CARPATHIAN J. MATH. **36** (2020), No. 2, 205 - 214
Online version at https://www.carpathian.cunbm.utcluj.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

*Dedicated to Prof. Billy E. Rhoades on the occasion of his $90^{th}$ anniversary*

# Fixed point results for single valued and set valued $P$-contractions and application to second order boundary value problems

ISHAK ALTUN, HATICE ASLAN HANÇER and ALI ERDURAN

ABSTRACT. In this paper, by considering the concept of set-valued nonlinear $P$-contraction which is newly introduced, we present some new fixed point theorems for set-valued mappings on complete metric space. Then by considering a single-valued case we provide an existence and uniqueness result for a kind of second order two point boundary value problem.

## 1. INTRODUCTION AND PRELIMINARIES

Throughout this paper $(M,\rho)$ will be a metric space. $\mathcal{P}_C(M)$ will be the collection of all nonempty closed subsets of $M$, $\mathcal{P}_{CB}(M)$ will be the collection of all nonempty closed and bounded subsets of $M$ and $\mathcal{P}_K(M)$ will be the collection of all nonempty compact subsets of $M$. For $A,B\in\mathcal{P}_C(M)$, let

$$h_\rho(A,B)=\max\left\{\sup_{\zeta\in A}\rho(\zeta,B),\sup_{\eta\in B}\rho(\eta,A)\right\},$$

where $\rho(\zeta,B)=\inf\{\rho(\zeta,\eta):\eta\in B\}$. Then $h_\rho$ is called generalized Pompeiu-Hausdorff distance on $\mathcal{P}_C(M)$. It is well known that $h_\rho$ is a metric on $\mathcal{P}_{CB}(M)$, which is called Pompeiu-Hausdorff metric induced by $\rho$. We can find detailed information about the Pompeiu-Hausdorff metric in [3, 8].

An element $\zeta\in M$ is said to be a fixed point of a set-valued mapping $T:M\to\mathcal{P}(M)$, if $\zeta\in T\zeta$. Nadler [11] initiated the study of fixed point theory for set-valued mappings on metric space by introducing the concept of set-valued contraction mapping. A mapping, which is $\mathcal{P}_{CB}(M)$ valued, is said to be set-valued contraction if there exists $L\in[0,1)$ such that $h_\rho(T\zeta,T\eta)\leq L\rho(\zeta,\eta)$ for all $\zeta,\eta\in M$ (see [11]). Therefore, Nadler [11] proved that every set-valued contraction on complete metric space has a fixed point. This result is an extension of Banach fixed point theorem to set-valued case. Also by considering a certain function $\varphi$, which depending on the distance the points $\zeta$ and $\eta$, instead of the constant $L$ in the definition of set-valued contraction, Reich [13] presented another fixed point theorem. However, in his result, since the mapping $T$ is $\mathcal{P}_K(M)$ valued, it is different from Nadler's result. Then, Mizoguchi and Takahashi [10] proved the following theorem which includes both Nadler's and Reich's results:

Received: 03.05.2020. In revised form: 11.06.2020. Accepted: 12.06.2020
2010 *Mathematics Subject Classification.* 54H25, 47H10.
Key words and phrases. *Fixed point, set-valued mapping, P-contraction, boundary value problem.*
Corresponding author: Ishak Altun; ishak.altun@tdtu.edu.vn

**Theorem 1.1.** *Let $(M,\rho)$ be a complete metric space and $T : M \to \mathcal{P}_{CB}(M)$ be a mapping. Assume that there exists a map $\varphi : (0,\infty) \to [0,1)$ such that*

$$\limsup_{r\to t^+} \varphi(r) < 1$$

*for all $t \in [0,\infty)$ and for any $\zeta, \eta \in M$ and $\zeta \neq \eta$*

$$h_\rho(T\zeta, T\eta) \leq \varphi(\rho(\zeta,\eta))\rho(\zeta,\eta).$$

*Then $T$ has a fixed point.*

The mapping $\varphi$ which is mentioned in the above theorem is called Mizoguchi-Takahashi type function (for short MT-function) in the literature.

