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Dedicated to Prof. Hong-Kun Xu on the occasion of his 60^{th} anniversary

Approximation of zeros of m-accretive mappings, with applications to Hammerstein integral equations

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ABSTRACT. An algorithm for approximating zeros of m-accretive operators is constructed in a uniformly smooth real Banach space. The sequence generated by the algorithm is proved to converge strongly to a zero of an m-accretive operator. In the case of a real Hilbert space, our theorem complements the celebrated proximal point algorithm of Martinet and Rockafellar for approximating zeros of maximal monotone operators. Furthermore, the convergence theorem proved is applied to approximate a solution of a Hammerstein integral equation. Finally, numerical experiments are presented to illustrate the convergence of our algorithm.

1. INTRODUCTION

Let E be a real normed space with dual space E^*. A mapping $A : E \to 2^E$ is called *accretive* if for each $x, y \in E, \eta \in Ax, \nu \in Ay$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle \eta - \nu, j(x-y) \rangle \geq 0,$$

where $J : E \to 2^{E^*}$ defined by $Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}$ is the normalized duality map on E. The mapping A is said to be *maximal accretive* if A is accretive and its graph is not included in the graph of any other accretive mapping. Also, the mapping A is said to be *m-accretive* if A is accretive and the following range condition holds: $R(I + \lambda A) = E$, for all $\lambda > 0$.

Interest in the study of accretive mappings stems from their usefulness in several areas such as in economics, differential equations, calculus of variation, and so on (see e.g., Berinde, [2], Browder, [5], Zeildler, [47]).

Consider, for example, the differential equation

$$\frac{du}{dt} + Au = 0, \tag{1.1}$$

where A is an accretive mapping on E. In several applications, equation (1.1) describes the evolution of physical phenomena which generate energy over time. At equilibrium state, $\frac{du}{dt} = 0$ and equation (1.1) reduces to

$$Au = 0, \tag{1.2}$$

whose solutions then correspond to the equilibrium state of the system described by equation (1.1). Such equilibrium states are very desirable in many applications, for example, in ecology, economics, physics and so on.

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In general, a fundamental problem in the study of accretive operators in Banach spaces is the following:

$$\text{find } u \in E \text{ such that } 0 \in Au. \tag{1.3}$$

Several existence theorems for equation (1.2) have been proved (see e.g., Browder [5]). Also, methods of approximating solutions of the inclusion $0 \in Au$ have been studied extensively. One of the classical methods for approximating solutions of inclusion in (1.3) where A is a maximal monotone operator *on a real Hilbert space* is the celebrated *proximal point algorithm* introduced by Martinet [34] and studied extensively by Rockafellar [41] and numerous other authors. The algorithm is given by: $x_0 \in H$,

$$x_{n+1} = \Big(I + \frac{1}{\lambda_n}A\Big)^{-1}x_n + e_n,\ n \geq 0, \tag{1.4}$$

where $\lambda_n > 0$ is a regularizing parameter. Rockafellar [41] proved that if the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is bounded from above, then the resulting sequence $\{x_n\}_{n=1}^{\infty}$ of the proximal point iterates converges *weakly* to a solution of (1.2), provided that a solution exists (see also Bruck and Reich [9]). Rockafellar [41] then posed the following question.

Question 1. *Does the proximal point algorithm always converge strongly?*

This question was resolved in the negative by Güler [27] who produced a proper closed convex function in the infinite dimensional Hilbert space l_2 for which the proximal point algorithm converges *weakly but not strongly*, (see also Bauschke *et al.* [3]). This naturally raised the following questions.

Question 2. *Can the proximal point algorithm be modified to guarantee strong convergence?*

Question 3. *Can another iterative algorithm be developed to approximate a solution of* (1.2), *assuming existence, such that the sequence of the algorithm converges strongly to a solution of* (1.2)*?*

In connection with Question 2, Solodov and Svaiter [44] proposed a modification of the proximal point algorithm which guarantees strong convergence in a real Hilbert space. Their algorithm is as follows.

