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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

Caristi type fixed point theorems using Száz principle in quasi-metric spaces

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ABSTRACT. In this paper, we use Száz maximum principle to prove generalizations of Caristi fixed point theorem in a preordered K-complete quasi metric space. Examples are given to support our results.

1. INTRODUCTION

The Brézis-Browder principle (see [3]) is one of the most useful tools in nonlinear analysis, it covers most of the maximality principles and leads to many existence results of fixed points obtained as maximal elements [3, 8].

Many authors gave extensions of this principle and obtained some generalized versions of Caristi's results. In 1982, M. Altman [2] generalized this principle in ordered sets and in 1984, M. Turinici [13] gave a maximality principle on quasi-ordered quasi-metric spaces which generalizes Altman's results.

In 2007, Á. Száz (see [12]) generalized the Brézis-Browder principle in abstract context and gave an abstract generalized Caristi fixed point theorem.

In this paper we use the Száz principle in a preorederd quasi-metric space in order to give more general versions of Caristi's fixed point theorem. For that, we introduce new classes of functions called K-functions, M-functions and A-functions, respectively, which generalize the notion of dominated function in Caristi's theorem.

Recall that Caristi's fixed point theorem states that any self map T of X has a fixed point, provided that (X, d) is complete and there exists a lower semicontinuous map φ from X in to the nonnegative real numbers such that for all x in X

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx).$$

Moreover, we give an improved fixed point result for a set valued mapping and we deduce some well known results as corollaries. We also study the problem of common fixed point for two set valued mappings. Throughout this paper we find many illustrative examples.

2. PRELIMINARIES

Let X be a nonempty set, a binary relation $\preceq$ on X is a preorder if it is reflexive and transitive in the sense that for all $x, y, z \in X$, one has $x \preceq x$ (reflexivity) and if $x \preceq y$ and $y \preceq z$ on has $x \preceq z$ (transitivity).

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A set X equipped with a preorder is called a preordered set. Moreover, x is called maximal element in X if $x \in X$ and for each $y \in X$, on has $x \preceq y$ implies $x = y$. If $(X, \preceq)$ is a preordered set, we define $S^+_\preceq(x) = \{z \in X \,;\, x \preceq z\}$.

Definition 2.1. [14] Let X be a nonempty set, a function $d : X \times X \to [0, \infty)$ is a quasi-distance if for each $x, y, z \in X$ we have :

$(i)\, d(x, y) = 0$ if and only if $x = y$; $(ii)\, d(x, z) \le d(x, y) + d(y, z)$ for each $x, y, z \in X$.

The pair (X, d) is called a quasi metric space.

Definition 2.2 ([5]). (i) A sequence $\{x_n\}_n$ in (X, d) is left K-Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, with $n_0 \le n \le m$, $d(x_n, x_m) < \varepsilon$;

(ii) (X, d) will be called left K-complete quasi metric if any left K-Cauchy sequence is left K-convergent to some $x \in X$ in the sense that $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 2.3. Let (X, d) be a left K-quasi metric space. A mapping $\varphi : X \to \mathbb{R}_+$ is said to be :

(i) lower semicontinuous at $\overline{x}$ if given any sequence $\{x_n\}_n$ in X, whenever $x_n \overset{d}{\to} \overline{x}$ and $\varphi(x_n) \to r$, then $\varphi(\overline{x}) \le r$.

(ii) lower semicontinuous if it is lower semicontinuous at every $x \in X$.

(iii) Upper semicontinuous if $-\varphi$ is lower semicontinuous.

Definition 2.4. [13]Let $(X, \preceq)$ be a preordered set.

(i) The preorder $\preceq$ will be called self-closed when the limit of any increasing left K-convergent sequence is an upper bound of the values set $\{x_n\}$.

(ii) The triplet $(X, d, \preceq)$ which $\preceq$ is self-closed, will be called left $\preceq$-K-complete quasi metric space if any increasing left K-Cauchy sequence is left K-convergent.

We conclude this section by recalling the Száz maximum principle.

Theorem 2.1. [12] *Let $(X, \preceq)$ be a preordered set and let $\Phi : X \times X \to \overline{\mathbb{R}}$ be a function satisfying the following statements:*

$(S1)$ $x \mapsto \sup_{y\in S^+_\preceq(x)} \Phi(x, y)$ *is decreasing;* $(S2)$ $-\infty < \sup_{y\in S^+_\preceq(x)} \Phi(x, y)$ *for all* $x \in X$;

$(S3)$ $\sup_{y\in S^+_\preceq(a)} \Phi(a, y) < \infty$ *for some* $a \in X$;

$(S4)$ *For every nondecreasing sequence* $\{x_n\}_n \subset X$ *with* $x_0 = a$, *there exists some* $x \in X$ *such that* $x_n \preceq x$ *for all* $n \in \mathbb{N}$ *and* $\liminf_{n\to\infty} \Phi(x_n, x_{n+1}) = 0$;

$(S5)$ $0 < \Phi(x, y)$ *for all* $x, y \in X$ *with* $x \prec y$. *Then, there exists a maximal element* $\hat{x} \in X$.

