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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

Attractive points of monotone further generalized hybrid mappings

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ABSTRACT. The aim of this paper is to introduce monotone further generalized mappings in a Hilbert space with partial order and study the existence and approximation results leading to attractive points for such mappings. Moreover, a numerical example is given to support our results and comparative study of the iterative processes has been done along with general discussion.

1. INTRODUCTION AND PRELIMINARIES

Numerous results in fixed point theory assume the closedness and convexity conditions of the underlying set. In 2011, Takahashi and Takeuchi [13] conceived the idea of attractive points of nonlinear mappings in Hilbert spaces and utilized it to eliminate these conditions in the famous Baillon's ergodic theorem [1] in Hilbert spaces. Takahashi and Yao [14] studied attractive points for hybrid mappings in a Hilbert space without the closedness condition. Khan [6] presented a new class of hybrid mappings called further generalized hybrid mappings and established some results for common attractive points. This has intrigued many mathematicians to study the existence and convergence of attractive points of nonlinear mappings [2, 3, 18].

Many fixed point iterative schemes have been developed and studied by several authors serving various purposes in the literature, see [9, 11, 12, 17]. Mann iteration [7, 9] is a widely popular iterative process but it is also known not to be strongly convergent in general. Strong convergence can be obtained by either applying stronger assumptions, or by modifying iteration schemes however more general schemes do not necessarily lead to better convergence results. In 2013, Khan [5] introduced a faster iterative scheme namely Picard-Man hybrid iterative process which was later extended by Thakur [17]. An alternative way of obtaining strong convergence is by Halpern's type iterative processes. An example of such a process is the inexact iterative process presented by Kanzow and Shehu [4]. Ran and Reurings [10] instigated the idea of fixed points of monotone mappings and established some fixed point results along with applications. Inspired by [6, 10], we introduce monotone further generalized mappings and study the existence of attractive points for such mappings. Some convergence results are established and numerical computations are presented to illustrate the validity of our results.
Now we recall some definitions and results to be used in main results. Let Y be a subset of a real Hilbert space H and $S : Y \to H$ be any mapping. Define the set of attractive points by $A(S) = \{z \in H : \|Sx - z\| \leq \|x - z\| \quad \text{for all } x \text{ in } Y\}$. Denote the strong convergence

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of the sequence $\{x_n\}$ by $x_n \to x$ and weak convergence of the sequence $\{x_n\}$ by $x_n \rightharpoonup x$. We know that for any $x, y, z, w \in H$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle; \tag{1.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2; \tag{1.2}$$

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2. \tag{1.3}$$

Let $x \in H$ and Y be closed and convex then there exists a unique nearest point $P_Y x$ in Y, that is,

$$\|x - P_Y x\| \leq \|x - y\|$$

for every $y \in Y$ where P_Y is known as metric projection of H onto Y. It is widely known that P_Y is firmly nonexpansive and satisfies the inequality

$$\langle x - P_Y x, P_Y x - u\rangle \geq 0,$$

for any $x \in H$ and $u \in Y$. Let l^∞ be the Banach space of bounded sequences under supremum norm and $\eta \in (l^\infty)^*$ (the dual of l^∞) then it is well known that η satisfies $\|\eta\| = \eta(1) = 1$ and $\eta_n(x_{n+1}) = \eta_n(x_n)$ for each $x = (x_1, x_2, x_3, ...) \in l^\infty$. This η is called Banach limit. It is also known that

$$\liminf_{n\to\infty} x_n \leq \eta_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

If $\lim_{n\to\infty} x_n$ exists and converges to some a then $\eta_n(x_n) = a$ [15].

Lemma 1.1. [15] *Suppose that x_n is a bounded sequence in H and η a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n | n \in \mathbb{N}\}$ so that $\eta_n\langle x_n, y\rangle = \langle z_0, y\rangle$.*

The following crucial lemma was established by Takahashi and Takeuchi [13].

Lemma 1.2. *Let Y be a nonempty subset of H and let S be a mapping from Y into H. Then, $A(S)$ is a closed and convex subset of H.*

Lemma 1.3. *Let $\{a_n\}$ be a sequence of non-negative real number satisfying the property $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\gamma_n + \beta_n$, where $\{\alpha_n\} \subseteq (0,1)$, $\{\beta_n\}$ is a sequence of nonnegative real numbers and $\{\gamma_n\}$ is a sequence of real numbers such that $\sum_{n=0}^\infty \alpha_n = \infty$, $\sum_{n=0}^\infty \beta_n < \infty$ and $\limsup_{n\to\infty} \gamma_n \leq 0$, for all $n \in \mathbb{N}$. Then $\{a_n\}$ converges to zero, as $n \to \infty$.*