Choose any $x_0 \in H$ and $\sigma \in [0,1)$. At iteration k, having x_k, choose $\mu_k > 0$, and find (y_k, v_k), an inexact solution of $0 \in Tx + \mu_k(x - x_k)$, with tolerance σ. Define

$$C_k = \{z \in H : \big\langle z - y_k, v_k\big\rangle \leq 0\}, \quad Q_k = \{z \in H : \big\langle z - x_k, x_0 - x_k\big\rangle \leq 0\}.$$

$$\text{Take} \qquad x_{k+1} = P_{C_k \cap Q_k}x_0,\ k \geq 1.$$

The authors themselves noted ([44], p. 195) that ". . . at each iteration, there are two subproblems to be solved. . . " : (i) find an inexact solution of the proximal point algorithm, and (ii) find the projection of x_0 onto $C_k \cap Q_k$. They also acknowledged that these two subproblems constitute a serious drawback in using their algorithm.

Kamimura and Takahashi [28] extended this work of Solodov and Svaiter [44] to the framework of Banach spaces that are both uniformly convex and uniformly smooth, where the operator A is maximal monotone. Reich and Sabach [40] extended this result to reflexive Banach spaces.

Lehdili and Moudafi [32] considered the technique of the proximal map and the Tikhonov regularization to introduce the so-called Prox-Tikhonov method which generates the sequence $\{x_n\}$ by the algorithm:

$$x_0 \in H, \qquad x_{n+1} = J_{\lambda_n}^{A_n} x_n,\ n \geq 0, \tag{1.5}$$

where $A_n := \mu_n I + A,\ \mu_n > 0$ and $J_{\lambda_n}^{A_n} := \big(I + \frac{1}{\lambda_n} A_n\big)^{-1}$. Using the notion of variational distance, Lehdili and Moudafi [32] proved strong convergence theorems for this algorithm and its perturbed version, under appropriate conditions on the sequences $\{\lambda_n\}$ and $\{\mu_n\}$.

Xu [45] also studied the recurrence relation (1.5) in a real Hilbert space. He used the technique of nonexpansive mappings to get convergence theorems for the perturbed version of the algorithm (1.5), under much relaxed conditions on the sequences $\{\lambda_n\}$ and $\{\mu_n\}$.

With respect to Question 3, Bruck [8] considered an iteration process of the *Mann-type* [33] and proved that the sequence of the process converges strongly to a solution of (1.2) in a real Hilbert space, where A is a maximal monotone map, provided *the initial vector is chosen in a neighbourhood of a solution of* (1.2). Chidume [13] extended this result to L_p spaces, $p \geq 2$ (see also Reich [37, 38, 39]). These results of Bruck [8] and Chidume [13] are not easy to use in any possible application because the neighborhood of a solution in which the initial vector must be chosen is not known precisely.

Still in response to Question 3, Chidume [12] recently proved the following theorem.

Theorem 1.1 (Chidume [12])**.** *Let E be a uniformly smooth real Banach space with modulus of smoothness ρ_E, and let $A : E \to 2^E$ be a multi-valued bounded m-accretive operator with $D(A) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $u_1 \in E$, define a sequence $\{u_n\}$ iteratively by,*

$$u_{n+1} = u_n - \lambda_n \eta_n - \lambda_n \theta_n (x_n - u_1)\big),\ \eta_n \in Ax_n,\ n \geq 1, \tag{1.6}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying certain conditions. There exists a constant $\gamma_0 > 0$ such that $\frac{\rho_E(\lambda_n)}{\lambda_n} \leq \gamma_0 \theta_n$. Then the sequence $\{u_n\}$ converges strongly to a zero of A.

Remark 1.1. This result of Chidume [12] resolves Question 3 in the affirmative but only for *m-accretive operators that bounded.* Consequently, the following question is of interest.

Question 4. *Can the requirement that A be bounded imposed in Theorem 1.1 be dispensed with?*

In this paper, we give an affirmative answer to this question thereby giving a complete solution to Question 3. This is achieved by means of a new important result concerning accretive operators, which was recently proved by Chidume *et al.* [14]. Furthermore, the convergence theorem proved is applied to approximate a solution of *a Hammerstein integral equation*. Finally, some numerical examples are presented to illustrate the strong convergence of the sequence of our algorithm.

2. Preliminaries

We shall use the following lemmas in the sequel.