3. RESULTS

Theorem 3.2. *Let $(X, d, \preceq)$ be a left $\preceq$-K-complete quasi metric space and $T : X \to 2^X$ a set valued map. If there exists a function $\Phi : X \times X \to \overline{\mathbb{R}}$ satistifying (S1), (S3), (S5) and*

$(S'4)$ for every nondecreasing sequence $\{x_n\}_n \subset X$ with $x_0 = a$, there exists some $x \in X$ such that $x_n \preceq x$ for all $n \in \mathbb{N}$ and $\sum_{n\ge0} \Phi(x_n, x_{n+1})$ is convergent;

(S6) for each $x \in X$ there exists $y \in Tx \cap S^+_\preceq(x)$ satisfying:

$$d(x, y) \le \Phi(x, y). \tag{3.1}$$

Then T has a fixed point in X.

Proof. It is obvious that $(S'4)$ implies $(S4)$. For each $x \in X$ there exists $y \in S^+_\preceq(x)$ such that

$$0 \le d(x, y) \le \Phi(x, y).$$

Hence,

$$-\infty < \Phi(x,y) \le \sup_{z\in S^{+}_{\preceq}(x)} \Phi(x,z)$$

which gives $(S2)$.

Theorem 2.1 implies that there exists a maximal element $\overline{x}$ in X. By $(S6)$, there exists $\overline{y} \in X$ such that

$$\overline{y} \in T\overline{x} \quad \text{and} \quad \overline{y} \in S^{+}_{\preceq}(\overline{x})$$

the maximality of $\overline{x}$ implies $\overline{x} \in T\overline{x}$. □

Let $(X, \preceq)$ be a preorder set and $\Phi : X \times X \to \overline{\mathbb{R}}$ a function such that :

(C1) N-superadditivity : $\Phi(x,x) = 0$ and $\Phi(x,y) + \Phi(y,z) \le \Phi(x,z)$ for each $x, y, z \in X$ with $x \preceq y$ and $y \preceq z$.

(C2) For each $x \in X$, $y \mapsto \Phi(x,y)$ is upper semicontinuous.

(C3) There exists $a \in X$ such that $\sup_{y\in S^{+}_{\preceq}(a)} \Phi(a,y) < \infty$.

(C4) There exists a function $\psi : X \to \mathbb{R}$ such that $\Phi(x,y) \le \psi(x)$ where $x \preceq y$ for each $x, y \in X$.

Definition 3.5. Let $(X, \preceq)$ be a preordered set, a real function $\Phi : X \times X \to \overline{\mathbb{R}}$ is said to be:

(i) K-function if (C1), (C2) and (C3) hold.

(ii) M-function if (C1), (C2) and (C4) hold.

Remark 3.1. Each M-function is a K-function. Indeed, let $a \in X$ then for each $y \in S^{+}_{\preceq}(a)$ we have $\Phi(a,y) \le \psi(a)$ which implies

$$\sup\nolimits_{y\in S^{+}_{\preceq}(a)} \Phi(a,y) \le \psi(a) < \infty.$$

Example 3.1. (1) Let (X,d) be a left K-quasi metric space and $\varphi : X \to \mathbb{R}_{+}$ a lower semicontinuous function. We define a preorder on X as follows

$$x \preceq y \Leftrightarrow d(x,y) \le \varphi(x) - \varphi(y)$$

then the function $\Phi : X \times X \to \overline{\mathbb{R}}$ defined by

$$\Phi(x,y) = \varphi(x) - \varphi(y)$$

satisfies (C1), (C2) and (C3). Indeed, since φ is lower semicontinuous $-\varphi$ is upper semicontinuous, then $y \mapsto \Phi(x,y)$ is upper semicontinuous and $\Phi(a,y) \le \varphi(a)$ for each $a \preceq y$ in X. That is

$$\sup_{y\in S^{+}_{\preceq}(a)} \Phi(a,y) < \infty.$$

(2) Let $X = \mathbb{R}$ and define the preorder $\preceq$ by

$$x \preceq y \iff y \le x.$$

Let $\varphi : X \to X$ be a continuous real valued function which is bounded above by some $M > 0$. Define Φ by

$$\Phi(x,y) = \int_{y}^{x} \varphi(t)\,dt$$

then $y \mapsto \Phi(x,y)$ is continuous, and $\Phi(x,y) \le (x-y)M \le xM = \psi(x)$ for each $x, y \in X$ with $x \preceq y$, that is Φ satisfies (C1), (C2) and (C4).

