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Dedicated to Prof. Hong-Kun Xu on the occasion of his 60^{th} anniversary

Existence and approximation of a fixed point of a fundamentally nonexpansive mapping in hyperbolic spaces

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ABSTRACT. We prove that a fundamentally nonexpansive mapping on a compact and convex subset of a hyperbolic space, has a fixed point. We also show that one−step iterative algorithm of two mappings is vital for the approximation of a common fixed point of two fundamentally nonexpansive mappings in a strictly convex hyperbolic space. Our results are new in metric fixed point theory and generalize several existing results.

1. INTRODUCTION AND PRELIMINARIES

A nonlinear structure for the iterative construction of a fixed point of certain classes of nonlinear mappings is a metric space embedded with a convex structure. Different notions of convexity in metric spaces exist in the literature (see, for example, Kohlenbach [13], Menger [15], Reich and Shafrir [18] etc).

A metric space X is a convex metric space in the sense of Menger [15] if any two points x, y in X are endpoints of a unique metric segment $[x, y]$, an isometric image of $[0, d(x, y)]$ and the unique point $z = tx \oplus (1-t)y$ on $[x, y]$ satisfies

$$d(x, z) = (1-t)d(x, y) \text{ and } d(z, y) = td(x, y) \text{ for } t \in [0, 1].$$

From the definition of convex metric space, we have that
(i) $0x \oplus 1y = y$ (ii) $1x \oplus 0y = x$ (iii) $tx \oplus (1-t)\,x = x$.

A convex metric space X is hyperbolic if

$$d\left(tx \oplus (1-t)y, tz \oplus (1-t)w\right) \le td\left(x, z\right) + (1-t)\, d\left(y, w\right)$$

for all $x, y, z, w \in X$ and $t \in [0, 1]$.

If $z = w$ in the hyperbolic inequality, it becomes

$$d\left(tx \oplus (1-t)y, z\right) \le td\left(x, z\right) + (1-t)\, d\left(y, z\right). \tag{1.1}$$

If $tx \oplus (1-t)y \in C$ for all $x, y \in C$ and $t \in [0, 1]$, then C is regarded as a convex subset of X. An important example of a nonlinear hyperbolic space [12] is the CAT(0) space X which is geodesically connected and every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane.

For fixed $a \in X$, $r > 0$ and $\varepsilon > 0$, set

$$\delta(r, \varepsilon) = \inf_{\substack{d(a,x)\le r, d(a,y)\le r,\\ d(x,y)\ge r\varepsilon}} \left(1 - \frac{1}{r} d\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right)\right), \quad x, y \in X.$$

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The space X is uniformly convex [23] if $\delta(r,\varepsilon) > 0$ and strictly convex [5] whenever $d(z,x) = d(z, tx \oplus (1-t)y) = d(z,y)$ for $x,y,z \in X, t \in (0,1)$,we must have that $x = y$.

From these definitions, it is easy to verify that a uniformly convex metric space X is strictly convex but the converse is not true in general. There exist strictly convex metric spaces which are not uniformly convex metric spaces.

Let T be a self-mapping on a subset C of a metric space X. A point $x \in C$ is a fixed point of T if $Tx = x$. Denote by $F(T)$, the set of all fixed points of T. The mapping T is (i) nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x, y \in C$ (ii) condition (C) mapping [25] if $\frac{1}{2}d(x,Tx) \leq d(x,y)$ implies $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in C$ (iii) fundamentally nonexpansive[6] if $d(T^2x,Ty) \leq d(Tx,y)$ for all $x,y \in C$ (iv) quasi-nonexpansive if $d(Tx,y) \leq d(x,y)$ for all $x \in C, y \in F(T)$.

For the above mappings, we have the following implications:

Nonexpansive mapping $\Longrightarrow$ condition(C) mapping $\Longrightarrow$ fundamentally nonexpansive mapping $\Longrightarrow$ quasi-nonexpansive mapping.

In 2014, Ghoncheh and Razani[6] introduced fundamentally nonexpansive mappings and they showed that these mappings are weaker than condition(C) mappings and stronger than quasi-nonexpansive mappings. In general, fundamentally nonexpansive mappings are discontinuous.

In the following example, we give one nonexpansive mapping and two fundamentally nonexpansive mappings with a common fixed point. Also the example clarify the above facts about fundamentally nonexpansive mappings.