Also, taking into account the following set $I_b^\zeta$, Feng and Liu [4] gave a fixed point theorem, which includes the Nadler's result as properly, for $\mathcal{P}_C(M)$ valued mappings:

$$I_b^\zeta = \{\eta \in T\zeta : b\rho(\zeta,\eta) \leq \rho(\zeta, T\zeta)\},$$

where $b \in (0,1)$. Then Klim and Wardowski [9] extended and generalized the Feng and Liu's result by considering a MT-type function as follows:

**Theorem 1.2** ([9]). *Let $(M,\rho)$ be a complete metric space and $T : M \to \mathcal{P}_C(M)$ be a mapping. Assume that the following conditions hold:*

*(i) $f(\zeta) = \rho(\zeta, T\zeta)$ is lower semicontinuous;*

*(ii) there exist $b \in (0,1)$ and $\varphi : [0,\infty) \to (0,b)$ such that*

$$\limsup_{r\to t^+} \varphi(r) < b$$

*for all $t \in [0,\infty)$ and there exists $\eta \in I_b^\zeta$ such that*

$$\rho(\eta, T\eta) \leq \varphi(\rho(\zeta,\eta))\rho(\zeta,\eta)$$

*for any $\zeta \in M$.*

*Then $T$ has a fixed point in $M$.*

On the other hand, one of the numerous generalizations of the Banach Contraction Principle for single valued mappings was given by Popescu [12]. Popescu considered the following new contraction for a self mapping $T$ of $M$: for all $\zeta, \eta \in M$

$$\rho(T\zeta, T\eta) \leq k\left[\rho(\zeta,\eta) + |\rho(\zeta,T\zeta) - \rho(\eta,T\eta)|\right], \tag{1.1}$$

where $0 \leq k < 1$. We will call $P$-contraction to mapping $T$ which satisfies (1.1). Briefly, Popescu proved that every $P$-contraction on complete metric space has a unique fixed point. Then recently taking into account some modified version of $P$-contraction for single valued mappings, some authors presented new fixed point results (see [1, 5, 6]). The concept of $P$-contraction was first taken into account by Hançer [7] in terms of set-valued mappings and a version of Feng-Liu's result was obtained.

In this study, we will present a nonlinear version of the concept of set-valued $P$-contractions by taking the $MT$-function into account and then we will present a fixed point theorem for such mappings, which is a new version of the Klim-Wardowski's result.

## 2. Fixed point result

We will begin with the following definition:

**Definition 2.1.** Let $(M,\rho)$ metric space and $T : M \to \mathcal{P}(M)$ be a set-valued mapping. If there exist $b \in (0,1)$ and $\theta : [0,\infty) \to (0,b)$ such that

$$\limsup_{r\to t^+} \theta(r) < b \tag{2.2}$$

for all $t \in [0, \infty)$ and there exists $\eta \in I_b^\zeta$ such that

$$\rho(\eta, T\eta) \leq \varphi(\rho(\zeta, \eta))[\rho(\zeta, \eta) + |\rho(\zeta, T\zeta) - \rho(\eta, T\eta)|], \tag{2.3}$$

for any $\zeta \in M$, where $\varphi(t) = \frac{\theta(t)}{2-\theta(t)}$ for $t \geq 0$, then $T$ is called set-valued nonlinear $P$-contraction mapping.

Now we present our one of the main results.

**Theorem 2.3.** *Let $(M, \rho)$ be a complete metric space and $T : M \to \mathcal{P}_C(M)$ be a set-valued nonlinear $P$-contraction mapping. If the function $f(\zeta) = \rho(\zeta, T\zeta)$ is lower semicontinuous, then $T$ has a fixed point.*

**Remark 2.1.** Since $\varphi(r)$ is positive and $0 < \theta(r) = \frac{2\varphi(r)}{1+\varphi(r)} < b < 1$ for $r \geq 0$, we have $0 < \varphi(r) < b < 1$ and also we have $\left(\frac{\varphi(r)}{1-\varphi(r)}\right)\left(\frac{1-b}{b}\right) < 1$ for $r \geq 0$.