(3) Let $X = [1,2]$ endowed by the usual order and define Φ by

$$\Phi(x,y) = \ln^{2}\left(\frac{y}{x}\right)$$

for all $x, y \in X$. It is clear that (C2) and (C3) hold. For (C1) we have

$$\Phi(x,x) = \ln^2\left(\frac{x}{x}\right) = \ln^2(1) = 0$$

and, for all $x \leq y \leq z$,

$$\ln^2\left(\frac{y}{x}\right) + \ln^2\left(\frac{z}{y}\right) \leq \left(\ln\left(\frac{y}{x}\right) + \ln\left(\frac{z}{y}\right)\right)^2 \leq \ln^2\left(\frac{z}{x}\right).$$

Then Φ is a K-funcion.

Let $\Phi : X \times X \to \overline{\mathbb{R}}$. Define a binary relation on $(X, \preceq)$ by

$$x \preceq_\Phi y \iff \begin{cases} d(x,y) \leq \Phi(x,y) \\ x \preceq y. \end{cases}$$

Lemma 3.1. *Let $(X, d, \preceq)$ be a preordered left K-quasi metric space and $\Phi : X \times X \to \overline{\mathbb{R}}$ a function satsifying condition (C1). Then $(X, d, \preceq_\Phi)$ is a preorder left K-quasi metric space.*

The following result is an improvement of the Caristi fixed point theorem in the setting of preordered left K-quasi metric space.

Theorem 3.3. *Let $(X, d, \preceq)$ be a left $\preceq$-K-complete quasi metric space, $T : X \to 2^X$ a set valued map. Assume that there exists a K-function $\Phi : X \times X \to \overline{\mathbb{R}}$ such that for each $x \in X$, one can find $y \in Tx$ with $d(x,y) \leq \Phi(x,y)$. Then, T has a fixed point in X.*

Proof. Consider an increasing sequence $\{x_n\}$ with respect to $\preceq_\Phi$ with $x_0 = a$ and prove that there exists $\overline{x} \in X$ such that $x_n \preceq_\Phi x$ for all $n \in \mathbb{N}$.

Since $x_n \preceq_\Phi x_{n+1}$, we have $d(x_n, x_{n+1}) \leq \Phi(x_n, x_{n+1})$ and then

$$\sum_{i=0}^{n} d(x_i, x_{i+1}) \leq \sum_{i=0}^{n} \Phi(x_i, x_{i+1}) \leq \Phi(a, x_{n+1}) \leq \sup_{y \in S^+_{\preceq_\Phi}(a)} \Phi(a, y)$$

it follows that $\sum_{i=0}^{n} \Phi(x_i, x_{i+1})$ is a convergent sequence and then

$$\lim_{n\to\infty} \Phi(x_n, x_{n+1}) = 0.$$

Thus $\{x_n\}_n$ is a left K-Cauchy sequence in X and then it $\{x_n\}_n$ left K-converge to some $\overline{x}$. Since $\preceq$ is self-closed we get $x_n \preceq \overline{x}$ for all $n \in \mathbb{N}$.

Note that for each $n, m \in \mathbb{N}$ with $n \leq m$ we have

$$x_n \preceq_\Phi x_m \Rightarrow d(x_n, x_m) \leq \Phi(x_n, x_m)$$

which by left K-convergent of $\{x_n\}_n$ and upper semicontinuity of $y \mapsto \Phi(x,y)$ gives

$$\limsup_{m\to\infty} \Phi(x_n, x_m) \leq \Phi(x_n, \overline{x}).$$

Then, $d(x_n, \overline{x}) \leq \Phi(x_n, \overline{x})$ i.e. $x_n \preceq_\Phi \overline{x}$ for all $n \in \mathbb{N}$ and then (S4) holds.

Define a function $\varphi : X \to [0, \infty]$ by

$$\varphi(x) = \sup_{y \in S^+_{\preceq}(x)} \Phi(x,y)$$

for each $x \in X$. It is clear that φ is nonnegative function. Moreover, for each $x \in X$ there is $y \in Tx$ such that $d(x,y) \leq \Phi(x,y)$, then $0 \leq \varphi(x)$ which gives (S2).

If $x \prec_\Phi y$ ($x \preceq_\Phi y$ and $x \neq y$) then $0 < d(x,y) \leq \Phi(x,y)$ which implies (S5).