Example 1.1. Take $C = [0,1] \subset \mathbb{R}$ with usual metric. Define $R, S, T, U : C \to C$ by

$$Rx = 1 - x,$$

$$Sx = \begin{cases} \frac{1}{2} & \text{if } x \neq 1 \\ \frac{3}{4} & \text{if } x = 1, \end{cases}$$

$$Tx = \begin{cases} \frac{1}{2} & \text{if } x \neq 1 \\ \frac{18}{25} & \text{if } x = 1 \end{cases}$$

and

$$Ux = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

We observe that
(i) R is nonexpansive, S and T are fundamentally nonexpansive with $F(R) = F(S) = F(T) = \left\{\frac{1}{2}\right\}$ but S and T are not condition(C) mappings (ii) U is quasi-nonexpansive but it is not fundamentally nonexpansive mapping.

Let C be a convex subset of a hyperbolic space X and $T_1, T_2 : C \to C$ nonlinear mappings.

(i) Mann's iterative algorithm[14]:

$$x_1 \in C, \; x_{n+1} = s_n T_1 x_n \oplus (1 - s_n) x_n \tag{1.2}$$

where $\{s_n\}$ is a sequence in the interval $(0, 1)$.

(ii) One−step iterative algorithm for two mappings [3]:

$$x_1 \in C,$$
$$x_{n+1} = s_n T_1 x_n \oplus (1 - s_n)\, y_n \tag{1.3}$$
$$y_n = \frac{t_n}{1 - s_n} T_2 x_n \oplus \left(1 - \frac{t_n}{1 - s_n}\right) x_n$$

where $\{s_n\}$ and $\{t_n\}$ are sequences in the interval $(\alpha, 1 - \alpha)$ for some $\alpha \in (0, 1)$ and $s_n + t_n < 1$.

(iii) Ishikawa's iterative algorithm[8]:

$$x_1 \in C,$$
$$x_{n+1} = s_n T_1 y_n \oplus (1 - s_n)\, x_n \tag{1.4}$$
$$y_n = t_n T_2 x_n \oplus (1 - t_n)\, x_n$$

where $\{s_n\}$ and $\{t_n\}$ are sequences in the interval $(\alpha, 1 - \alpha)$ for some $\alpha \in (0, 1)$.

When $T_2 = I$(the identity mapping), both (1.3) and (1.4) reduce to Mann's iterative algorithm (1.2).

For some properties of fixed point set of fundamentally nonexpansive mappings in Banach spaces, CAT(0) spaces and hyperbolic spaces, we refer the reader to [16, 20, 26]. The interested reader in the study of common fixed points of nonlinear mappings may consult the references [1, 7, 22].

Keeping in mind the relationship between a strictly convex metric space and a uniformly convex metric space, we obtain an analogue of Schu's Lemma[21] in strictly convex metric spaces and apply it to study two algorithms in strictly convex metric spaces. The algorithm (1.3) approximates common fixed point of two fundamentally nonexpansive mappings in a strictly convex hyperbolic space which cannot be acheived through algorithm (1.4). However, algorithm (1.4) is useful to approximate the common fixed point of a nonexpansive mapping and a fundamentally nonexpansive mapping in the same setting of a strictly convex hyperbolic space.

For our main section, we shall need the following lemmas.

Lemma 1.1. [25] *A fundamentally nonexpansive mapping T on a subset C of a metric space X, is quasi-nonexpansive and*

$$d(x, Ty) \leq 3d(Tx, x) + d(x, y)$$

for all $x, y \in C$.

Lemma 1.2. [11] *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a hyperbolic space X converging, respectively, to x and y, and $\{s_n\}$ a sequence in $[0, 1]$ converging to s.Then $s_n x_n \oplus (1 - s_n)\, y_n$ converges to $sx \oplus (1 - s)\, y$.*

The following is an analogue of Lemma 2.2 [24] whose proof carries over in the setting of a hyperbolic space without any change.

Lemma 1.3. *Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in a hyperbolic space X and $\{t_n\}$ a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$. If $u_{n+1} = t_n v_n \oplus (1 - t_n)\, u_n$ for all $n \geq 1$ and $\limsup_{n\to\infty} [d(v_{n+1}, v_n) - d(u_{n+1}, u_n)] \leq 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

2. Fixed point theorems

We start with the following fixed point theorem.