*Proof of Theorem 2.3.* Suppose that $T$ has no fixed point, so $\rho(\zeta, T\zeta) > 0$ for each $\zeta \in M$. Since $T\zeta \in \mathcal{P}_C(M)$ for any $\zeta \in M$, then $I_b^\zeta$ is nonempty for any constant $b \in (0, 1)$. Also note that since $\rho(\zeta, T\zeta) > 0$ then $\zeta \notin I_b^\zeta$ for each $\zeta \in M$. Let $\zeta_0 \in M$ be an arbitrary point. By the assumption, there exists $\zeta_1 \in I_b^{\zeta_0}, \zeta_1 \neq \zeta_0$ such that

$$\rho(\zeta_1, T\zeta_1) \leq \varphi(\rho(\zeta_0, \zeta_1))[\rho(\zeta_0, \zeta_1) + |\rho(\zeta_0, T\zeta_0) - \rho(\zeta_1, T\zeta_1)|],$$

and, for $\zeta_1 \in M$, there exists $\zeta_2 \in I_b^{\zeta_1}, \zeta_1 \neq \zeta_2$ such that

$$\rho(\zeta_2, T\zeta_2) \leq \varphi(\rho(\zeta_1, \zeta_2))[\rho(\zeta_1, \zeta_2) + |\rho(\zeta_1, T\zeta_1) - \rho(\zeta_2, T\zeta_2)|].$$

Continuing this process, we can construct an iterative sequence $\{\zeta_n\}$ such that $\zeta_{n+1} \in I_b^{\zeta_n}, \zeta_{n+1} \neq \zeta_n$ such that

$$\begin{aligned} \rho(\zeta_{n+1}, T\zeta_{n+1}) \quad &\leq \quad \varphi(\rho(\zeta_n, \zeta_{n+1}))[\rho(\zeta_n, \zeta_{n+1}) \\ &\quad + |\rho(\zeta_n, T\zeta_n) - \rho(\zeta_{n+1}, T\zeta_{n+1})|], \end{aligned} \tag{2.4}$$

for $n = 0, 1, 2, \ldots$. Now, if there exists $m \in \mathbb{N}$ such that

$$\rho(\zeta_{m+1}, T\zeta_{m+1}) \geq \rho(\zeta_m, T\zeta_m),$$

then from (2.4) we have (note that, $b\rho(\zeta_m, \zeta_{m+1}) \leq \rho(\zeta_m, T\zeta_m)$ since $\zeta_{m+1} \in I_b^{\zeta_m}$)

$$\begin{aligned} \rho(\zeta_{m+1}, T\zeta_{m+1}) \quad &\leq \quad \varphi(\rho(\zeta_m, \zeta_{m+1}))[\rho(\zeta_m, \zeta_{m+1}) \\ &\quad + |\rho(\zeta_m, T\zeta_m) - \rho(\zeta_{m+1}, T\zeta_{m+1})|] \\ &= \quad \varphi(\rho(\zeta_m, \zeta_{m+1}))[\rho(\zeta_m, \zeta_{m+1}) \\ &\quad + \rho(\zeta_{m+1}, T\zeta_{m+1}) - \rho(\zeta_m, T\zeta_m)] \end{aligned}$$