Let $x \preceq_\Phi y$ then for each $z \in S^+_{\preceq}(y)$ we get

$$\Phi(x,y) + \Phi(y,z) \leq \Phi(x,z)$$

and since $\Phi(x, y) \geq 0$ we have

$$\Phi(y, z) \leq \Phi(x, z)$$

then

$$\sup_{z \in S^+_{\preceq}(y)} \Phi(y, z) \leq \sup_{z \in S^+_{\preceq}(x)} \Phi(x, z)$$

i.e. $\varphi(y) \leq \varphi(x)$ which implies that φ is nonincreasing function, that is (S1) holds.

All assumptions of Száz principle hold, then X has a maximal element $\hat{x}$. By hypothesis, there exists $\hat{y} \in T\hat{x}$ such that $\hat{x} \preceq_\Phi \hat{y}$ which leads to $\hat{x} = \hat{y}$ and then $\hat{x} \in T\hat{x}$. □

Corollary 3.1. *Under assumptions of Theorem 3.3, with $T : X \to X$ a single valued map satisfying $d(x, Tx) \leq \Phi(x, Tx)$ for all $x \in X$, there exists $x \in X$ such that $Tx = x$.*

The next example gives an application of our precedent result where the function Φ is not of the Caristi's form.

Example 3.2. Let $X = [1, e^2]$ endowed by the quasi-distance d defined by

$$d(x, y) = \begin{cases} 1 & if\, x > y \\ y - x & if\, x \leq y \end{cases}$$

for each $x, y \in X$ and $\Phi : X \times X \to \overline{\mathbb{R}}$ defined by

$$\Phi(x, y) = ln^3\left(\frac{y}{x}\right) + (y - x)$$

It is clear that Φ is a K-function with respect to the usual order. Indeed, we show only (C1), since (C2) and (C3) result immediately from definition of Φ. For each $x, y, z \in X$ with $x \leq y \leq z$ we get

$$\Phi(x, y) + \Phi(y, z) = ln^3\left(\frac{y}{x}\right) + (y - x) + ln^3\left(\frac{z}{y}\right) + (z - y) \leq \left(ln\left(\frac{y}{x}\right) + ln\left(\frac{z}{y}\right)\right)^3$$

$$+(z - x) \leq ln^3\left(\frac{z}{x}\right) + (z - x) = \Phi(x, z).$$

Let T be a mapping from X to X defined by

$$Tx = e\sqrt{x}$$

if $x \in X$, that is $x \leq e^2$ then $x \leq Tx$ and

$$d(x, Tx) = e\sqrt{x} - x; \quad \Phi(x, Tx) = ln^3\left(\frac{e\sqrt{x}}{x}\right) + (e\sqrt{x} - x)$$

since $ln^3\left(\frac{e\sqrt{x}}{x}\right) \geq 0$ for all $x \in X$ we get

$$d(x, Tx) \leq \Phi(x, Tx)$$

that is $x \preceq_\Phi Tx$ for all $x \in X$. Hence, all assumptions of Theorem 3.3. hold, then T has a fixed point, namely $Te^2 = e^2$.

Since an M-function is a K-function, we obtain:

Corollary 3.2. *If X, $\preceq_\Phi$ and T as above and $\Phi : X \times X \to \overline{\mathbb{R}}$ a M-function, then T has a fixed point in X.*

For the K-function Φ and the preorder as in Example 3.1-(1) we get :

Corollary 3.3. ([4, 7, 10]) *Let (X,d) be a left K-complete quasi metric space and $\varphi : X \to [0,\infty)$ a lower semicontinuous function. If the mapping $T : X \to X$ satisfies for each $x \in X$ the condition :*

$$d(x,Tx) \le \varphi(x) - \varphi(Tx),$$

then T has a fixed point in X.

Proof. We show only that the preorder defined on Example 3.1-(1) is self closed. Suppose $\{x_n\}_n$ is an increasing sequence in $(X,\preceq)$ which left K-converge to $\overline{x}$.

Then $\{\varphi(x_n)\}_n$ is a decreasing sequence in $\mathbb{R}_+$, so there exists $r \in \mathbb{R}$ such that $\varphi(x_n) \searrow r$, since φ is lower semicontinuous we have $\varphi(\overline{x}) \le r$ and it follows that

$$d(x_n,\overline{x}) = \lim_{m\to\infty} d(x_n,x_m) \le \lim_{m\to\infty} (\varphi(x_n) - \varphi(x_m)) \le \varphi(x_n) - r \le \varphi(x_n) - \varphi(\overline{x}).$$

Therefore $\overline{x}$ is an upper bound for $\{x_n\}_n$ in $(X,\preceq)$. Which ends the proof. □

4. Caristi's theorems for two set valued mappings

In this section we obtain an extended version of Caristi's fixed point theorem for two set valued mappings in quasi-metric space (X,d).