Theorem 2.1. *Let C be a nonempty, compact and convex subset of a hyperbolic space X. If $T : C \to C$ is a fundamentally nonexpansive mapping, then T has a fixed point.*

Proof. Define a sequence $\{Tu_n\}$ in C by:

$$u_1 \in C, Tu_{n+1} = \frac{1}{2}T^2 u_n \oplus \frac{1}{2}Tu_n.$$

Since T is fundamentally nonexpansive, therefore

$$d\left(T^2u_{n+1}, T^2u_n\right) \leq d\left(Tu_{n+1}, Tu_n\right) \text{ for all } n \geq 1.$$

With the help of Lemma 1.3, we see that

$$\lim_{n\to\infty} d\left(T^2u_n, Tu_n\right) = 0. \tag{2.5}$$

Since C is compact, there is a subsequence $\{Tu_{n_i}\}$ of $\{Tu_n\}$ such that $Tu_{n_i} \to u \in C$.

The inequality

$$d\left(T^2u_{n_i}, u\right) \leq d\left(T^2u_{n_i}, Tu_{n_i}\right) + d\left(Tu_{n_i}, u\right)$$

together with (2.5) implies that

$$\lim_{i\to\infty} d\left(T^2u_{n_i}, u\right) = 0. \tag{2.6}$$

Finally the inequality

$$\begin{aligned} d\left(Tu, u\right) &\leq d\left(Tu, T^2u_{n_i}\right) + d\left(T^2u_{n_i}, Tu_{n_i}\right) + d\left(Tu_{n_i}, u\right) \\ &\leq d\left(T^2u_{n_i}, Tu_{n_i}\right) + 2d\left(Tu_{n_i}, u\right), \end{aligned}$$

together with (2.5)-(2.6) provides that $Tu = u$. □

Since every nonexpansive mapping is fundamentally nonexpansive, we have the following result as a consequence of Theorem 2.1.

Corollary 2.1. *Let C be a nonempty, compact and convex subset of a hyperbolic space X. If $T : C \to C$ is a nonexpansive mapping, then T has a fixed point.*

In case, Lemma 1.3 is unknown to us, we employ Banach contraction principle (BCP) to obtain the following fixed point theorem.

Theorem 2.2. *Let C be a nonempty, compact and convex subset of a complete hyperbolic space X. If $T : C \to C$ is a nonexpansive mapping, then T has a fixed point.*

Proof. Set $T_r x = \frac{1}{r}x_0 \oplus \left(1 - \frac{1}{r}\right)Tx$ for a fixed $x_0 \in C$ and $r \geq 1$. Then

$$\begin{aligned} d\left(T_r x, T_r y\right) &= d\left(\frac{1}{r}x_0 \oplus \left(1 - \frac{1}{r}\right)Tx, \frac{1}{r}x_0 \oplus \left(1 - \frac{1}{r}\right)Ty\right) \\ &\leq \left(1 - \frac{1}{r}\right) d\left(Tx, Ty\right) \\ &\leq \left(1 - \frac{1}{r}\right) d\left(x, y\right). \end{aligned}$$

This gives that T_r is a contraction for each $r \geq 1$. Therefore by (BCP), each T_r has a unique fixed point z_r in C. That is, $T_r z_r = z_r$. Since C is compact, there exists a subsequence $\{z_{r_i}\}$ of $\{z_r\}$ such that $z_{r_i} \to z$.

Since C is bounded and

$$\begin{aligned} d\left(z_r, Tz_r\right) &= d\left(\frac{1}{r}x_0 \oplus \left(1 - \frac{1}{r}\right)Tz_r, Tz_r\right) \\ &\leq \frac{1}{r}d\left(x_0, Tz_r\right) \to 0, \end{aligned}$$

therefore

$$d(z_r, Tz_r) \to 0.$$

Finally we have

$$\begin{aligned} d(z, Tz) &\leq d(z, z_{r_i}) + d(Tz_{r_i}, Tz) \\ &\leq d(z, z_{r_i}) + d(z_{r_i}, z) \\ &\leq 2d(z, z_{r_i}) \to 0, \end{aligned}$$

providing that $Tz = z$. □

The following theorem is an interesting generalization of BCP obtained by Prešić [17]:

Theorem 2.3. *Let X be a complete metric space, n a positive integer, $s_1, s_2, s_3, ..., s_n \in \mathbb{R}_+$, $\sum_{i=1}^n s_i = s < 1$ and $f : X^n \to X$ a mapping satisfying*

$$d(f(x_0, x_1, ..., x_{n-1}), f(x_1, x_2, ..., x_n)) \leq \sum_{i=1}^n s_{i+1} d(x_i, x_{i+1})$$

for all $x_0, x_1, ..., x_n \in X$. Then f has a unique fixed point y, that is, there exists a unique $y \in X$ such that $f(y, y, ..., y) = y$ and the sequence defined by

$$x_{r+1} = f(x_{r-n+1}, ..., x_r), \quad r = n-1, n, n+1, ...$$

converges to y for any $x_0, x_1, ..., x_{n-1} \in X$.

We note that Theorem 2.3 becomes historical BCP when $n = 1$.

Some generalizations of Theorem 2.3 has been obtained in [2, 19].

Let X be a metric space, n a positive integer, $s_1, s_2, s_3, ..., s_k \in \mathbb{R}_+, \sum_{i=1}^n s_i = s \leq 1$. A mapping $f : X^n \to X$ satisfying

$$d(f(x_0, x_1, ..., x_{n-1}), f(x_1, x_2, ..., x_n)) \leq \sum_{i=1}^n s_{i+1} d(x_i, x_{i+1})$$

for all $x_0, x_1, ..., x_n \in X$, is called a Prešić nonexpansive mapping [4].

Here we state and prove a fixed point theorem for Prešić nonexpansive mappings on the product of hyperbolic spaces.

Theorem 2.4. *Let C be a nonempty, compact and convex subset of a hyperbolic space X, n a positive integer, and let $f : X^n \to X$ a Prešić nonexpansive mapping. Then f has a fixed point, that is, there exists $y \in X$ such that $f(y, y, ..., y) = y$.*

Proof. Define $T : C \to C$ by

$$T(x) = f(x, x, ..., x), \ x \in C.$$

For any $w, z \in C$, we have

$$\begin{aligned} d(T(w), T(z)) &= d(f(w, w, ..., w), f(z, z, ..., z)) \\ &\leq d(f(w, w, ..., w), f(w, ..., w, z)) \\ &\quad + d(f(w, ..., w, z), f(w, ..., w, z, z)) \\ &\quad + ... + d(f(w, z, ..., z), f(z, z, ..., z)) \\ &\leq s_n d(w, z) + s_{n-1} d(w, z) + ... + s_1 d(w, z) \\ &= \sum_{i=1}^n s_i d(w, z) \\ &= s d(w, z) \\ &\leq d(w, z). \end{aligned}$$

This shows that T is nonexpansive. By Theorem 2.1, there exists $y \in C$ such that $T(y) = f(y, y, ..., y) = y$. □

3. Convergence theorems

Khan et al. [10] established an analogue of Lemma 1.3 of Schu [21] in uniformly convex hyperbolic spaces which is of crucial importance in the fixed point approximation of certain nonlinear mappings. Below, we obtain a version of Lemma 1.3 [21] in a strictly convex hyperbolic space.

Lemma 3.4. *Let C be a nonempty, compact and convex subset of a strictly convex hyperbolic space X. Let $q \in C$ and $\{a_n\}$ be a sequence in $[\alpha, \beta]$ for some $\alpha, \beta \in (0,1)$. If $\{u_n\}$ and $\{v_n\}$ are sequences in C such that $\limsup_{n\longrightarrow\infty} d(u_n, q) \leq c$, $\limsup_{n\longrightarrow\infty} d(v_n, q) \leq c$ and $\lim_{n\longrightarrow\infty} d(a_n u_n \oplus (1-a_n) v_n, q) = c$ for some $c \geq 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

Proof. Suppose on the contrary that $\limsup_{n\longrightarrow\infty} d(u_n, v_n) \neq 0$. Since C and $[\alpha, \beta]$ are compact, there exists subsequences $\{u_{n_k}\}$ of $\{u_n\}$, $\{v_{n_k}\}$ of $\{v_n\}$ and $\{a_{n_k}\}$ of $\{a_n\}$ such that $u_{n_k} \to u \in C, v_{n_k} \to v \in C$ with $u \neq v$ and $a_{n_k} \to a \in [\alpha, \beta]$.