Lemma 1.4. [16] *Let Y be a nonempty closed convex subset of H. Let P be a projection from H onto Y. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in Y$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in Y$.*

Lemma 1.5. [8] *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity such that there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_k} < \Gamma_{n_{k+1}}$ for all $k \in \mathbb{N}$. The sequence $\{\phi(n)\}_{n\geq n_0}$ of integers is defined as $\phi(n) = \max\{m \leq n : \Gamma_{n_m} < \Gamma_{n_{m+1}}\}$ where $n_0 \in \mathbb{N}$ such that $\{m \leq n_0 : \Gamma_{n_m} < \Gamma_{n_{m+1}} \neq \Phi\}$. Then, the following hold:*

(1) $\phi(n_0) \leq \phi(n_0+1) \leq \cdots$ and $\phi(n) \to \infty$;
(2) $\Gamma_{\phi(n)} \leq \Gamma_{\phi(n+1)}$ and $\Gamma_n \leq \Gamma_{\phi(n+1)}$ for all $n \geq n_0$.

Throughout this paper we assume that H is a real Hilbert space endowed with partial order $\preccurlyeq$ and Y is a nonempty subset of H. For $x, y \in H$ define the sets $\Delta_1 = \{(x,y) : x \preccurlyeq y\}$ and $\Delta_2 = \{(x,y) : y \preccurlyeq x\}$. Then we say two elements x, y in H are comparable if either (x,y) belongs to Δ_1 or (x,y) belongs to Δ_2 . A mapping $S : H \to H$ is said to be monotone if $S(x) \preccurlyeq S(y)$ where $(x,y) \in \Delta_1$. We now present the definition of monotone further generalized mappings.

Definition 1.1. Let $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$ and $\epsilon \geq 0$. A mapping $S : Y \to H$ is called monotone further generalized hybrid mapping, if S is monotone and any one of the following hold:

(1) $\alpha+\beta > 0$ and $\alpha\|Sx-Sy\|^2+\beta\|x-Sy\|^2+\gamma\|y-Sx\|^2+\delta\|x-y\|^2+\epsilon\|x-Sx\|^2 \leq 0$ whenever $(x, y) \in \Delta_1$;

(2) $\alpha+\gamma > 0$ and $\alpha\|Sx-Sy\|^2+\beta\|y-Sx\|^2+\gamma\|x-Sy\|^2+\delta\|x-y\|^2+\epsilon\|y-Sy\|^2 \leq 0$ whenever $(x, y) \in \Delta_2$.

Now, we construct some examples of monotone further generalized hybrid mappings which also give some insight into the relationship of fixed and attractive points.

Example 1.1. Let $Y = [1, 2] \subset \mathbb{R}$ where $\mathbb{R}$ with the usual order $\preccurlyeq$ and assume that $(x, y) \in \Delta_1$. Define $S : Y \to Y$ by $Sx = \frac{1+x}{2}$ for $x \neq 2$ and 1 for $x = 2$. Then S is a monotone further generalized hybrid mapping with $\alpha = 3$, $\beta = \delta = -1$ and $\gamma = \epsilon = 0$. Indeed, if $x, y \in [1, 2)$ then

$$\begin{aligned} |x - Sy|^2 + |x - y|^2 &= \left|x - \frac{1+y}{2}\right|^2 + |x - y|^2 \\ &\geq \frac{3}{4}|x - y|^2 = 3|Sx - Sy|^2. \end{aligned}$$

If $x \in [1, 2)$ and $y = 2$ we get

$$|x - Sy|^2 + |x - y|^2 = |x - 1|^2 + |x - y|^2 \geq \frac{3}{4}|x - 1|^2 = 3|Sx - Sy|^2.$$

Through easy calculations, we can see that $A(S) = (-\infty, 1]$ as $|z - Sx| \leq |x - z|$ must be satisfied for each $x \in Y$. Observe that $x = 1$ is the fixed point of S thus $F(S) \subset A(S)$. It is worth mentioning that if Y was not closed then the fixed point would have failed to existed.

Example 1.2. Let $Y = (0, 1) \subset \mathbb{R}$ where $\mathbb{R}$ with the usual order $\preccurlyeq$ and assume that $(x, y) \in \Delta_1$. Define $S : Y \to Y$ by $Sx = \frac{1}{2}$ for $0 < x < \frac{1}{2}$ and $\frac{3}{4}$ for $\frac{1}{2} \leq x < 1$. Then S is a monotone further generalized hybrid mapping with $\alpha = 1$, $\gamma = \delta = \frac{-1}{4}$ and $\beta = \epsilon = 0$. In fact, for $x \in (0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1)$ we have

$$|Sx - Sy|^2 - \frac{1}{4}|y - Sx|^2 - \frac{1}{4}|x - y|^2 \leq |\frac{1}{4}|^2 - \frac{1}{4}\left|y - \frac{1}{2}\right|^2 - \frac{1}{4}|x - y|^2 \leq 0.$$

In case of $x, y \in (0, \frac{1}{2})$ and $x, y \in [\frac{1}{2}, 1)$, the inequality is obvious. By easy calculations, we find that $A(S) = [\frac{3}{4}, \infty)$ as $|z - Sx| \leq |x - z|$ must be satisfied for each $x \in Y$. Notice that $x = \frac{3}{4}$ is the fixed point of S thus $F(S) \subset A(S)$.