Definition 4.6. Let $(X,\preceq)$ be a preordered set. A real valued function $\Phi : X \times X \to \overline{\mathbb{R}}$ will be called an A-function if we have (C3) and the following conditions :

($C'1$) $\Phi(x,y) = -\Phi(y,x)$ for all $x,y \in X$ and

$$\Phi(x,y) + \Phi(y,z) \le \Phi(x,z),$$

for all $x \preceq y \preceq z$.

(C5) The range of the map $\varphi : x \mapsto \sup_{y\in S^+_{\preceq}(x)} \Phi(x,y)$ is closed.

(C6) For all $x,y \in X$, we have $[\Phi(x,y) \ge 0 \Rightarrow x \preceq y]$.

We define a second preorder on $(X,\preceq)$ as follows

$$x \preceq^{\Phi} y \iff \Phi(x,y) \ge 0$$

for each $x,y \in X$.

The following is a Caristi-Type fixed point theorem, where the lower semicontinuity assumption of the dominated function is not needed.

Theorem 4.4. *Let $(X,d,\preceq)$ be a preordered left K-complete quasi metric space, $T,S : X \to 2^X$ two set valued maps. If there exists an A-function $\Phi : X \times X \to \overline{\mathbb{R}}$ such that for each $x \in X$ there exist $y \in Tx \cap S^+_{\preceq}(x)$ and $z \in Sx \cap S^+_{\preceq}(x)$ such that*

$$d(x,y) \le \Phi(x,z),$$

then there exists $\hat{x} \in X$ such that $\hat{x} \in T\hat{x} \cap S\hat{x}$.

Proof. It is obvious clear that $\left(X,d,\preceq^{\Phi}\right)$ is a preorder left K-complete quasi metric space and the function

$$\varphi : x \mapsto \sup_{z\in S^+_{\preceq^{\Phi}}(x)} \Phi(x,z)$$

is nonincreasing. Indeed, if $x \preceq^{\Phi} y$ we get by ($C'1$)

$$\Phi(y,z) \le \Phi(x,z)$$

for all $z \in S^+_{\preceq^{\Phi}}(y)$, then

$$\sup_{z\in S^+_{\preceq^{\Phi}}(y)} \Phi(y,z) \le \sup_{z\in S^+_{\preceq^{\Phi}}(y)} \Phi(x,z)$$

and since $S^{+}_{\preceq^{\Phi}}(y) \subset S^{+}_{\preceq^{\Phi}}(x)$, we have

$$\sup_{z \in S^{+}_{\preceq^{\Phi}}(y)} \Phi(y,z) \leq \sup_{z \in S^{+}_{\preceq^{\Phi}}(x)} \Phi(x,z) \iff \varphi(y) \leq \varphi(x).$$

Let $\{x_n\}$ be an increasing sequence w.r.t. $\preceq^{\Phi}$ where $x_0 = a$. Since $x_n \preceq^{\Phi} x_{n+1}$ we obtain $\Phi(x_n, x_{n+1}) \geq 0$. By (C6), we obtain $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Moreover, since $x \mapsto \varphi(x)$ is decreasing, we obtain

$$0 \leq \cdots \leq \varphi(x_{n+1}) \leq \varphi(x_n) \leq \varphi(a)$$

for each $n \in \mathbb{N}$. Then the real sequence $\{\varphi(x_n)\}_n$ is convergent in $[0,\infty]$ to some $r \geq 0$. Since $\varphi(X)$ is closed, we obtain $r \in \varphi(X)$ and then there exists $\overline{x} \in X$ such that

$$r = \varphi(\overline{x}) \leq \varphi(x_n),$$

for all $n \in \mathbb{N}$.

We claim by contradiction that $x_n \preceq^{\Phi} \overline{x}$ for all $n \in \mathbb{N}$. Suppose there exists $n_0 \in \mathbb{N}$ such that $\Phi(x_{n_0}, \overline{x}) < 0$. By condition $(C'1)$, $\Phi(\overline{x}, x_{n_0}) > 0$ and for all $z \in X$ such that $x_{n_0} \preceq^{\Phi} z$ we get

$$\Phi(\overline{x}, x_{n_0}) + \Phi(x_{n_0}, z) \leq \Phi(\overline{x}, z)$$

hence

$$\Phi(\overline{x}, x_{n_0}) + \sup_{z \in S^{+}_{\preceq^{\Phi}}(x_{n_0})} \Phi(x_{n_0}, z) \leq \sup_{z \in S^{+}_{\preceq^{\Phi}}(\overline{x})} \Phi(\overline{x}, z)$$

which implies

$$\varphi(x_{n_0}) < \Phi(\overline{x}, x_{n_0}) + \varphi(x_{n_0}) \leq \varphi(\overline{x})$$

and since $\varphi(\overline{x}) \leq \varphi(x_n)$ for all $n \in \mathbb{N}$, this is a contradiction.