With the help of Lemma 1.2, we have that

$$\begin{aligned} c &= \lim_{n\longrightarrow\infty} d(a_n u_n \oplus (1-a_n) v_n, q) \\ &= \lim_{k\longrightarrow\infty} d(a_{n_k} u_{n_k} \oplus (1-a_{n_k}) v_{n_k}, q) \\ &= d(au \oplus (1-a) v, q). \end{aligned}$$

Moreover,

$$d(u, q) = \limsup_{k\longrightarrow\infty} d(u_{n_k}, q) = \limsup_{n\longrightarrow\infty} d(u_n, q) \leq c$$

and

$$d(v, q) = \limsup_{k\longrightarrow\infty} d(v_{n_k}, q) = \limsup_{n\longrightarrow\infty} d(v_n, q) \leq c.$$

If

$$d(u, q) < c,$$

then

$$\begin{aligned} c &= d(au \oplus (1-a) v, q) \\ &\leq ad(u, q) + (1-a) d(v, q) < c, \end{aligned}$$

a contradiction. Therefore $d(u, q) = c$. Similarly $d(v, q) = c$.

That is,

$$d(u, q) = d(v, q) = d(au \oplus (1-a) v, q) = c.$$

By the strict convexity of X, we have $u = v$, a contradiction. Hence, $\lim_{n\to\infty} d(u_n, v_n) = 0$. □

Now we state and prove our convergence theorems.

Theorem 3.5. *Let C be a nonempty compact and convex subset of a strictly convex hyperbolic space X. Let $T_1, T_2 : C \to C$ be fundamentally nonexpansive mappings such that $F(T_1) \cap F(T_2) \neq \phi$, then the iterative algorithm $\{x_n\}$ defined in (1.3), converges strongly to a common fixed point of T_1 and T_2.*

Proof. Let $x \in F(T_1) \cap F(T_2)$. Then

$$\begin{aligned} d(x_{n+1}, x) &= d(s_n T_1 x_n \oplus (1-s_n) y_n, x) \\ &\le s_n d(T_1 x_n, x) + (1-s_n) d\left(\frac{t_n}{1-s_n} T_2 x_n \oplus \left(1 - \frac{t_n}{1-s_n}\right) x_n, x\right) \\ &\le s_n d(x_n, x) + t_n d(T_2 x_n, x) + (1-s_n-t_n) d(x_n, x) \\ &\le s_n d(x_n, x) + t_n d(x_n, x) + (1-s_n-t_n) d(x_n, x) \\ &= d(x_n, x). \end{aligned}$$

That is, $\lim_{n\to\infty} d(x_n, x)$ exists. Assume that $\lim_{n\to\infty} d(x_n, x) = c$. If $c = 0$, the theorem is finished. Suppose that $c > 0$. Since $\limsup_{n\longrightarrow\infty} d(T_1 x_n, x) \le c$, $\limsup_{n\longrightarrow\infty} d(y_n, x) \le c$ and $\lim_{n\to\infty} d(s_n T_1 x_n \oplus (1-s_n) y_n, x) = c$, therefore by Lemma 3.4 (with $u_n = T_1 x_n, v_n = x_n, q = x, a_n = s_n$), we get that

$$\lim_{n\to\infty} d(T_1 x_n, y_n) = 0. \tag{3.7}$$

Now the estimate

$$\begin{aligned} d(x_{n+1}, T_1 x_n) &= d(s_n T_1 x_n \oplus (1-s_n) y_n, T_1 x_n) \\ &\le (1-s_n) d(y_n, T_1 x_n) \\ &\le (1-\alpha) d(y_n, T_1 x_n) \end{aligned}$$

together with (3.7) implies that

$$\lim_{n\to\infty} d(x_{n+1}, T_1 x_n) = 0. \tag{3.8}$$

By the triangle inequality, we have

$$d(x_{n+1}, x) \le d(x_{n+1}, T_1 x_n) + d(T_1 x_n, y_n) + d(y_n, x).$$

Taking $\liminf$ on both sides in the above inequality, we have

$$c \le \liminf_{n\to\infty} d(y_n, x) \le \limsup_{n\to\infty} d(y_n, x) \le c.$$