2. Main results

First, we establish existence results for attractive points of monotone further generalized mappings.

Theorem 2.1. *Let $S : Y \to Y$ be a monotone further generalized hybrid mapping then $A(S)$ is nonempty if and only if there exists $z \in Y$ such that $\{S^n z\}$ is bounded where $X(Y, \Delta_1) = \{x \in Y | (Sx, x) \in \Delta_1$ and $(z, x) \in \Delta_1\}$ is nonempty.*

Proof. Since $X(Y, \Delta_1)$ is nonempty there exists $x \in Y$ such that $(Sx, x) \in \Delta_1$ and $(z, x) \in \Delta_1$ then $S^n x \preccurlyeq S^{n-1}x \preccurlyeq \ldots \preccurlyeq S^2 x \preccurlyeq Sx \preccurlyeq x$ and $S^n z \preccurlyeq S^n x$. Therefore, $(S^n z, x) \in \Delta_1$ for all $n \in \mathbb{N}$. Since S is monotone further generalized hybrid mapping we have

$$\alpha\|S^{n+1}z - Sx\|^2 + \beta\|S^n z - Sx\|^2 + \gamma\|x - S^{n+1}z\|^2 + \delta\|S^n z - x\|^2 + \epsilon\|S^n z - S^{n+1}z\|^2 \leq 0$$

for $n \in \mathbb{N}$. Applying Banach limit in the above inequality we obtain

$$(\alpha+\beta)\eta_n\|S^n z - Sx\|^2 + (\gamma+\delta)\eta_n\|S^n z - x\|^2 \leq 0. \tag{2.4}$$

Further, using $\|Sx - S^n z\|^2 = \|x - S^n z\|^2 + \|Sx - x\|^2 - 2\langle x - Sx, x - S^n z\rangle$, (2.4) becomes

$$(\alpha+\beta)\eta_n\bigg(\|x - S^n z\|^2 + \|Sx - x\|^2 - 2\langle x - Sx, x - S^n z\rangle\bigg) + (\gamma+\delta)\eta_n\|S^n z - x\|^2 \leq 0$$

which implies

$$(\alpha+\beta)\|Sx - x\|^2 - 2\eta_n(\alpha+\beta)\langle x - Sx, x - S^n z\rangle + (\alpha+\beta+\gamma+\delta)\eta_n\|S^n z - x\|^2 \leq 0.$$

From $\alpha+\beta+\gamma+\delta \geq 0$ we have

$$(\alpha+\beta)\|Sx - x\|^2 - 2\eta_n(\alpha+\beta)\langle x - Sx, x - S^n z\rangle \leq 0. \tag{2.5}$$

Since there exists $p \in H$ by Lemma 1.1 such that $\eta_n\langle y, S^n z\rangle = \langle y, p\rangle$ from (2.5) we obtain

$$(\alpha+\beta)\|Sx - x\|^2 - 2(\alpha+\beta)\langle x - Sx, x - p\rangle \leq 0. \tag{2.6}$$

Now, taking $y = Sx$, $x = z$ and $w = p$ in (1.3) yields

$$2\langle x - Sx, x - p\rangle = \|x - p\|^2 + \|Sx - x\|^2 + \|Sx - p\|^2.$$

Thus, (2.6) becomes $(\alpha+\beta)\|Sx - x\|^2 - (\alpha+\beta)\bigg(\|x - p\|^2 + \|Sx - x\|^2 - \|Sx - p\|^2\bigg) \leq 0$ which further implies

$$(\alpha+\beta)\bigg(\|Sx - p\|^2 - \|x - p\|^2\bigg) \leq 0.$$

From $\alpha+\beta > 0$ we have

$$\|p - Sx\|^2 \leq \|x - p\|^2,$$

for all $x \in Y$. Therefore, $p \in A(S)$. The converse is obvious. □

Theorem 2.2. *Let $S : Y \to Y$ be a monotone further generalized hybrid mapping then $A(S)$ is nonempty if and only if there exists $z \in Y$ such that $\{S^n z\}$ is bounded where $X(Y, \Delta_2) = \{x \in Y | (Sx, x) \in \Delta_2$ and $(z, x) \in \Delta_2\}$ is nonempty.*

Proof. The proof is similar to last theorem. □

Now we present convergence theorems for finding attractive points of our mappings in a Hilbert space endowed with partial order. First, we prove the following lemma which will help us in establishing our main results.