and so from Remark 2.1 we have

$$\begin{aligned}
\rho(\zeta_{m+1},T\zeta_{m+1}) &\leq \frac{\varphi(\rho(\zeta_m,\zeta_{m+1}))}{1-\varphi(\rho(\zeta_m,\zeta_{m+1}))}\rho(\zeta_m,\zeta_{m+1}) \\
&\quad -\frac{\varphi(\rho(\zeta_m,\zeta_{m+1}))}{1-\varphi(\rho(\zeta_m,\zeta_{m+1}))}\rho(\zeta_m,T\zeta_m) \\
&= \frac{\varphi(\rho(\zeta_m,\zeta_{m+1}))}{1-\varphi(\rho(\zeta_m,\zeta_{m+1}))}[\rho(\zeta_m,\zeta_{m+1})-\rho(\zeta_m,T\zeta_m)] \\
&\leq \frac{\varphi(\rho(\zeta_m,\zeta_{m+1}))}{1-\varphi(\rho(\zeta_m,\zeta_{m+1}))}[\frac{1}{b}\rho(\zeta_m,T\zeta_m)-\rho(\zeta_m,T\zeta_m)] \\
&\leq \left(\frac{\varphi(\rho(\zeta_m,\zeta_{m+1}))}{1-\varphi(\rho(\zeta_m,\zeta_{m+1}))}\right)\left(\frac{1-b}{b}\right)\rho(\zeta_m,T\zeta_m) \\
&< \rho(\zeta_m,T\zeta_m) \\
&\leq \rho(\zeta_{m+1},T\zeta_{m+1}),
\end{aligned}$$

which is a contradiction. Therefore $\rho(\zeta_{n+1},T\zeta_{n+1}) < \rho(\zeta_n,T\zeta_n)$ for all $n \in \mathbb{N}$. Thus, we have from (2.4)

$$\rho(\zeta_{n+1},T\zeta_{n+1}) \leq \varphi(\rho(\zeta_n,\zeta_{n+1}))[\rho(\zeta_n,\zeta_{n+1})+\rho(\zeta_n,T\zeta_n)-\rho(\zeta_{n+1},T\zeta_{n+1})]$$

and so

$$\begin{aligned}
\rho(\zeta_{n+1},T\zeta_{n+1}) &\leq \frac{\varphi(\rho(\zeta_n,\zeta_{n+1}))}{1+\varphi(\rho(\zeta_n,\zeta_{n+1}))}[\rho(\zeta_n,\zeta_{n+1})+\rho(\zeta_n,T\zeta_n)] \\
&\leq \frac{2\varphi(\rho(\zeta_n,\zeta_{n+1}))}{1+\varphi(\rho(\zeta_n,\zeta_{n+1}))}\rho(\zeta_n,\zeta_{n+1}).
\end{aligned}$$

Now since $\zeta_{n+2} \in I_b^{\zeta_{n+1}}$ we have

$$\begin{aligned}
b\rho(\zeta_{n+1},\zeta_{n+2}) &\leq \rho(\zeta_{n+1},T\zeta_{n+1}) \\
&\leq \frac{2\varphi(\rho(\zeta_n,\zeta_{n+1}))}{1+\varphi(\rho(\zeta_n,\zeta_{n+1}))}\rho(\zeta_n,\zeta_{n+1}),
\end{aligned}$$

and so

$$\begin{aligned}
\rho(\zeta_{n+1},\zeta_{n+2}) &\leq \frac{2\varphi(\rho(\zeta_n,\zeta_{n+1}))}{b(1+\varphi(\rho(\zeta_n,\zeta_{n+1})))}\rho(\zeta_n,\zeta_{n+1}) \\
&= \frac{\theta(\rho(\zeta_n,\zeta_{n+1}))}{b}\rho(\zeta_n,\zeta_{n+1}).
\end{aligned}$$

Therefore, we obtain

$$\rho(\zeta_{n+1},\zeta_{n+2}) < \rho(\zeta_n,\zeta_{n+1}), \tag{2.5}$$

for all $n \in \mathbb{N}$, since $\theta(\rho(\zeta_n,\zeta_{n+1})) < b$. By (2.5), there exists $\delta \geq 0$ such that $\lim_{n\to\infty}\rho(\zeta_n,\zeta_{n+1}) = \delta^+$. Hence from (2.2) there exists $q \in [0,b)$ such that

$$\limsup_{n\to\infty}\frac{2\varphi(\rho(\zeta_n,\zeta_{n+1}))}{1+\varphi(\rho(\zeta_n,\zeta_{n+1}))} = \limsup_{n\to\infty}\theta(\rho(\zeta_n,\zeta_{n+1})) = q.$$