In other hand, we have

$$\sum_{i=0}^{n} \Phi(x_i, x_{i+1}) \leq \Phi(a, x_{n+1}) < \infty$$

then $\sum_{i=0}^{n} \Phi(x_i, x_{i+1})$ is a convergent sequence since it is increasing and bounded above by $\sup_{y \in S^{+}_{\preceq^{\Phi}}(a)} \Phi(a,y) < \infty$, so $\lim_{n\to\infty} \Phi(x_n, x_{n+1}) = 0$.

(S2) holds obviously. And since $x \prec^{\Phi} y$ implies that $\Phi(x,y) > 0$ we have (S5). Therefore, all assumptions of Száz principle hold, hence there exists a maximal element $\hat{x} \in X$. It follows that there exist $\hat{y} \in T\hat{x}$ and $\hat{z} \in S\hat{x}$. Since $\hat{x} \preceq^{\Phi} \hat{z}$, we obtain $\hat{x} = \hat{z}$. And then $\Phi(\hat{x}, \hat{z}) = 0$ thus $d(\hat{x}, \hat{y}) = 0$, which implies $\hat{x} \in S\hat{x}$ and $\hat{x} \in T\hat{x}$. The proof is completed. □

Example 4.3. Let X be the space of all nonexpansive mappings from $[0,1]$ to itself, i.e. $f \in X$ if for all $x, y \in [0,1]$ we have $|f(x) - f(y)| \leq |x - y|$.

Let $f, g \in X$ and let $d : X \times X \to [0,\infty)$ be the mapping defined by :

$$d(f,g) = \begin{cases} \sup_{x \in [0,1]} g(x) - f(x) & \text{if } f \leq g \\ 1 & \text{otherwise.} \end{cases}$$

Then d is a quasi-metric. Next, we define a preorder on X as follows :

$$f \preceq g \Leftrightarrow f(1) \leq g(1). \tag{4.2}$$

Now, we have to show that $(X, d, \preceq)$ is preordered left K-complete quasi metric space. Let (f_n) be a left K-Cauchy sequence in X. Let $0 < \varepsilon < 1$, there exists $n_0 \in \mathbb{N}$ such for all $m \geq n \geq n_0$ we have

$$d(f_n, f_m) < \varepsilon$$

i.e. $f_n \leq f_m$ and $\sup_{x\in[0,1]} f_m(x) - f_n(x) < \varepsilon$, thus $0 \leq f_m(x) - f_n(x) < \varepsilon$ for all $x \in [0,1]$ and for all $m \geq n \geq n_0$. Hence $(f_n(x))_n$ is a real increasing sequence in $[0,1]$. Then, there exists $f : [0,1] \to [0,1]$ such that for all $x \in [0,1]$,

$$f_n(x) \to f(x) \quad \text{and} \quad f_n(x) \leq f(x), \text{ for all } n \geq n_0,$$

then for all $n \geq n_0$ we have $0 \leq f(x) - f_n(x) < \varepsilon$; which implies that $\sup_{x\in[0,1]} f(x) - f_n(x) < \varepsilon$, then

$$\lim_n d(f_n, f) = 0$$

i.e. $\{f_n\}$ is left K-converge to f. Now, for $x, y \in [0,1]$, $|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \leq |f(x) - f_n(x)| + |x - y| + |f_n(y) - f(y)|$, so letting $n \to \infty$, f is nonexpansive mapping. Then $(X, d, \preceq)$ is preordered left K-complete quasi metric space.

Let $\Phi(f, g) = g(1) - f(1)$. Then Φ satisfies obviously $(C3)$, $(C'1)$ and $C6)$. Let us prove $(C5)$. We have for all $f \in X$,

$$\varphi(f) = \sup\{g(1) - f(1) : f \preceq g\} = \sup\{g(1) - f(1) : f(1) \leq g(1)\} = 1 - f(1)$$

thus $\varphi(X) \subseteq [0,1]$. Now, let $y \in [0,1]$, and set $f(x) = 1 - y$ for each $x \in [0,1]$, then f is in X and we have

$$\varphi(f) = y$$

i.e. $\varphi(X) \supseteq [0,1]$, and since $[0,1]$ is closed we have $(C5)$.