Lemma 2.1 (Xu and Roach, [46])**.** *Let E be a uniformly smooth real Banach space. Then, there exist constants D and C such that for all $x,\ y \in E,\ j(x) \in J(x)$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + D \max\Big\{\|x\| + \|y\|, \frac{1}{2}C\Big\}\rho_E(\|y\|),$$

where ρ_E denotes the modulus of smoothness of E.

Lemma 2.2 (see e.g., Chidume, [11]). *Let E be a normed real linear space. Then, the following inequality holds:*

(2.7) $$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall\, j(x+y) \in J(x+y),\ \forall\, x,\ y \in E.$$

Lemma 2.3 (Fitzpatrick, Hess and Kato, [26]). *Let E be a real reflexive Banach space, $A : D(A) \subset E \to E$ be an accretive mapping. Then, A is locally bounded at any interior point of $D(A)$.*

Lemma 2.4 (Chidume and Idu, [18]). *Let $q > 1$ and let X, Y be real uniformly smooth spaces. Let $E = X \times Y$ with the norm $\|z\|_E = (\|u\|_X^q + \|v\|_Y^q)^{\frac{1}{q}}$, for arbitrary $z = [u, v] \in E$. Let $E^* = X^* \times Y^*$ denote the dual space of E. For arbitrary $x = [x_1, x_2] \in E$, define the mapping $j_q^E : E \to E^*$ by*

$$j_q^E(x) = j_q^E[x_1, x_2] := [j_q^X(x_1), j_q^Y(x_2)],$$

so that for arbitrary $z_1 = [u_1, v_1],\ z_2 = [u_2, v_2]$ in E, the duality pairing $\langle \cdot, \cdot \rangle$ is given by

$$\langle z_1, j_q^E \rangle := \langle u_1, j_q^X(u_2)\rangle + \langle v_1, j_q^Y(v_2)\rangle.$$

Then,

- (a) *E is uniformly smooth,*
- (b) *j_q^E is single-valued duality mapping on E.*

Lemma 2.5 (Chidume *et al.*, [14]). *Let E be a smooth and reflexive real Banach space and $A : E \to 2^E$ be an accretive map with $0 \in intD(A)$. Then, for any $M > 0$, there is exists $C > 0$ such that:*

- (i) $(y, v) \in G(A)$;
- (ii) $\langle v, j(x) - j(x-y)\rangle \leq M(2\|x\| + \|y\|)$;
- (iii) $\|y\| \leq M,\ \|x\| \leq M$; *imply* $\|v\| \leq C$.

In this lemma, int$D(A)$ denotes the interior of the domain of A and $G(A)$ denotes the graph of A.

Since this lemma is yet to appear, for completeness, we give its proof here.

Proof. By Lemma 2.3, A is locally bounded at 0. This implies that there exist $r > 0,\ M^* > 0$ such that $B_E(0, r) := \{x \in E : \|x\| \leq r\} \subset int\ D(A)$ and

$$\|u\| \leq M^*,\ \forall\, x \in B_E(0, r),\ u \in Ax.$$

Let $M > 0$, $x \in B_E(0, r)$ and $y \in D(A)$. Assume that $\|y\| \leq M$ and $v \in Ay$ such that

$$\langle v, Jx - J(x-y)\rangle \leq M(2\|x\| + \|y\|).$$

By the accretivity of A, $\langle v - u, J(y-x)\rangle \geq 0, \forall u \in Ax$. This implies that

$$\langle v, J(x-y)\rangle \leq \langle u, J(x-y)\rangle \leq M^*(\|y\| + r).$$

Furthermore,

$$\begin{aligned}\langle v, Jx\rangle &= \langle v, J(x-y)\rangle + \langle v, Jx - J(x-y)\rangle \\ &\leq M^*(\|y\| + r) + M(2\|x\| + \|y\|) \\ &\leq M^*(M + r) + M(2r + M).\end{aligned}$$

This implies that

$$|\langle v, Jx\rangle| \leq M^*(M + r) + M(2r + M), \qquad \forall x \in B_E(0, r).$$

For $f \in B_{E^*}(0,1)$, by the reflexivity and smoothness of E, there exists $x \in B_E(0,r)$ such that $Jx = rf$. So,

$$|\langle v, f\rangle| = \frac{1}{r}|\langle v, Jx\rangle| \leq \frac{1}{r}\Big(M^*(M+r) + M(2r+M)\Big).$$