That is,

$$\lim_{n\to\infty} d\left(\frac{t_n}{1-s_n} T_2 x_n \oplus \left(1 - \frac{t_n}{1-s_n}\right) x_n, x\right) = c.$$

Again by Lemma 3.4 (with $u_n = T_2 x_n, v_n = x_n, q = x, a_n = \frac{t_n}{1-s_n}$), we have

$$\lim_{n\to\infty} d(x_n, T_2 x_n) = 0. \tag{3.9}$$

Further note that

$$\begin{aligned} d(x_{n+1}, x_n) &\le d(x_{n+1}, T_1 x_n) + d(T_1 x_n, y_n) + d(y_n, x_n) \\ &\le d(x_{n+1}, T_1 x_n) + d(T_1 x_n, y_n) + \frac{t_n}{1-s_n} d(T_2 x_n, x_n) \\ &< d(x_{n+1}, T_1 x_n) + d(T_1 x_n, y_n) + d(T_2 x_n, x_n). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.10}$$

As a direct use of (3.8),(3.10) in the inequality

$$d(x_n, T_1 x_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, T_1 x_n),$$

we get that

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = 0. \tag{3.11}$$

Since C is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to w \in C$. Next we show that w is a common fixed point of T_1 and T_2.

Choosing $x = x_{n_i}, y = w, T = T_j\ (j = 1, 2)$ in Lemma 1.1 and applying (3.9) and (3.11), we have that

$$d(x_{n_i}, T_j w) \leq 3d(T_j x_{n_i}, x_{n_i}) + d(x_{n_i}, w) \to 0.$$

That is, $x_{n_i} \to T_j w$. Therefore $T_1 w = w = T_2 w$. Since $\lim_{n\to\infty} d(x_n, w)$ exists and $x_{n_i} \to w$, therefore $x_n \to w$. □

Theorem 3.6. *Let C be a nonempty compact and convex subset of a strictly convex hyperbolic space X. Let $T_1, T_2 : C \to C$ be nonexpansive and fundamentally nonexpansive mappings, respectively, such that $F(T_1) \cap F(T_2) \neq \phi$, then the Ishikawa's iterative algorithm $\{x_n\}$ defined in (1.4) converges strongly to a common fixed point of T_1 and T_2.*

Proof. Let $x \in F(T_1) \cap F(T_2)$. Then

$$\begin{aligned} d(x_{n+1}, x) &= d(s_n T_1 y_n \oplus (1 - s_n) x_n, x) \\ &\leq s_n d(T_1 y_n, x) + (1 - s_n) d(x_n, x) \\ &\leq s_n d(y_n, x) + (1 - s_n) d(x_n, x) \\ &= s_n d(t_n T_2 x_n \oplus (1 - t_n) x_n, x) + (1 - s_n) d(x_n, x) \\ &\leq s_n [t_n d(T_2 x_n, x) + (1 - t_n) d(x_n, x)] \\ &\quad + (1 - s_n) d(x_n, x). \\ &\leq s_n [t_n d(x_n, x) + (1 - t_n) d(x_n, x)] \\ &\quad + (1 - s_n) d(x_n, x) \\ &= d(x_n, x). \end{aligned}$$

Hence $\lim_{n\to\infty} d(x_n, p)$ exists. Assume that $\lim_{n\to\infty} d(x_n, x) = c$. If $c = 0$, we have done. Suppose that $c > 0$. Since $\limsup_{n\longrightarrow\infty} d(T_1 y_n, x) \leq c$ and $\lim_{n\to\infty} d(s_n T_1 y_n \oplus (1 - s_n) x_n, x) = c$, therefore by Lemma 3.4 (with $u_n = T_1 y_n, v_n = x_n, q = x, a_n = s_n$), we get that

$$\lim_{n\to\infty} d(x_n, T_1 y_n) = 0. \tag{3.12}$$

Next

$$d(x_n, x) \leq d(x_n, T_1 y_n) + d(T_1 y_n, x) \leq d(x_n, T_1 y_n) + d(y_n, x)$$

gives that

$$c \leq \liminf_{n\to\infty} d(y_n, x) \leq \limsup_{n\to\infty} d(y_n, x) \leq c.$$

That is,

$$\lim_{n\to\infty} d(y_n, x) = c$$

which is expressible as

$$\lim_{n\to\infty} d(t_n T_2 x_n \oplus (1 - t_n) x_n, x) = c.$$

Moreover,

$$\limsup_{n\to\infty} d(x_n, x) = c \text{ and } \limsup_{n\to\infty} d(T_2 x_n, x) \leq c.$$