For, $T, S : X \to 2^X$, we take for all $f \in X$:

$$T(f) = \{g \in X : f \leq g\} \quad \text{and} \quad S(f) = \{h \in X : f \leq \sqrt{h}\}.$$

So, for each $f \in X$; let $g = f$ and $h = 1$ then, $d(f, g) = 0$ and $\Phi(f, h) = 1 - f(1)$, that is

$$d(f, g) \leq \Phi(f, h).$$

All assumptions of Theorem 4.4 are satsified. Note that $f \equiv 1$ is a common fixed point for T and S.

Corollary 4.4. *If there exists an A-function $\Phi : X \times X \to \overline{\mathbb{R}}$ such that for each $x \in X$*

$$(*) \qquad (x \preceq Tx \quad \text{and} \quad x \preceq Sx) \Rightarrow d(x, Tx) \leq \Phi(x, Sx),$$

then T and S have a common fixed point.

Example 4.4. Let $X = \left[\frac{1}{2}, 1\right]$, consider the usual order in $\mathbb{R}$ and the quasi-metric defined by:

$$d(x, y) = \begin{cases} y - x & \text{if } x \leq y \\ 1 & \text{if } x > y \end{cases}$$

It is shown in [10, 11, 5] that (X, d) is left K-complete quasi metric space. Define the maps T and S as follows :

$$Tx = \sqrt{x} \quad \text{and} \quad Sx = \sqrt[3]{x}$$

for all $x \in \left[\frac{1}{2}, 1\right]$. It is obvious that $T(X) \subset X$, $S(X) \subset X$ and for each $x \in X$ we have $x \leq Tx$ and $x \leq Sx$.

Let $f : X \to \mathbb{R}$ be the continuous function defined by $f(x) = \frac{1}{x^2}$ and consider the antiderivative F of f which vanishes at $\dfrac{1}{2}$ that is

$$F(x) = \int_{\frac{1}{2}}^{x} f(t)\, dt = \frac{-1}{x} + 2$$

We consider for each $x, y \in X$ the map Φ defined by

$$\Phi(x,y) = \int_x^y f(t)\,dt = F(y) - F(x).$$

Note that for each $x, y \in X$ we have

$$x \leq y \Leftrightarrow x \preceq^{\Phi} y.$$

Φ is an A-function. Indeed, (C$'$1) is satisfied since it is as Example 3.1-(2), also (C3) holds since for each $x \in X$,

$$\varphi(x) = \sup_{y \in S^{+}_{\leq}(x)} \Phi(x,y) < \infty.$$

Let show that the range of $\left[\frac{1}{2}, 1\right]$ by φ is closed subset on $\overline{\mathbb{R}}$. By definition we have

$$\varphi(x) = \sup\nolimits_{y\in[x,1]} \Phi(x,y) = \sup\nolimits_{y\in[x,1]} (F(y) - F(x)) = \sup\nolimits_{y\in[x,1]} \left(\tfrac{1}{x} - \tfrac{1}{y}\right)$$

then $\varphi(x) = \frac{1}{x} - 1$ and $\varphi\left(\left[\frac{1}{2}, 1\right]\right) = [0,1]$ is closed subset. For each $x \in X$ we have

$$d(x,Tx) = Tx - x = \sqrt{x} - x$$

and

$$\Phi(x,Sx) = \int_x^{Sx} f(t)\,dt = F(Sx) - F(x) = \tfrac{1}{x} - \tfrac{1}{\sqrt[3]{x}}$$

then $d(x,Tx) \leq \Phi(x,Sx)$, i.e. the inequality $(*)$ holds.

All assumptions of Corollary 4.4 are satisfied, then T and S admit a common fixed point; it is obvious that $1 = T(1) = S(1)$ satisfies the requirement.

Corollary 4.5. *Under the assumptions of Theorem 4.4 with $S = T$, the mapping T has a fixed point.*

From Theorem 4.4, we obtain an improvment of [8, Theorem 2.3] and [9, Theorem 3] without lower semicontinuity.

Theorem 4.5. *Under the assumptions of Theorem 4.4, with S and T are single valued mappings such that $S = T^p$ for some $p \in \mathbb{N}^*$. If, for each $x \in X$ we have $x \preceq Tx$ and*

$$d(x,Tx) \leq \Phi(x,T^p x),$$

then T has a fixed point.

Recall that two integres $p, q \in \mathbb{N}$ are coprime if there exist two positive integers n and m such that $np = mq + 1$ or $mq = np + 1$.

Theorem 4.6. *Let $(X, d, \preceq)$ be a preordered left K-complete quasi metric space, Φ an A-function, p and q two coprime positive integers. Assume that T is a self map of X satisfying for all $x \in X$:*
(i) $x \preceq T^p x$ and $x \preceq T^q x$; $\quad (ii)$ $\max\{d(x,T^p x),\, d(x,T^q x)\} \leq \Phi(x,T^{pq}x)$.
Then, T has a fixed point.