Therefore, for any $b_0 \in (q,b)$, there exists $n_0 \in \mathbb{N}$ such that

$$\theta(\rho(\zeta_n,\zeta_{n+1})) < b_0 \tag{2.6}$$

for all $n > n_0$. Consequently, from (2.4) and (2.6), for $n > n_0$,we have (note that, $b\rho(\zeta_m, \zeta_{m+1}) \leq \rho(\zeta_m, T\zeta_m)$ since $\zeta_{m+1} \in I_b^{\zeta_m}$)

$$\begin{aligned}
\rho(\zeta_{n+1}, \zeta_{n+2}) &\leq \frac{\theta(\rho(\zeta_n, \zeta_{n+1}))}{b}\rho(\zeta_n, \zeta_{n+1}) \\
&\leq \frac{\theta(\rho(\zeta_n, \zeta_{n+1}))}{b}\frac{\theta(\rho(\zeta_{n-1}, \zeta_n))}{b}\rho(\zeta_{n-1}, \zeta_n) \\
&\vdots \\
&\leq \frac{\theta(\rho(\zeta_n, \zeta_{n+1}))}{b}\frac{\theta(\rho(\zeta_{n-1}, \zeta_n))}{b}\cdots\frac{\theta(\rho(\zeta_0, \zeta_1))}{b}\rho(\zeta_0, \zeta_1) \\
&= \frac{1}{b^n}\{\theta(\rho(\zeta_n, \zeta_{n+1}))\theta(\rho(\zeta_{n-1}, \zeta_n))\cdots\theta(\rho(\zeta_0, \zeta_1))\}\rho(\zeta_0, \zeta_1) \\
&\leq \frac{\rho(\zeta_0, \zeta_1)}{b^n}\{b_0^{n-n_0}\theta(\rho(\zeta_{n_0-1}, \zeta_{n_0}))\cdots\theta(\rho(\zeta_0, \zeta_1))\} \\
&= \left(\frac{b_0}{b}\right)^{n-n_0}\frac{\theta(\rho(\zeta_{n_0-1}, \zeta_{n_0}))\cdots\theta(\rho(\zeta_0, \zeta_1))}{b^{n_0}}\rho(\zeta_0, \zeta_1).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\sum_{n=n_0}^{\infty}\rho(\zeta_{n+1}, \zeta_{n+2}) &\leq \sum_{n=n_0}^{\infty}\left(\frac{b_0}{b}\right)^{n-n_0}\frac{\theta(\rho(\zeta_{n_0-1}, \zeta_{n_0}))\cdots\theta(\rho(\zeta_0, \zeta_1))}{b^{n_0}}\rho(\zeta_0, \zeta_1) \\
&= \frac{\theta(\rho(\zeta_{n_0-1}, \zeta_{n_0}))\cdots\theta(\rho(\zeta_0, \zeta_1))}{b^{n_0}}\rho(\zeta_0, \zeta_1)\sum_{n=n_0}^{\infty}\left(\frac{b_0}{b}\right)^{n-n_0} \\
&< \infty.
\end{aligned}$$

This shows that $\sum_{n=n_0}^{\infty}\rho(\zeta_{n+1}, \zeta_{n+2})$ is convergent and so $\{\zeta_n\}$ is a Cauchy sequence. According to the completeness of $\zeta$, there exists $\xi \in M$ such that $\{\zeta_n\}$ converges to $\xi$. Since $f$ is lower semicontinuous, we find that

$$\begin{aligned}
\rho(\xi, T\xi) &= f(\xi) \\
&\leq \liminf f(\zeta_n) \\
&= \liminf \rho(\zeta_n, T\zeta_n) \\
&\leq \liminf \rho(\zeta_n, \zeta_{n+1}) = 0,
\end{aligned}$$

which contradict to the assumption that $T$ has no fixed point. Therefore $T$ has a fixed point. □

If the mapping $T$ is $\mathcal{P}_K(M)$ valued in Theorem 2.3, then the constant $b$ can be relaxed as 1. Hence we present the following result.