Lemma 2.6. *Let $S : Y \to Y$ be a monotone further generalized hybrid mapping. Suppose that $\{x_n\}$ is a bounded sequence in Y and there exists $y \in Y$ such that either $(x_n, y) \in \Delta_1$ or $(x_n, y) \in \Delta_2$. If $x_n \rightharpoonup z$ and $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ then $z \in A(S)$.*

Proof. Assume that $(x_n, y) \in \Delta_1$ then since S is a monotone further generalized hybrid mapping we have

$$\alpha\|Sx_n - Sy\|^2 + \beta\|x_n - Sy\|^2 + \gamma\|Sx_n - y\|^2 + \delta\|x_n - y\|^2 + \epsilon\|x_n - Sx_n\|^2 \leq 0$$

for $n \in \mathbb{N}$. From (1.3) we obtain

$$\alpha\bigg(\|x_n - Sy\|^2 + \|x_n - Sx_n\|^2 - 2\langle x_n - Sx_n, x_n - Sy\rangle\bigg) + \beta\|x_n - Sy\|^2$$

$$+\gamma\bigg(\|x_n - y\|^2 + \|Sx_n - x_n\|^2 - 2\langle x_n - Sx_n, x_n - y\rangle\bigg) + \delta\|x_n - y\|^2 + \epsilon\|x_n - Sx_n\|^2 \leq 0$$

Since

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0,$$

applying Banach limit yields

$$(\alpha+\beta)\eta_n\|x_n - Sy\|^2 + (\gamma+\delta)\eta_n\|x_n - y\|^2 \le 0,$$

for all $y \in Y$. As $\alpha+\beta+\gamma+\delta \ge 0$ and $\alpha+\beta > 0$ we obtain $\eta_n\|x_n - Sy\|^2 \le \eta_n\|x_n - y\|^2$. Similarly, if $(x_n, y) \in \Delta_2$ then

$$\alpha\|Sy - Sx_n\|^2 + \beta\|y - Sx_n\|^2 + \gamma\|Sy - x_n\|^2 + \delta\|y - x_n\|^2 + \epsilon\|y - Sy\|^2 \le 0.$$

Since

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0,$$

by applying Banach limit we obtain

$$(\alpha+\gamma)\eta_n\|x_n - Sy\|^2 + (\beta+\delta)\|x_n - y\|^2 + \epsilon\|y - Sy\|^2 \le 0,$$

for all $y \in Y$. As $\epsilon \ge 0$, $\alpha+\beta+\gamma+\delta \ge 0$ and $\alpha+\gamma > 0$ we obtain

$$\eta_n\|x_n - Sy\|^2 \le \eta_n\|x_n - y\|^2.$$

Thus, for both cases

$$\eta_n\|x_n - Sy\|^2 \le \eta_n\|x_n - y\|^2.$$

Further, since $x_n \rightharpoonup z$, from

$$\|x_n - Sy\|^2 = \|x_n - y\|^2 + \|y - Sy\|^2 + 2\langle x_n - y, y - Sy\rangle$$

we have

$$\|y - Sy\|^2 + 2\langle z - y, y - Sy\rangle \le 0.$$

Using (1.3) we obtain

$$\|z - Sy\|^2 - \|z - y\|^2 \le 0$$

which implies

$$\|z - Sy\| \le \|z - y\|,$$

for all $y \in Y$. Hence, $z \in A(S)$. □

Now we obtain a strong convergence result for approximation of attractive points of our mapping using an inexact iterative process [4].

Theorem 2.3. *Let Y be convex and $S : Y \to Y$ a monotone further generalized hybrid mapping such that $A(S) \neq \phi$. Let $x_1 \in H$ and $\{x_n\}$ be a sequence in H generated by*

$$x_{n+1} = \delta_n u + \alpha_n x_n + \beta_n Sx_n + r_n \tag{2.7}$$

where $u \in Y$ denotes a fixed vector, r_n represents the residual. The nonnegative real numbers $\alpha_n, \beta_n, \delta_n$ be such that $\alpha_n + \beta_n + \delta_n \le 1$ for $n \ge 1$ and $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$, $\sum_{n=1}^{\infty}(1 - (\alpha_n + \beta_n + \delta_n)) < \infty$ and $\sum_{n=1}^{\infty} \|r_n\| < \infty$. Suppose there exists $y \in Y$ such that $(x_n, y) \in \Delta_1$ or $(x_n, y) \in \Delta_2$ then $\{x_n\} \to \overline{x} = P_{A(S)}u$.