Proof. By (i) and transitivity of $\preceq$ we get $x \preceq T^{pq}x$ and by (ii) we obtain $d(x,(T^p)x) \leq \Phi(x,(T^p)^q x)$, so using Theorem 4.5 where $(T^p)^q = S$ we conclude that there exists $\bar{x} \in X$ such that $T^p\bar{x} = \bar{x}$.

Next we show that T^q admit the same fixed point $\bar{x}$, for this concider the subset $K := \{\bar{x}, T\bar{x}, \ldots, T^{p-1}\bar{x}\}$ constituted by the first p-elements of the orbit under T of $\bar{x}$, it follows immediately that is non-empty stable by T (i.e. $T(K) \subseteq K$) and because K is a finite subset of X, we deduce that it is a left K-complete space.

By (ii), we have also for all $x \in K$, $d(x,T^q x) \leq \Phi(x,T^{pq}x)$ and Theorem 4.5 shows that T^q has at least a fixed point in K, that is, there is a positive integer $j \in \{0, 1, \ldots, p-1\}$

such that $T^q(T^j\bar{x}) = T^j\bar{x}$ and since $0 \leq j \leq p-1$ there is some positive integer $k > 0$ with $j + k = p$, thus $T^k(T^q(T^j\bar{x})) = T^k(T^j\bar{x})$ i.e.

$$T^{q+k+j}\bar{x} = T^{q+p}\bar{x} = T^q(T^p\bar{x}) = T^q\bar{x} = T^{k+j}\bar{x} = T^p\bar{x} = \bar{x},$$

then $T^q\bar{x} = \bar{x}$ also a fixed point for T^q.

Note that for all $n \in \mathbb{N}$, we have $T^{np}\bar{x} = \bar{x}$, since p and q are coprime positive integers there exist two positive integers r and l such that $rp = lq + 1$ or $rq = lp + 1$. Assume that $rp = lq + 1$ and as T^p and T^q have a common fixed point then

$$\bar{x} = T^{rp}\bar{x} = T^{lq+1}\bar{x} = T(T^{lq}\bar{x}) = T\bar{x}$$

which complete the proof. □

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References

[1] Ali-Akbari, M., Honari, B., Pourmahdian, M. and Rezaii, M. M., *The space of formal balls and models of quasi-metric spaces*, Math. Structures Comput. Sci., **19** (2009), No. 2, 337–355

[2] Altman, M., *A generalization of the Brézis-Browder principle on ordered sets*, Nonlinear Analysis : Theory, Methods & Applications, **6** (1982), No. 2, 157–165

[3] Brézis H. and Browder, F. E., *A general principle on ordered sets in nonlinear functional analysis*, Advances in Math. , **21** (1976), No. 3, 355–364

[4] Caristi, J., *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc., **215** (1976), 241–251

[5] Cobzaş S., *Completeness in quasi-metric spaces and Ekeland variational principle*, Topology Appl., **158** (2011), No. 8, 1073–1084

[6] Gaba, Y. U., *An order theoretic approach in fixed point theory*, Math. Sci. (Springer), **8** (2014), No. 3, 87–93

[7] Karapinar, K. and Romaguera, S., *On the weak form of Ekeland's Variational Principle in quasi-metric spaces*, Topology Appl., **2015** (2105), No. 184, 54–60

[8] Kirk, W. A. and Saliga, L. M., *The Brézis-Browder order principle and extensions of Caristi's theorem*, Nonlinear Analysis : Theory, Methods & Applications, **47** (2001), No. 4, 2765–2778

[9] Lazaiz, S., Chaira, K., Aamri, M. and Marhrani, El-M., *Some remarks on Caristi type fixed point theorem*, International Journal of Pure and Applied Mathematics, **104** (2015), No. 4, 585–597

[10] Romaguera, S. and Tirado, P., *A characterization of Smyth complete quasi-metric spaces via Caristi's fixed point theorem*, Fixed Point Theory Appl., **2015** (2015), No. 1, 1–13

[11] Reilly, I. L., Subrahmanyam, P. V. and Vamanamurthy, M. K.,*Cauchy sequences in quasi-pseudo-metric spaces*, Monatshefte for Mathematik, **93** (1982), No. 2, 127–140

[12] Száz, Á., *An improved Altman type generalization of the Brézis-Browder ordering principle*, Math. Commun., **12** (2007), No. 2, 155–161

[13] Turinici, M., *A generalization of Altman's ordering principle*, Proc. Amer. Math. Soc., **90** (1984), No. 2, 128–132

[14] Wilson, W. A., *On quasi-metric spaces*, Amer. J. Math., **53** (1931), No. 3, 675–684

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