So again by Lemma 3.4 (with $u_n = T_2 x_n, v_n = x_n, q = x, a_n = t_n$), we have

$$\lim_{n\to\infty} d(x_n, T_2 x_n) = 0. \tag{3.13}$$

Now observe that

$$\begin{aligned} d(x_n, T_1 x_n) &\leq d(x_n, T_1 y_n) + d(T_1 y_n, T_1 x_n) \\ &\leq d(x_n, T_1 y_n) + d(y_n, x_n) \\ &= d(x_n, T_1 y_n) + d(t_n T_2 x_n \oplus (1 - t_n) x_n, x_n) \\ &\leq d(x_n, T_1 y_n) + t_n d(T_2 x_n, x_n) \\ &\leq d(x_n, T_1 y_n) + (1 - \alpha) d(T_2 x_n, x_n). \end{aligned}$$

Taking $\limsup_{n\to\infty}$ on both sides in the above inequality and using (3.12)-(3.13), we achieve that

$$\limsup_{n\to\infty} d(x_n, T_1 x_n) = 0.$$

That is,

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = 0. \tag{3.14}$$

By the compactness of C, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to w \in C$. Since T_1 is nonexpansive and (3.14) holds, therefore

$$\begin{aligned} d(T_1 w, w) &\leq d(T_1 w, T_1 x_{n_i}) + d(T_1 x_{n_i}, x_{n_i}) + d(x_{n_i}, w) \\ &\leq 2d(x_{n_i}, w) + d(T_1 x_{n_i}, x_{n_i}) \to 0. \end{aligned}$$

That is, $T_1 w = w$. To get $T_2 w = w$, we use Lemma 1.1 with $x = x_{n_i}, y = w, T = T_2$. As $\lim_{n\to\infty} d(x_n, w)$ exists and $x_{n_i} \to w$, therefore $x_n \to w$. □

As a direct consequence of Theorem 3.6, we have the following result.

Theorem 3.7. *Let C be a nonempty compact and convex subset of a strictly convex hyperbolic space X. Let $f : C^n \to C$ be a Prešić nonexpansive mapping and $T : C \to C$ a fundamentally nonexpansive mapping such that $F(f) \cap F(T) \neq \phi$. Define $\{x_n\}$ as under:*

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= s_n f(y_n, y_n, ..., y_n) \oplus (1 - s_n) x_n \\ y_n &= t_n T x_n \oplus (1 - t_n) x_n \end{aligned}$$

where $s_n, t_n \in [\alpha, \beta]$ for some $\alpha, \beta \in (0, 1)$. Then $\{x_n\}$ converges strongly to a common fixed point of f and T.

Remark 3.1. We observe that:
(1) Our results are new in strictly convex hyperbolic spaces which also include uniformly convex hyperbolic spaces, $CAT(0)$ spaces, Hilbert spaces and uniformly(strictly) convex Banach spaces, simultaneously.
(2) The iterative algorithm(1.3) is better than the iterative algorithm(1.4) in the sense that (1.3) approximates a common fixed point of two discontinuous mappings, namely, fundamentally nonexpansive mappings while (1.4) requires one mapping to be nonexpansive which is continuous.
(3) Lemma 3.4 sets an analogue of Lemma 2.5 [10], Theorem 3.5 generalizes Theorem 2.5[3] for two fundamentally nonexpansive mappings and Theorem 3.6 improves Theorem 2[9] and Theorems 4.2-4.3 [27] in the setting of strictly convex hyperbolic spaces.

Remark 3.2. The essentials of hypotheses in our Theorems 3.5-3.6 are natural in view of mappings R, S, T in Example 1.1 with $s_n = 2^{-n}, t_n = 3^{-n}$.