Therefore,

$$\sup_{\|f\|_{E^*}\leq 1} |\langle v, f\rangle| \leq \frac{1}{r}\Big(M^*(M+r) + M(2r+M)\Big).$$

This completes proof of Lemma 2.5. □

3. Main results

In Theorem 3.2 below, $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \theta_n = 0,\ \{\theta_n\}$ is decreasing;

(ii) $\lim\limits_{n\to\infty} \left[\dfrac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n\theta_n}\right] = 0,$ and $\lambda_n\theta_n < 1;$ (iii) $\dfrac{\rho_E(M_0\lambda_n)}{M_0\lambda_n} \leq \gamma_0\theta_n,$

for some constants $\gamma_0 > 0$ and $M_0 > 0$; where ρ_E is the modulus of smoothness of E.

Prototypes for $\{\lambda_n\}$ and $\{\theta_n\}$ are: $\lambda_n = \dfrac{1}{(n+1)^a}$ and $\theta_n = \dfrac{1}{(n+1)^b}$, where $a+b<1$ and $0<b<a$ (see e.g., Chidume and Idu, [18]).

We now prove the following theorem.

Theorem 3.2. *Let E be a uniformly smooth real Banach space and $A : E \to 2^E$ be a multi-valued m-accretive operator with $D(A) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = x_n - \lambda_n u_n - \lambda_n\theta_n(x_n - x_1),\ u_n \in Ax_n,\ n \geq 1. \tag{3.8}$$

Then, the sequence $\{x_n\}$ converges strongly to a solution of the inclusion $0 \in Au$.

Proof. First, we show that $\{x_n\}$ is bounded. Let x^* be a solution of the inclusion $0 \in Au$. Then, there exists $r > 0$ such that $x_1 \in B(x^*, \frac{r}{2}) := \{x \in E : \|x - x^*\| \leq \frac{r}{2}\}$. Define $B = B(x^*, r)$. Then, for any $x \in B$, we have that $\|x\| \leq r + \|x^*\|$.

Let $x,\ y \in E$ and $u_y \in Ay$ be arbitrary. Since A is locally bounded at $0 \in E = int(D(A))$, there exist $\delta > 0$, $K > 0$ such that $\|u_y\| \leq K$, for all $y \in B(0,\delta)$. Therefore, we obtain

$$\begin{aligned}\langle u_y, j(x) - j(x-y)\rangle &\leq \|u_y\|\|j(x) - j(x-y)\|\\ &\leq K\|j(x) - j(x-y)\|, \quad \text{for } y \in B(0,\delta)\\ &\leq K(2\|x\| + \|y\|), \quad \text{for } \|y\| \leq \delta.\end{aligned}$$

Define

$M := \max\{r+\|x^*\|, \delta, K\}$. So, $\|y\| \leq M,\ \|x\| \leq M$ and $\langle u_y, j(x)-j(x-y)\rangle \leq M(2\|x\|+\|y\|)$, which implies, by Lemma 2.5, that there exists $L > 0$ such that $\|u_y\| \leq L$.

Now, define the following constants:

$$M_0 := \sup\{\|u_x + \theta(x - x_1)\| : x \in B, u_x \in Ax; 0 < \theta < 1\} + 1.$$

$$M_1 := \sup\Big\{D\max\Big\{\|x - x^*\| + \lambda M_0, \frac{C}{2}\Big\} : x \in B, \lambda \in (0,1)\Big\}.$$

$$\gamma_0 := \frac{1}{2}\min\Big\{1, \frac{r^2}{4M_1M_0}\Big\},$$

where D and C are the constants in Lemma 2.1.

Claim: $x_n \in B,\ \forall\, n \geq 1$.