**Theorem 2.4.** *Let $(M, \rho)$ complete metric space and $T : M \to \mathcal{P}_K(M)$ be a set-valued mapping. Assume that the following conditions hold:*
*(i) $f(\zeta) = \rho(\zeta, T\zeta)$ is lower semicontinuous;*
*(ii) there exists $\theta : [0, \infty) \to (0, 1)$ such that*

$$\limsup_{r\to t^+}\theta(r) < 1$$

*for all $t \in [0, \infty)$ and for any $\zeta \in M$ there exists $\eta \in I_1^{\zeta}$ such that*

$$\rho(\eta, T\eta) \leq \varphi(\rho(\zeta, \eta))[\rho(\zeta, \eta) + |\rho(\zeta, T\zeta) - \rho(\eta, T\eta)|],$$

*where $\varphi(t) = \frac{\theta(t)}{2-\theta(t)}$ for $t \geq 0$. Then $T$ has a fixed point in $M$.*

*Proof.* Suppose that $T$ has no fixed point, so $\rho(\zeta, T\zeta) > 0$ for each $\zeta \in M$. Since $T\zeta \in \mathcal{P}_K(M)$ for any $\zeta \in M$, then there exists $\eta \in T\zeta$ such that $\rho(\zeta,\eta) = \rho(\zeta, T\zeta)$. Therefore $\eta \in I_1^\zeta$, that is, $I_1^\zeta$ is nonempty. Also since $\rho(\zeta, T\zeta) > 0$, then $\zeta \neq \eta$.

Let $\zeta_0 \in M$ be an arbitrary point. By the assumption, there exists $\zeta_1 \in T\zeta_0, \zeta_1 \neq \zeta_0$ such that

$$\rho(\zeta_0, \zeta_1) = \rho(\zeta_0, T\zeta_0),$$

and

$$\rho(\zeta_1, T\zeta_1) \leq \varphi(\rho(\zeta_0, \zeta_1))[\rho(\zeta_0, \zeta_1) + |\rho(\zeta_0, T\zeta_0) - \rho(\zeta_1, T\zeta_1)|].$$

Similarly, for $\zeta_1 \in M$, there exists $\zeta_2 \in T\zeta_1, \zeta_1 \neq \zeta_2$ such that

$$\rho(\zeta_1, \zeta_2) = \rho(\zeta_1, T\zeta_1),$$

and

$$\rho(\zeta_2, T\zeta_2) \leq \varphi(\rho(\zeta_1, \zeta_2))[\rho(\zeta_1, \zeta_2) + |\rho(\zeta_1, T\zeta_1) - \rho(\zeta_2, T\zeta_2)|].$$

Continuing this way we can construct a sequence $\{\zeta_n\}$ in $M$ such that $\zeta_{n+1} \in T\zeta_n, \zeta_{n+1} \neq \zeta_n$,

$$\rho(\zeta_n, \zeta_{n+1}) = \rho(\zeta_n, T\zeta_n)$$

and

$$\rho(\zeta_{n+1}, T\zeta_{n+1}) \leq \varphi(\rho(\zeta_n, \zeta_{n+1}))[\rho(\zeta_n, \zeta_{n+1}) + |\rho(\zeta_n, T\zeta_n) - \rho(\zeta_{n+1}, T\zeta_{n+1})|].$$

Since $\varphi(\rho(\zeta_n, \zeta_{n+1})) < 1$, then using the analogous method like in the proof of Theorem 2.3, we obtain that $\{\zeta_n\}$ is Cauchy sequence. According to the completeness of $M$, there exists $\xi \in M$ such that $\{\zeta_n\}$ converges to $\xi$. Since $f$ is lower semicontinuous, we find that

$$\begin{aligned} 0 &\leq \rho(\xi, T\xi) = f(\xi) \\ &\leq \liminf f(\zeta_n) \\ &= \liminf \rho(\zeta_n, T\zeta_n) \\ &\leq \liminf \rho(\zeta_n, \zeta_{n+1}) = 0, \end{aligned}$$

which contradict to the assumption that $T$ has no fixed point. Therefore $T$ has a fixed point. □

The following corollaries can be obtained from Theorem 2.3 and Theorem 2.4.

