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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

Fixed point theorems and convergence theorems for some monotone generalized nonexpansive mappings

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ABSTRACT. We present some new coincidence fixed point theorems for generalized multi-valued weak Γ-contraction mappings. Our outcomes extend several recent results in the framework of complete metric spaces endowed with a graph. Two illustrative examples are included and some consequences are derived.

1. INTRODUCTION

Recall that monotone Lipschitzian mappings saw an uptick interest after the publication of an extension of Banach Contraction Principle to partially ordered metric spaces by Ran and Reurings [13] (see also Turinici [16]). Right thereafter, many authors tried to develop a metric fixed point theory for monotone Lipschitzian mappings, see for example [1, 4, 12]. The case of mappings which are not Lipschitzian in the traditional definition did not get much attention. In this work, we initiate such discussion by looking at the class of mappings introduced by Suzuki [15]. It is worth remembering that the difficulty in doing this lies in the correct formulation of the monotone counterpart. There are examples where the authors may be giving the wrong formulation when trying to extend to the monotone case [1, 6].

An interesting reference with many applications of the fixed point theory of monotone mappings is the excellent book by Carl and Heikkilä [7]. For more on metric fixed point theory, the reader may consult the book of Khamsi and Kirk [9], and for the geometry of Banach spaces, we recommend the book of Beauzamy [5].

2. BASIC DEFINITIONS AND PRELIMINARIES

The concept of monotone Lipschitzian mappings involve two structures: a partial order and a metric distance. We have to keep in mind that most of the spaces involved in applications have these two natural structures with interesting natural intertwining properties.

Throughout, we assume that $(X, \|.\|, \preceq)$ is a partially ordered Banach space whose order intervals are convex and closed. Recall that order intervals are any of the following subsets

$$[a, \to) = \{x \in X;\ a \preceq x\},\ (\leftarrow, b] = \{x \in X;\ x \preceq b\},\ and\ [a, b] = [a, \to) \cap (\leftarrow, b],$$

for any $a, b \in X$. Recall that $x \in X$ and $y \in X$ are said to be comparable if $x \preceq y$ or $y \preceq x$.

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Definition 2.1. Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space. Let K be a nonempty subset of X. A map $T : K \to K$ is said to be monotone if for any $x, y \in K$ such that $x \preceq y$, we have $T(x) \preceq T(y)$. A fixed point of T is any element $x \in K$ such that $T(x) = x$. The set of fixed points of T will be denoted by $Fix(T)$.

Suzuki in [15] introduced the concept of mapping satisfying condition (C) which we adapt here for monotone mappings.

Definition 2.2. Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space. Let K be a nonempty subset of X. A map $T : K \to K$ is said to be monotone (C)-nonexpansive if T is monotone and for any comparable elements $x, y \in K$, the following holds:

$$\text{(C)} \qquad \frac{1}{2}\,\|x - T(x)\| \le \|x - y\| \quad \textit{implies} \quad \|T(x) - T(y)\| \le \|x - y\|.$$

It is easy to check that monotone nonexpansive mappings (as introduced in [3]) are monotone (C)-nonexpansive. Moreover, if T is monotone (C)-nonexpansive and $z \in Fix(T)$, then for any x comparable to z, we have $0 = \frac{1}{2}\,\|z - T(z)\| \le \|x - z\|$ which will imply $\|T(x) - z\| \le \|x - z\|$. In other words, we have a monotone quasi-nonexpansive behavior.

Lemma 2.1. *Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space. Let K be a nonempty subset of X and $T : K \to K$ be a monotone (C)-nonexpansive mapping. The following hold:*

(i) If x and $T(x)$ are comparable, we have $\|T(x) - T^2(x)\| \le \|x - T(x)\|$.
(ii) If x, y and $T(x)$ are pairwise comparable, then either $\frac{1}{2}\|x - T(x)\| \le \|x - y\|$ or $\frac{1}{2}\|T(x) - T^2(x)\| \le \|T(x) - y\|$ holds.
(iii) If x, y and $T(x)$ are pairwise comparable, then either $\|T(x) - T(y)\| \le \|x - y\|$ or $\|T^2(x) - T(y)\| \le \|T(x) - y\|$ holds.