We prove this by induction. By construction, $x_1 \in B$. Assume $x_n \in B$ for some $n \geq 1$. We prove $x_{n+1} \in B$. Using the recursion formula (3.8), Lemma 2.1, condition (iii), and denoting $0 \in Ax^*$ by 0^*, we compute as follows:

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|x_n - x^* - \lambda_n(u_n + \theta_n(x_n - x_1))\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\lambda_n\langle u_n + \theta_n(x_n - x_1), j(x_n - x^*)\rangle \\
&\quad + D\max\Big\{\|x_n - x^*\| + \lambda_n\|u_n + \theta_n(x_n - x_1)\|, \frac{C}{2}\Big\} \times \\
&\quad \rho_E(\lambda_n\|u_n + \theta_n(x_n - x_1)\|) \\
&\leq \|x_n - x^*\|^2 - 2\lambda_n\langle u_n - 0^*, j(x_n - x^*)\rangle - 2\lambda_n\theta_n\langle x_n - x_1, j(x_n - x^*)\rangle \\
&\quad + M_1\rho_E(\lambda_n\|u_n + \theta_n(x_n - x_1)\|) \\
&\leq \|x_n - x^*\|^2 - 2\lambda_n\theta_n\|x_n - x^*\|^2 + \lambda_n\theta_n(\|x^* - x_1\|^2 + \|x_n - x^*\|^2) \\
&\quad + M_1\rho_E(\lambda_n\|u_n + \theta_n(x_n - x_1)\|) \\
&\leq (1-\lambda_n\theta_n)\|x_n - x^*\|^2 + \lambda_n\theta_n\|x^* - x_1\|^2 + M_1\frac{\rho_E(\lambda_n M_0)}{\lambda_n M_0}\lambda_n M_0 \\
&\leq (1-\lambda_n\theta_n)\|x_n - x^*\|^2 + \lambda_n\theta_n\|x^* - x_1\|^2 + M_1\gamma_0\lambda_n\theta_n M_0 \\
&\leq \Big(1 - \frac{1}{2}\lambda_n\theta_n\Big)r^2 \leq r^2.
\end{aligned}$$

Hence, $x_n \in B, \quad \forall\, n \geq 1$, and so $\{x_n\}$ is bounded. The rest of the proof of the strong convergence of $\{x_n\}$ to a zero of A now follows the same method as in the proof of Theorem 3.2 in [12]. □

4. Applications to Hammerstein integral equations

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form

$$u(x) + \int_\Omega k(x,y)f(y,u(y))dy = w(x), \tag{4.9}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions.

If we define an operator K by $K(v) := \int_\Omega \kappa(x,y)v(y)dy;\ x \in \Omega$, and the so-called *super position* or *Nemytskii* operator by $Fu(y) := f(y,u)$, then, equation (4.9) can put in the form

$$u + KFu = 0. \tag{4.10}$$

Without loss of generality, we have taken $w \equiv 0$. Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear part posses Green's function can, as a rule, be transformed into the form of equation (4.9) (see e.g., Pascali and Sburian [36], Chapter IV). Equations of Hammerstein-type also play a special role in the theory of optimal control systems and in automation and network theory (see e.g., Dolezale [24]).

Several existence and uniqueness theorems have been proved for equations of Hammerstein type (see, e.g., Brezis and Browder [4], Browder and Gupta [7], Browder *et al.* [6] Chepanovich [10], De Figueiredo and Gupta [23]).

Iterative methods for approximating solutions of problem (4.10) have been studied (see e.g., Brezis and Browder [4], Chidume and Zegeye [15, 16], Chidume and Djitte [17], Ofoedu and Onyi [35], Shehu [43], Chidume and Idu [18], Djitte and Sene [25], Chidume and Bello [19], Chidume *et al.* [21, 22] and the references therein).

In this section, we shall apply Theorem 3.2 to approximate a solution of equation (4.10). The following definition and lemmas will be needed in what follows.

Definition 4.2. Let C be a nonempty subset of a real normed space, X. A mapping $T : C \to X$ is said to be *pseudocontractive* if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq ||x-y||^2, \ \forall\, x, y \in C, \tag{4.11}$$

where $J : X \to 2^{X^*}$ is the normalized duality map.

The class of pseudocontractive mappings is easily seen to include the nonexpansive mappings. Interest in the study of pseudocontractive mappings stems from the fact that a mapping T is pseudocontractive if and only if $(I-T)$ is accretive (see, e.g., Kato, [29]).

Definition 4.3. Let C be a nonempty subset of a real normed space, X. The set C is said to have the fixed point property for nonexpansive self-mappings if every nonexpansive mapping, $T : C \to C$ has a fixed point.

