| \\ \le\;& c\left(\|w_n - v^h\| + \sum\nolimits_{j=1}^{n}\|w_j^{hk} - w_j\|k + c^*Tk + \|u_0^{hk} - u_0\| + \sqrt{\|v^h - w_n\|}\right). \end{aligned} \tag{5.48}$$

Let v^h be the moving least-squares approximation of w_n. Taking $|\mu| = 0$ in Lemma 5.5, one has

$$\|w_n - v^h\| \le C_1 h^{m+1}|w_n|_{m+1}, \quad \|u_0^{hk} - u_0\| \le C_1 h^{m+1}|u_0|_{m+1},$$

where C_1 is a constant appeared in Lemma 5.5. Letting $\hat{C} = C_1|w_n|_{m+1}$, we have

$$\|w_n - w_n^{hk}\| \le c\left(\hat{C}h^{m+1} + c^*Tk + \sqrt{\hat{C}h^{m+1}} + \sum\nolimits_{j=1}^{n}\|w_j^{hk} - w_j\|k\right). \tag{5.49}$$

Thus, it follows from (5.49) and Lemma 5.6 that

$$\max_{1\le n\le N}\|w_n - w_n^{hk}\| \le C\left(h^{m+1} + k + h^{\frac{m+1}{2}}\right), \tag{5.50}$$

where the constant C depends on $M, L_A, L_B, L_j, L_j^*, T$ and $\|u_{tt}\|_{L^2([0,T],V)}$.

Next we show the second conclusion. Similar to the proof of (5.46), we can show that

$$\|u_n^{hk} - u_n\| \le \sum_{j=1}^{n}\|w_j^{hk} - w_j\|k + \|u_0^{hk} - u_0\| + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|w_j - w(s)\|ds. \tag{5.51}$$

As the proof process of (5.48), it follows from (5.51) that

$$\|u_n^{hk} - u_n\| \le \sum_{j=1}^{n} \|w_j^{hk} - w_j\|k + \|u_0^{hk} - u_0\| + c^*Tk. \tag{5.52}$$

From (5.52) and Lemma 5.6, we obtain

$$\begin{aligned}
\max_{1\le n\le N} \|u_n^{hk} - u_n\| &\le \sum_{j=1}^{n} \|w_j^{hk} - w_j\|k + \|u_0^{hk} - u_0\| + c^*Tk \\
&\le T \max_{1\le n\le N} \|w_j^{hk} - w_j\| + \hat{C}h^{m+1} + c^*Tk \\
&\le \tilde{C}\left(h^{m+1} + k + h^{\frac{m+1}{2}}\right),
\end{aligned}$$

where the constant $\tilde{C}$ depends on $M, L_A, L_B, L_j, L_j^*, T$ and $\|u_{tt}\|_{L^2([0,T],V)}$. □

Remark 5.3. If A is linear and $j(u,v) = I_K(v) = \begin{cases} 0, & v \in K; \\ +\infty, & v \notin K, \end{cases}$ then Theorem 5.3 degenerates to Theorem 4.1 of [7].

Remark 5.4. We would like to point out that Theorem 5.3 improves Theorem 3.4 of Ding et al. [9] in the following aspects: (i) the parabolic evolutionary variational inequality is extended to the parabolic evolutionary quasivariational inequality; (ii) The linearity of the operator A is dropped; (iii) The error estimates are more reasonable. In fact, we give the error estimates of $\max_{1\le n\le N}\|w_n - w_n^{hk}\|$ and $\max_{1\le n\le N}\|u_n^{hk} - u_n\|$ in Theorem 5.3 of this paper, while Theorem 3.4 of Ding [9] obtained the error estimate of

$$\max_{1\le n\le N}\left(\|u_n - u_n^{hk}\| + \sum_{0\le n\le N-1} \|u_{n+1} - u_n + u_n^{hk} - u_{n+1}^{hk}\|\right).$$

6. A concluding remark

This paper focuses on the study of a class of parabolic evolutionary quasivariational inequalities in real Hilbert spaces. The existence and uniqueness of the solution for the parabolic evolutionary quasivariational inequality is proved under some mild conditions. The error estimate for the parabolic evolutionary quasivariational inequality is also investigated by using the forward Euler difference scheme and the element-free Galerkin spatial approximation.

It is well known that variational-hemivariational inequality problems arise in the study of various nonlinear boundary value problems which can be used to describe many mathematical models in Physics, Mechanics and Engineering Sciences [20, 22]. Thus, it would be important and interesting to discritize variational-hemivariational inequalities by employing the element-free Galerkin method in the future work.

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