Proof. For (i), we use the simple fact $1/2\|x - T(x)\| \le \|x - T(x)\|$. Clearly (iii) follows directly from (ii). Thus, we only need to show that (ii) holds. Assume that the conclusion of (ii) fails. Then we must have

$$\|x - y\| < \frac{1}{2}\|x - T(x)\| \quad \textit{and} \quad \|T(x) - y\| < \frac{1}{2}\|T(x) - T^2(x)\|,$$

which combined with (i) implies

$$\begin{aligned} \|x - T(x)\| &\le \|x - y\| + \|T(x) - y\| \\ &< \frac{1}{2}\|x - T(x)\| + \frac{1}{2}\|T(x) - T^2(x)\| \\ &\le \frac{1}{2}\|x - T(x)\| + \frac{1}{2}\|x - T(x)\| \\ &= \|x - T(x)\|, \end{aligned}$$

which is a contradiction. Therefore the conclusion of (ii) holds. □

A direct implication of Lemma 2.1 is the following simple fact:

Lemma 2.2. *Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space. Let K be a nonempty subset of X and $T : K \to K$ be a monotone (C)-nonexpansive mapping. If x, y and $T(x)$ are pairwise comparable, then we have*

$$\|x - T(y)\| \le 3\|x - T(x)\| + \|x - y\|.$$

We close this section by the following important result discovered by Goebel and Kirk [8] (see also [14]):

Lemma 2.3. *Let $(X, \|.\|)$ be a Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in X and $\lambda \in (0,1)$. Assume that $x_{n+1} = \lambda\, y_n + (1-\lambda)\, x_n$ and $\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|$, for any $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$ holds.*

3. Main results

The following lemmas will be crucial throughout and deal with the behavior of the Krasnoselskij iteration [10] associated to monotone (C)-nonexpansive mappings.

Lemma 3.4. *Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space. Let K be a nonempty bounded convex subset of X and $T : K \to K$ be a monotone (C)-nonexpansive mapping. Let $x_0 \in K$ such that x_0 and $T(x_0)$ are comparable. Fix $\lambda \in [1/2, 1)$. Define the sequence $\{x_n\}$ by the successive iteration*

$$x_{n+1} = \lambda T(x_n) + (1-\lambda)x_n,$$

for any $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \|x_n - T(x_n)\| = 0$ holds, i.e. $\{x_n\}$ is an approximate fixed point sequence of T.

Proof. Note that since order intervals are convex, if $x_0 \preceq T(x_0)$ (resp. $T(x_0) \preceq x_0$), then $\{x_n\}$ is monotone increasing (resp. monotone decreasing). In fact, one can easily prove by induction that if $x_0 \preceq T(x_0)$, then we have

$$x_n \preceq x_{n+1} \preceq T(x_n) \preceq T(x_{n+1}),$$

for any $n \in \mathbb{N}$. Since

$$\frac{1}{2}\, \|x_n - T(x_n)\| \leq \lambda\, \|x_n - T(x_n)\| = \|x_n - x_{n+1}\|,$$

and x_n is comparable to x_{n+1}, the monotone (C)-nonexpansiveness of T implies

$$\|T(x_n) - T(x_{n+1})\| \leq \|x_n - x_{n+1}\|$$

for any $n \in \mathbb{N}$. Using Lemma 2.3, we conclude that $\lim\limits_{n\to\infty} \|x_n - T(x_n)\| = 0$. □

If we assume compactness to the assumptions of Lemma 3.4, we get the first convergence result for an iteration associated to a monotone (C)-nonexpansive mapping. Recall that a map $T : K \to K$ is said to be compact if $T(K)$ has a compact closure.

Theorem 3.1. *Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space. Let K be a nonempty bounded closed convex subset of X and $T : K \to K$ be a compact monotone (C)-nonexpansive mapping. Let $x_0 \in K$ such that x_0 and $T(x_0)$ are comparable. Fix $\lambda \in [1/2, 1)$. Define the sequence $\{x_n\}$ by the successive iteration*

$$x_{n+1} = \lambda T(x_n) + (1-\lambda)x_n,$$

for any $n \in \mathbb{N}$. Then $\{x_n\}$ converges to a point $z \in K$ which is a fixed point of T, i.e., $z \in Fix(T)$.

Proof. Without loss of generality, we may assume $x_0 \preceq T(x_0)$. From Lemma 3.4, we know that $\lim\limits_{n\to\infty} \|x_n - T(x_n)\| = 0$. Since T is compact, there exists a point z and a subsequence $\{T(x_{\phi(n)})\}$ which converges to z. Note that since K is closed, we have $z \in K$. Clearly, the subsequence $\{x_{\phi(n)}\}$ also converges to z. Since

$$x_n \preceq x_{n+1} \preceq T(x_n) \preceq T(x_{n+1}),$$

and order intervals are closed, we conclude that $x_n \preceq T(x_n) \preceq z$, for any $n \in \mathbb{N}$. Lemma 2.2 implies

$$\|x_{\phi(n)} - T(z)\| \leq 3\|T(x_{\phi(n)}) - x_{\phi(n)}\| + \|x_{\phi(n)} - z\|$$

for any $n \in \mathbb{N}$. Therefore $\{x_{\phi(n)}\}$ also converges to $T(z)$. Hence $T(z) = z$, i.e., z is a fixed point of T. The quasi-nonexpansiveness of T and z is comparable to the sequence $\{x_n\}$ imply