This is the case, for instance, if C is weakly compact, convex and has normal structure (see, e.g., Kirk, [30]).

Lemma 4.6 (Barbu, [1])**.** *Let E be a real Banach space, A be an m-accretive set of $E \times E$ and let $B : E \to E$ be a continuous, m-accretive operator with $D(B) = E$. Then, $A + B$ is m-accretive.*

Lemma 4.7 (Chidume and Idu, [18])**.** *Let E be a uniformly smooth real Banach space and $X = E \times E$. Let $F, K : E \to E$ be m-accretive mappings. Let $A : X \to X$ be defined by $A[u,v] = [Fu - v, Kv + u]$. Then, A is m-accretive.*

Remark 4.2. We remark that for A defined in Lemma 4.7, $[u^*, v^*]$ is a zero of A if and only if u^* solves equation (4.10), with $v^* = Fu^*$.

For pseudocontractive mappings, we have the following theorem.

Theorem 4.3 (Kirk, [31])**.** *Let X be a real Banach space, and let C be a nonempty, closed and convex subset of X which possesses the fixed point property for nonexpansive self-mappings. Let $T : C \to C$ be a Lipschitz pseudocontractive mapping. Then, T has a fixed point.*

We now prove the following existence theorem.

Theorem 4.4. *Let X be a real Banach space, and let C be a nonempty, closed and convex subset of X which possesses the fixed point property for nonexpansive self-mappings. Let $F, K : C \to C$ be m-accretive and Lipschitz mappings with $D(K) = D(F) = C$. Let $E := C \times C$ and $A : E \to E$ be defined by $A[u,v] := [Fu - v, u + Kv]$. Then, the equation $u + KFu = 0$ has a solution in C.*

Proof. Define $S, Q : E \to E$ by

$$S[u,v] := [Fu, Kv] \qquad\qquad Q[u,v] := [-v, u].$$

Then, $A = S + Q$. Moreover, S is m-accretive, Q is m-accretive and continuous. By Lemma 4.6, A is m-accretive. By Kato [29], $T := I - A$ is pseudocontractive. Furthermore,

T is Lipschitz. By Lemma 4.3, T has a fixed point in C. This fixed point is a solution of $u + KFu = 0$. □

We now prove the following theorem.

Theorem 4.5. *Let X be a uniformly smooth real Banach space. Let $F,\ K : X \to X$ be m-accretive mappings. Let $E := X \times X$ and $A : E \to E$ be defined by $A[u,v] := [Fu - v, Kv + u]$. For arbitrary $x_1,\ z_1 \in E$, define the sequence $\{z_n\}$ in E by*

$$z_{n+1} = z_n - \lambda_n A z_n - \lambda_n \theta_n (z_n - x_1),\ n \geq 1. \tag{4.12}$$

Assume that the equation $u + KFu = 0$ has a solution. Then, the sequence $\{z_n\}_{n=1}^{\infty}$ converges strongly to a solution of $u + KFu = 0$.

Proof. By Lemma 2.4, E is uniformly smooth, and by Lemma 4.7, A is m-accretive. Hence, the conclusion follows from Theorem 3.2 and Remark 4.2. □

Theorem 4.5 can also be stated as follows.

Theorem 4.6. *Let X be a uniformly smooth real Banach space and let $F,\ K : X \to X$ be m-accretive mappings. For $(x_1, y_1),\ (u_1, v_1) \in X \times X$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E, by*

$$u_{n+1} = u_n - \lambda_n (Fu_n - v_n) - \lambda_n \theta_n (u_n - x_1),\ n \geq 1,$$

$$v_{n+1} = v_n - \lambda_n (Kv_n + u_n) - \lambda_n \theta_n (v_n - y_1),\ n \geq 1.$$

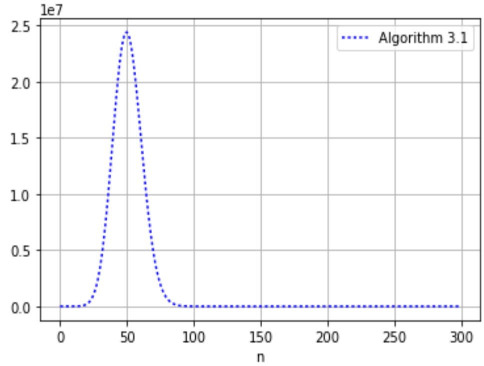
Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