Proof. Let $x_1 \in Y$. Suppose $x_* \in A(S)$ and since $\alpha_n + \beta_n \le 1 - \delta_n$ then from (2.7) we have

$$\|x_{n+1} - x_*\| \le \max\{\|u - x_*\|, \|x_n - x_*\|\} + (1 - \alpha_n - \beta_n - \delta_n)\|x_*\| + \|r_n\|. \tag{2.8}$$

Then by induction,

$$\|x_{n+1} - x_*\| \le \max\{\|u - x_*\|, \|x_n - x_*\|\} + \sum_{k=1}^{n} r_k + \|x_*\| \sum_{k=1}^{n}(1 - \alpha_k - \beta_k - \delta_k) \tag{2.9}$$

for all $n \in \mathbb{N}$. Therefore,

$$\|x_{n+1} - x_*\| \leq \max\{\|u - x_*\|, \|x_1 - x_*\|\} + \sum_{k=1}^{n} r_k + \|x_*\| \sum_{k=1}^{n} (1 - \alpha_k - \beta_k - \delta_k) \tag{2.10}$$

for all $n \in \mathbb{N}$. Since

$$\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n - \delta_n) < \infty$$

and

$$\sum_{n=1}^{\infty} r_k < \infty,$$

then from (2.10) we obtain that $\{x_n\}$ is bounded. Since $A(S)$ is nonempty, closed, and convex the projection $P_{A(S)}u$ exists. Let $\overline{x} = P_{A(S)}u$. Now, we consider the following cases:

Case 1. Let $\lim_{n\to\infty} \|x_n - \overline{x}\|$ exists. Then from (2.7) and (1.1) we have

$$\begin{aligned} \|x_{n+1} - \overline{x}\|^2 \leq& (\alpha_n + \beta_n)^2 \|x_n - \overline{x}\|^2 - \alpha_n\beta_n \|x_n - Sx_n\|^2 + \\ & 2\langle \delta_n(u - \overline{x}) + r_n - (1 - \alpha_n - \beta_n - \delta_n)\overline{x}, x_{n+1} - \overline{x}\rangle \\ \leq& (1 - \delta_n)\|x_n - \overline{x}\|^2 - \alpha_n\beta_n\|x_n - Sx_n\|^2 + \\ & 2\delta_n\langle u - \overline{x}, x_{n+1} - \overline{x}\rangle + 2\langle r_n - (1 - \alpha_n - \beta_n - \delta_n)\overline{x}, x_{n+1} - \overline{x}\rangle. \end{aligned} \tag{2.11}$$

Since $\{x_n\}$ is bounded, we have

$$\begin{aligned} \alpha_n\beta_n\|x_n - Sx_n\|^2 \leq& \|x_n - \overline{x}\|^2 - \|x_{n+1} - \overline{x}\|^2 + \delta_n M_5 + \\ & (1 - \alpha_n - \beta_n - \delta_n)M_6 + \|r_n\|M_7 \end{aligned} \tag{2.12}$$

for $M_5, M_6, M_7 > 0$. Since $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty}(1 - (\alpha_n + \beta_n + \delta_n)) < \infty$ and $\sum_{n=1}^{\infty} \|r_n\| < \infty$ we obtain

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{2.13}$$

As $\{x_n\}$ is bounded there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}\langle u - \overline{x}, x_n - \overline{x}\rangle = \lim_{i\to\infty}\langle u - \overline{x}, x_{n_i} - \overline{x}\rangle$ where $\{x_{n_i}\}$ converges weakly. Let $x_{n_i} \rightharpoonup p$ then by (2.13) and Lemma 2.6, $p \in A(S)$. From property of projection we obtain $\limsup_{n\to\infty}\langle u - \overline{x}, x_n - \overline{x}\rangle = \lim_{i\to\infty}\langle u - \overline{x}, x_{n_i} - \overline{x}\rangle = \limsup_{n\to\infty}\langle u - \overline{x}, p - \overline{x}\rangle \leq 0$. Now, (2.11) implies

$$\begin{aligned} \|x_{n+1} - \overline{x}\|^2 \leq& (1 - \delta_n)\|x_n - \overline{x}\|^2 + 2\delta_n\langle u - \overline{x}, x_{n+1} - \overline{x}\rangle \\ & + (1 - \alpha_n - \beta_n - \delta_n)M_6 + \|r_n\|M_7. \end{aligned}$$

Then, Lemma 1.3 and conditions $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n - \delta_n) < \infty$ and $\sum_{n=1}^{\infty} \|r_n\| < \infty$ yield $\lim_{n\to\infty} \|x_n - \overline{x}\| = 0$ that is x_n converges to $\overline{x} \in P_{A(S)}u$.