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REFERENCES

[1] Ceng, L. C., *Approximation of common solutions of a split inclusion problem and a fixed-point problem*, J. Appl. Numer. Optim., **1** (2019), 1–12

[2] Cirić, L. B. and Prešić, S. B., *On Prešić type generalization of the Banach contraction mapping principle*, Acta Math. Univ. Comenian. (N.S.), **76** (2007), No. 2, 143–147

[3] Fukhar-ud-din, H., *One step iterative scheme for a pair of nonexpansive mappings in a convex metric space*, Hacet. J. Math. Stat., **44** (2015), 1023 – 1031

[4] Fukhar-ud-din, H., Berinde, V. and Khan, A. R., *Fixed point approximation of Prešić nonexpansive mappings in product of CAT(0) spaces*, Carpathian J. Math., **32** (2016), No. 3, 315–322

[5] Fukhar-ud-din, H. and Khamsi, M. A., *Approximating common fixed points in hyperbolic spaces*, Fixed Point Theory Appl., 2014, 2014:113, 15 pp.

[6] Ghoncheh, S. J. H. and Razani, A., *Fixed point theorems for some generalized nonexpansive mappings in Ptolemy spaces*, Fixed Point Theory Appl., 2014, 2014:76

[7] Gunduz, B., Khan, S. H. and Akbulut, S., *Common fixed points of two finite families of nonexpansive mappings in Kohlenbach hyperbolic spaces*, J. Nonlinear Funct. Anal., 2015 (2015), Article ID 15

[8] Ishikawa, S., *Fixed point by a new iteration method*, Proc. Amer. Math. Soc., **44** (1974), 147–150

[9] Khan, S. H. and Fukhar-ud-din, H., *Weak and strong convergence of a scheme with errors for two nonexpansive mappings*, Nonlinear Anal., **61** (2005), 1295–1301

[10] Khan, A. R., Fukhar-ud-din, H. and Khan, M. A. A., *An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces*, Fixed Point Theory Appl., 2012, 2012:54

[11] Khan, A. R., Fukhar-ud-din, H. and Shukri, S. A., *Implicit Ishikawa type algorithms in hyperbolic spaces*, Palest. J. Math., **6** (2017), No. 1, 101–110

[12] Khan, A. R., Khamsi, M. A. and Fukhar-ud-din, H., *Strong convergence of a general iteration scheme in CAT(0) spaces*, Nonlinear Anal., **74** (2011), 783–791

[13] Kohlenbach, U., *Some logical metatheorems with applications in functional analysis*, Trans. Am. Math. Soc., **357** (2005), No. 1, 89–128

[14] Mann, W. R., *Mean value methods in iterations*, Proc. Amer. Math. Soc., **4** (1953), 506–510

[15] Menger, K., Untersuchungen über allgemeine Metrik, Math. Ann., **100** (1928), 75–163

[16] Moosaei, M., *On fixed points of fundamentally nonexpansive mappings in Banach spaces*, Int. J. Nonlinear Anal. Appl., **7** (2016), 219–224

[17] Prešić, S. B., *Sur une classe d' inéquations aux différences finites et sur la convergence de certaines suites*, Publ. Inst. Math. (Beograd) (N.S.), **5** (1965), No. 19, 75–78

[18] Reich, S. and Shafrir, I., *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods, **15** (1990), 537–558

[19] Rus, I., *An iterative method for the solution of the equation* $x = f(x, x, ..., x)$, Rev. Anal. Numer. Theor. Approx., **10** (1981), No. 1, 95–100

[20] Salahifard, H., Vaezpour, S. M. and Dhompongsa, S., *Fixed point theorems for some generalized nonexpansive mappings in CAT(0) spaces*, J. Nonlinear Anal. Optim., **4** (2013), 241–248

[21] Schu, J., *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc., **43** (1991), 153–159

[22] Takahashi, W., *Weak and strong convergence theorems for families of nonlinear and nonself mappings in Hilbert spaces*, J. Nonlinear Var. Anal., **1** (2017), 1–23

[23] Shimizu, T. and Takahashi, W., *Fixed points of multivalued mappings in certain convex metric spaces*, Topol. Methods Nonlinear Anal., **8** (1996), 197–203

[24] Suzuki, T., *Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces* , Fixed Point Theory Appl., 2005 (2005), Article ID 685918

[25] Suzuki, T., *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **341** (2008), 1088–1095

[26] Suanoom, C. and Klin-eam, C., *Remark on fundamentally nonexpansive mappings in hyperbolic spaces*, J. Nonlinear Sci. Appl., **9** (2016), 1952–1956

[27] Takahashi, W. and Tamura, T., *Convergence theorems for a pair of nonexpansive mappings*, J. Convex Anal., **5** (1998), 45–56

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