$$\begin{array}{rcl} \|x_{n+1} - z\| & \leq & \lambda \, \|T(x_n) - z\| + (1-\lambda) \, \|x_n - z\| \\ & \leq & \lambda \, \|x_n - z\| + (1-\lambda) \, \|x_n - z\| \\ & = & \|x_n - z\|, \end{array}$$

for any $n \in \mathbb{N}$. In other words, the sequence $\{\|x_n - z\|\}$ is a decreasing sequence of positive numbers of which a subsequence goes to 0. Hence $\lim\limits_{n\to\infty} \|x_n - z\| = 0$, i.e., $\{x_n\}$ converges to z. □

Next we discuss the case when compactness is dropped. As was done for nonexpansive mappings, we call upon Opial to help weaken this strong assumption.

Definition 3.3. Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space.

(i) [11] We say that X satisfies the weak-Opial property if for any sequence $\{x_n\}$ in X which converges weakly to x, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for any $y \neq x$.

(ii) [2] We say that X satisfies the monotone weak-Opial property if for any monotone sequence $\{x_n\}$ in X which converges weakly to x, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for any $y \neq x$ and y is greater or less than all the elements of the sequence $\{x_n\}$.

Usually when we try to weaken the compactness assumption, we use the Opial property. Then we say that ℓ_p, $p > 1$, and any Hilbert space satisfy this property for the weak topology. Of course, as it was noted by Opial, the Banach spaces $L_p([0,1)]$, for $p > 1$, fails the Opial property for the weak topology. This is bad since these spaces are uniformly convex. This is the reason for the introduction of the monotone weak-Opial property by the authors in [2]. Amazingly, they proved that $L_p([0,1)]$, for $p > 1$, satisfy the monotone weak-Opial property.

Recall that a map $T : K \to K$ is said to be weakly compact if $T(K)$ has a weakly compact closure. Moreover, note that since the order intervals are closed and convex, if a monotone sequence $\{x_n\}$ has a subsequence which weakly converges to some z, then $\{x_n\}$ weakly converges to z and $x_n \preceq z$ (resp. $z \preceq x_n$) if $\{x_n\}$ is monotone increasing (resp. monotone decreasing).

Theorem 3.2. *Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space which satisfies the monotone weak-Opial property. Let K be a nonempty bounded closed convex subset of X and $T : K \to K$ be a weakly compact monotone (C)-nonexpansive mapping. Let $x_0 \in K$ such that x_0 and $T(x_0)$ are comparable. Fix $\lambda \in [1/2, 1)$. Define the sequence $\{x_n\}$ by the successive iteration*

$$x_{n+1} = \lambda T(x_n) + (1-\lambda)x_n,$$

for any $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a point $z \in K$ which is a fixed point of T.

Proof. Without loss of generality, we may assume $x_0 \preceq T(x_0)$. From Lemma 3.4, we know that $\lim\limits_{n\to\infty} \|x_n - T(x_n)\| = 0$. Since T is weakly compact, there exists a point z and a subsequence $\{T(x_{\phi(n)})\}$ which converges weakly to z. Note that since K is closed and convex, we have $z \in K$. Note that $\{x_{\phi(n)}\}$ also weakly converges to z as well. Since $\{x_n\}$ is monotone increasing, then $\{x_n\}$ weakly converges to z as well as $\{T(x_n)\}$. Note that we have $x_n \preceq T(x_n) \preceq z$, which implies by monotonicity of T that $x_n \preceq T(z)$ for any $n \in \mathbb{N}$. Assume that $T(z) \neq z$, then by the monotone weak-Opial property, we have

$$\liminf_{n\to\infty} \|x_n - z\| \ < \ \liminf_{n\to\infty} \|x_n - T(z)\|.$$

On the other hand, Lemma 2.2 implies

$$\|x_n - T(z)\| \leq 3\|T(x_n) - x_n\| + \|x_n - z\|$$

for any $n \in \mathbb{N}$. Hence

$$\liminf_{n\to\infty} \|x_n - T(z)\| \ \leq \ \liminf_{n\to\infty} \|x_n - z\|.$$

This contradiction forces $T(z) = z$, i.e., z is a fixed point of T. □

If we combine the two above theorems, we get an existence fixed point theorem for monotone (C)-nonexpansive mappings.

Theorem 3.3. *Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space. Let K be a nonempty convex subset of X and $T : K \to K$ be a monotone (C)-nonexpansive mapping. Let $x_0 \in K$ such that x_0 and $T(x_0)$ are comparable. Assume either of the following assumptions holds*

(1) K is compact;

(2) K is weakly compact and X satisfies the monotone weak-Opial property.

Then T has a fixed point.

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