Case 2. Let $\Gamma_n = \|x_n - \overline{x}\|^2$ for $n \geq 1$ and define $\phi : \mathbb{N} \to \mathbb{N}$ by $\phi(n) = \max\{k \leq n : \Gamma_{n_k} \leq \Gamma_{n_{k+1}}\}$. From (2.12), we have $\|x_{\phi(n)} - Sx_{\phi(n)}\| \to 0$. Now,

$$\begin{aligned} \|x_{\phi(n)+1} - x_{\phi(n)}\| \leq& \delta_{\phi(n)}\|u - x_{\phi(n)}\| + \beta_{\phi(n)}\|Sx_{\phi(n)} - x_{\phi(n)}\| \\ & + \|r_{\phi(n)} - (1 - \alpha_{\phi(n)} - \beta_{\phi(n)} - \delta_{\phi(n)})x_{\phi(n)}\| \end{aligned} \tag{2.14}$$

Again, since $\{x_n\}$ is bounded for all n and conditions $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty}(1-(\alpha_n+\beta_n+\delta_n)) < \infty$ and $\sum_{n=1}^{\infty} \|r_n\| < \infty$ are satisfied, we obtain $\|x_{\phi(n)+1} - x_{\phi(n)}\| \to 0$ as $n \to \infty$. Since $\{x_{\phi(n)}\}$ is bounded there exists a subsequence which converges weakly to some $p \in A(S)$. As in Case 1, $\limsup_{n\to\infty}\langle u - \overline{x}, x_{\phi(n)+1} - \overline{x}\rangle \leq 0$ which implies

$$\begin{aligned} \|x_{\phi(n)+1} - \overline{x}\|^2 \leq& (1 - \delta_{\phi(n)})\|x_{\phi(n)} - \overline{x}\|^2 + 2\delta_{\phi(n)}\langle u - \overline{x}, x_{\phi(n)} - \overline{x}\rangle \\ & + (1 - \alpha_{\phi(n)} - \beta_{\phi(n)} - \delta_{\phi(n)})M_6 + \|r_{\phi(n)}\|M_7. \end{aligned}$$

Then, using Lemma 1.3 and conditions $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n-\delta_n)<\infty$ and $\sum_{n=1}^{\infty}\|r_n\|<\infty$ we obtain $\lim_{n\to\infty}\|x_{\phi(n)}-\overline{x}\|=0$. By (2.14) we have $\lim_{n\to\infty}\|x_{\phi(n)+1}-\overline{x}\|=0$ or $\lim_{n\to\infty}\Gamma_{\phi(n)+1}=0$. Using Lemma 1.5 we have $\Gamma_n\leq\Gamma_{\phi(n)+1}$, therefore $\lim_{n\to\infty}\|x_n-\overline{x}\|=0$. Thus we have our conclusion. □

Now, we prove a weak convergence theorem for attractive points via iteration process introduced in [17].

Theorem 2.4. *Let Y be convex and $S:Y\to Y$ a monotone further generalized hybrid mapping with $A(S)\neq\phi$. Suppose $\{x_n\}$ is defined by ,*

$$\begin{cases} x_{n+1}=Sy_n,\\ y_n=S((1-\alpha_n)x_n+\alpha_n z_n),\\ z_n=(1-\beta_n)x_n+\beta_n Sx_n, \end{cases} \tag{2.15}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are such that $0<a\leq\alpha_n,\beta_n\leq b<1$. Suppose there exists $y\in Y$ such that $(x_n,y)\in\Delta_1$ or $(x_n,y)\in\Delta_2$ then $\{x_n\}\rightharpoonup p\in A(S)$. Moreover, $p=\lim_{n\to\infty}P_{A(S)}x_n$.

Proof. Let $u\in A(S)$. Then from (2.15) we have $\|x_{n+1}-u\|\leq\|y_n-u\|$. Further, we have

$$\begin{aligned}\|y_n-u\|&\leq\|(1-\alpha_n)x_n+\alpha_n z_n-u\|\\ &\leq(1-\alpha_n)\|x_n-u\|+\alpha_n\|z_n-u\|,\end{aligned} \tag{2.16}$$

$$\|z_n-u\|\leq(1-\beta_n)\|x_n-u\|+\beta_n\|Sx_n-u\|=\|x_n-u\|. \tag{2.17}$$

Therefore from (2.16) and (2.17) we have

$$\|y_n-u\|\leq\|x_n-u\|\quad\text{and}\quad\|x_{n+1}-u\|\leq\|x_n-u\|. \tag{2.18}$$

Hence, $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-u\|$ exists. From (2.17) we have

$$\limsup_{n\to\infty}\|z_n-u\|\leq\limsup_{n\to\infty}\|x_n-u\|\quad\text{and}\quad\limsup_{n\to\infty}\|Sx_n-u\|\leq\limsup_{n\to\infty}\|x_n-u\|. \tag{2.19}$$