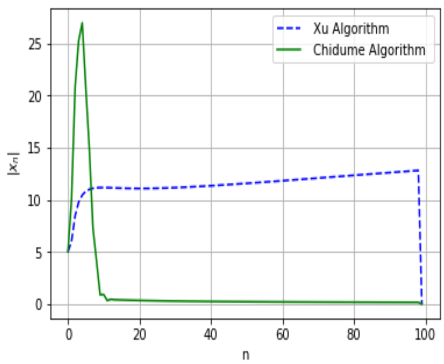
5. NUMERICAL EXPERIMENT

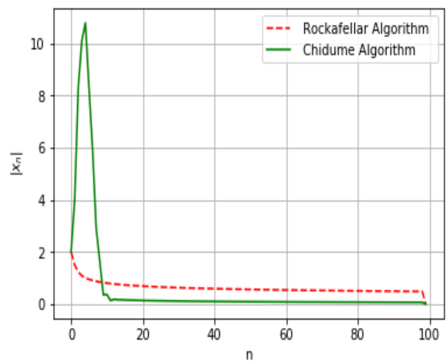
In this section, we shall numerically demonstrate the convergence of the sequence generated by the algorithm proposed in this paper. We shall also compare the convergence of our algorithm with that of the proximal point algorithm and some of its modifications.

Example 5.1. Let $E = \mathbb{R}$ and $Ax = 4x$. Then, A is accretive and $0 \in A^{-1}(0)$. Taking $\lambda_n = \frac{1}{(n+1)^{0.2}}$, and $\theta_n = \frac{1}{(n+1)^{0.25}}$ we obtain the following table and graph of $|x_n|$ against number of iterations, where $\{x_n\}$ is the sequence generated algorithm for approximating solutions of $Au = 0$, assuming existence.

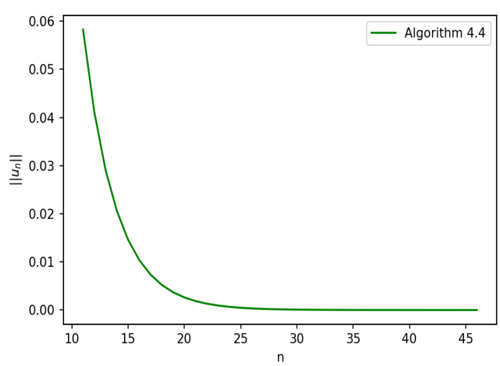
No of iterations	Initial Points	$\lvert x_n \rvert$	Time (s)
189	2	0.12506344	0.1206655502319336
198	2	0.1247684	0.1018977165222168
600	0.5	0.02405017	0.10132288932800293
944	-0.5	0.02157754	0.09952473640441895
1999	1.5	0.05406199	0.12050509452819824

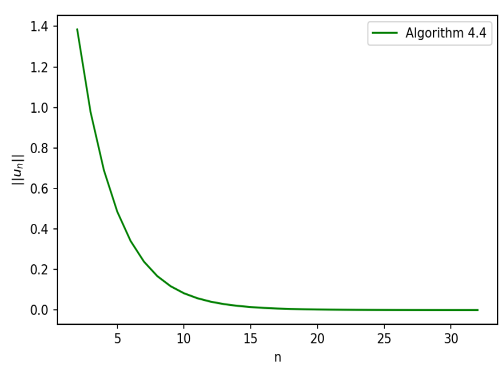






Example 5.2. Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Let $F = \begin{bmatrix} 3 & 1 \\ -1 & 8 \end{bmatrix}$ and $K = \begin{bmatrix} 7 & -2 \\ 2 & 5 \end{bmatrix}$. Taking $\lambda_n = \frac{1}{(n+1)^{0.2}}$, and $\theta_n = \frac{1}{(n+1)^{0.25}}$, and the initial points $u = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we obtain the following graph of $|u_n|$ against number of iterations, where $\{u_n\}$ is the sequence generated by Algorithm (4.12) for approximating solutions of $u + KFu = 0$, assuming existence.





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