Now,

$$\|x_{n+1}-u\|\leq(1-\alpha_n)\|x_n-u\|+\alpha_n\|z_n-u\| \tag{2.20}$$

which implies $\frac{\|x_{n+1}-u\|-\|x_n-u\|}{\alpha_n}\leq\|z_n-u\|-\|x_n-u\|$. Since $\alpha_n\in(0,1)$ we have

$$\begin{aligned}\|x_{n+1}-u\|-\|x_n-u\|&\leq\frac{\|x_{n+1}-u\|-\|x_n-u\|}{\alpha_n}\\ &\leq\|z_n-u\|-\|x_n-u\|.\end{aligned}$$

Therefore, $\|x_{n+1}-u\|\leq\|z_n-u\|$. Furthermore,

$$\liminf_{n\to\infty}\|x_{n+1}-u\|\leq\liminf_{n\to\infty}\|z_n-u\|. \tag{2.21}$$

From (2.19) and (2.21) we have $\lim_{n\to\infty}\|z_n-u\|=\lim_{n\to\infty}\|x_n-u\|$ and hence

$$\lim_{n\to\infty}\|x_n-u\|-\lim_{n\to\infty}\|z_n-u\|=0. \tag{2.22}$$

Now,

$$\begin{aligned}\|z_n-u\|^2&\leq(1-\beta_n)\|x_n-u\|^2+\beta_n\|Sx_n-u\|^2-\beta_n(1-\beta_n)\|x_n-Sx_n\|^2\\ &\leq\|x_n-u\|^2-\beta_n(1-\beta_n)\|x_n-Sx_n\|^2.\end{aligned}$$

Then, $\beta_n(1-\beta_n)\|x_n-Sx_n\|^2\leq\|x_n-u\|^2-\|z_n-u\|^2$ which implies

$$\|x_n-Sx_n\|^2\leq\frac{1}{a(1-b)}(\|x_n-u\|^2-\|z_n-u\|^2). \tag{2.23}$$

Hence (2.22) and (2.23) yield $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$. Now, from Lemma 2.6 we obtain $z \in A(S)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $\{x_{n_i}\} \rightharpoonup z_1$ and $\{x_{n_j}\} \rightharpoonup z_2$. Since $z_1, z_2 \in A(S)$ then $\lim_{n\to\infty}\|x_n - z_1\|^2$ and $\lim_{n\to\infty}\|x_n - z_2\|^2$ exist. Put $a = \lim_{n\to\infty}(\|x_n - z_1\|^2 - \|x_n - z_2\|^2)$. Then from $\{x_{n_i}\} \rightharpoonup z_1$ and $\{x_{n_j}\} \rightharpoonup z_2$ we obtain

$$a = 2\langle z_1, z_2 - z_1\rangle + \|z_1\|^2 - \|z_2\|^2 \quad \text{and} \quad a = 2\langle z_2, z_2 - z_1\rangle + \|z_1\|^2 - \|z_2\|^2$$

which implies $2\langle z_2 - z_1, z_2 - z_1\rangle = 0$. Hence, $z_2 = z_1$ and $\{x_n\}$ converges weakly to $z \in A(S)$. Now, we show that $z = \lim_{n\to\infty} P_{A(S)}x_n$. Since $\|x_{n+1} - z\| \leq \|x_n - z\|$ and $A(S)$ is closed and convex then from Lemma 1.4, $P_{A(S)}x_n$ converges strongly to some $q \in A(S)$. By property of projection $\langle x_n - P_{A(S)}x_n, P_{A(S)}x_n - p\rangle \geq 0$ for all $p \in A(S)$. Taking $n \to \infty$ we obtain $\langle z - q, q - p\rangle \geq 0$ for all $p \in A(S)$, in particular $\langle z - q, q - z\rangle \geq 0$, which implies $p = z = \lim_{n\to\infty} P_{A(S)}x_n$. □

3. Numerical results

For numerical results first we construct an example of a monotone further generalized hybrid mapping. Let $Y = (0, 2) \subset \mathbb{R}$ where $\mathbb{R}$ is endowed with the usual order $\preccurlyeq$. Clearly Y is a convex subset of X and without loss of generality suppose that $(x, y) \in \Delta_1$. Define $S : Y \to Y$ by $Sx = \frac{x^2}{2}$ if $0 < x < \frac{1}{2}$, $Sx = \frac{1}{4}$ if $\frac{1}{2} \leq x < 1$ and $1 - \frac{x}{2}$ if $1 \leq x < 2$ for all $x \in Y$. If $x, y \in (0, \frac{1}{2})$ then

$$\|Sx - Sy\|^2 \leq \left\|\frac{x^2}{2} - \frac{y^2}{2}\right\|^2 \leq \frac{1}{4}\|x^2 - y^2\|^2 \leq \frac{1}{4}\|x - y\|^2\|x + y\|^2 \leq \frac{1}{2}\|y - Sx\|^2 + \frac{1}{4}\|x - y\|^2.$$

If $x \in (0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1)$, then since $\frac{3}{8} < \|y - Sx\| < 1$, we have

$$\|Sx - Sy\|^2 \leq \left\|\frac{x^2}{2} - \frac{1}{4}\right\|^2 \leq \frac{1}{2}(\frac{3}{8})^2 \leq \frac{1}{2}\|y - Sx\|^2 + \frac{1}{4}\|x - y\|^2.$$

If $x \in (0, \frac{1}{2})$ and $y \in [1, 2)$ then since $\frac{7}{8} < \|y - Sx\| < 2$, we have

$$\|Sx - Sy\|^2 \leq \left\|\frac{x^2}{2} - (1 - \frac{y}{2})\right\|^2 \leq \frac{1}{2}(\frac{7}{8})^2 \leq \frac{1}{2}\|y - Sx\|^2 + \frac{1}{4}\|x - y\|^2.$$

If $x \in [\frac{1}{2}, 1)$ and $y \in [1, 2)$, then since $\frac{3}{4} < \|y - Sx\| < \frac{7}{4}$, we have

$$\|Sx - Sy\|^2 \leq \left\|\frac{1}{4} - (1 - \frac{y}{2})\right\|^2 \leq \frac{1}{2}(\frac{3}{4})^2 \leq \frac{1}{2}\|y - Sx\|^2 + \frac{1}{4}\|x - y\|^2.$$

For $x \leq y \in [\frac{1}{2}, 1)$ and $x, y \in [1, 2)$, the inequality is obvious. Hence, the mapping S is a monotone further generalized hybrid mapping with $\alpha = 1$, $\delta = \frac{1}{4}$, $\gamma = \frac{1}{2}$ and $\beta = \epsilon = 0$. It can be seen that $A(S) = (-\infty, 0]$. Set $u = 1$, $\alpha_n = \beta_n = \frac{1}{4}$, $\delta_n = \frac{1}{n+500}$ and $r_n = 0$ in inexact iterative process (2.7).

For initial points $x_1 = 0.25, 1, 1.5$ (2.7) converges to attractive point 0, see Figure 1. Note that the choice of initial points affects the rate of convergence of iterative process (2.7). Now we compare the convergence of the iteration (2.15) with the Ishikawa [18], S [2] and Picard-Mann (P-M) iteration [5] by numerical experiments for different choices of initial points and parameters.

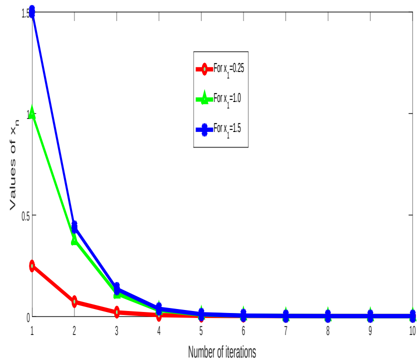
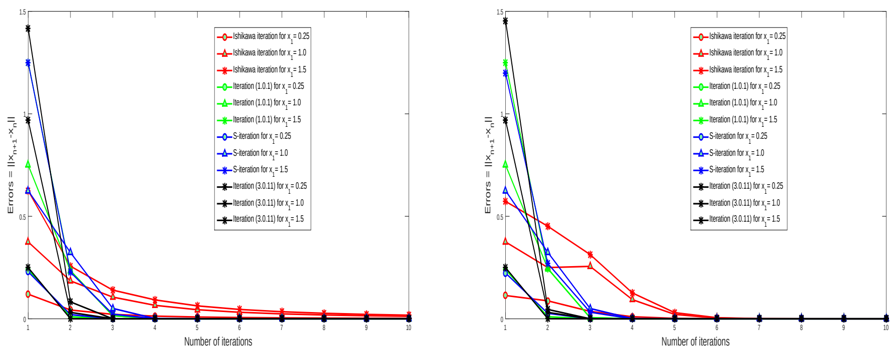


FIGURE 1. Convergence of modified inexact Krasnoselskii-Mann iteration



(A) $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{n}{n+1}$ (B) $\alpha_n = \frac{n}{n+1}$ and $\beta_n = \frac{n}{n+5}$

FIGURE 2. Comparison of errors for Ishikawa iteration, P-M iteration, S-iteration and the iteration (2.15)

Figure A and B represent the comparison of errors $\|x_{n+1} - x_n\|$ for the four iterative processes. It is clear that the errors converge to zero faster for iterative process (2.15) as compared to other iterative processes.

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