**Corollary 2.1** (Theorem 2 in [7]). *Let $(M, \rho)$ complete metric space and $T : M \to \mathcal{P}_C(M)$ be a set-valued mapping. If for any $\zeta \in M$ there exists $\eta \in I_b^\zeta$ such that*

$$\rho(\eta, T\eta) \leq c[\rho(\zeta, \eta) + |\rho(\zeta, T\zeta) - \rho(\eta, T\eta)|],$$

*where $c$ is a positive real number satisfying $\frac{2c}{b(1+c)} < 1$. Then $T$ has a fixed point in $M$ provided that $f(\zeta) = \rho(\zeta, T\zeta)$ is lower semicontinuous.*

*Proof.* If we consider a constant map $\varphi(t) = c$ in Theorem 2.3, then the proof is complete. □

**Corollary 2.2** (Theorem 3 in [7] ). *Let $(M, \rho)$ complete metric space and $T : M \to \mathcal{P}_K(M)$ be a set-valued mapping. If for any $\zeta \in M$ there exists $\eta \in I_1^\zeta$ such that*

$$\rho(\eta, T\eta) \leq c[\rho(\zeta, \eta) + |\rho(\zeta, T\zeta) - \rho(\eta, T\eta)|]$$

*where $c$ is a positive real number satisfying $c < 1$. Then $T$ has a fixed point in $M$ provided that $f(\zeta) = \rho(\zeta, T\zeta)$ is lower semicontinuous.*

The following result can also be obtained from Theorem 2.3.

**Theorem 2.5.** *Let $(M,\rho)$ complete metric space and $T : M \to \mathcal{P}_C(M)$ be a set-valued mapping. Assume that the following conditions hold:*
*(i) $f(\zeta) = \rho(\zeta, T\zeta)$ is lower semicontinuous;*
*(ii) there exists a function $\theta : [0,\infty) \to (0,1)$ such that*

$$\sup_{t\in[0,\infty)} \left\{ \limsup_{r\to t^+} \theta(r) \right\} < 1 \tag{2.7}$$

*and for any $\zeta \in M$ and $\eta \in T\zeta$ such that*

$$\rho(\eta, T\eta) \leq \varphi(\rho(\zeta,\eta))[\rho(\zeta,\eta) + |\rho(\zeta, T\zeta) - \rho(\eta, T\eta)|],$$

*where $\varphi(t) = \frac{\theta(t)}{2-\theta(t)}$ for $t \geq 0$. Then $T$ has a fixed point in $M$.*

*Proof.* By the assumption (2.7), there exists $b(t) \in (0,1)$ for all $t \geq 0$ such that

$$\limsup_{r\to t^+} \theta(r) = b(t)$$

and

$$\sup_{t\in[0,\infty)} b(t) = b < 1.$$

Since $T\zeta$ is closed for every $\zeta \in M$, then $I_b^\zeta$ is nonempty. Therefore, for any $\zeta \in M$ there exists $\eta \in I_b^\zeta$ such that (2.3) holds. Thus from Theorem 2.3, $T$ has a fixed point. □

If we take the mapping $T$ as a single-valued mapping in Theorem 2.4, we can obtain the following corollary. Note that in this case we do not need the lower-semicontinuity of $f$. Moreover, we can obtain the uniqueness of the fixed point.

**Corollary 2.3.** *Let $(M,\rho)$ complete metric space and $T : M \to M$ be a mapping. Assume that there exists $\theta : [0,\infty) \to (0,1)$ satisfying*

$$\limsup_{r\to t^+} \theta(r) < 1$$

*for all $t \in [0,\infty)$ such that for any $\zeta, \eta \in M$*

$$\rho(T\zeta, T\eta) \leq \varphi(\rho(\zeta,\eta))[\rho(\zeta,\eta) + |\rho(\zeta, T\zeta) - \rho(\eta, T\eta)|],$$

*where $\varphi(t) = \frac{\theta(t)}{2-\theta(t)}$ for $t \geq 0$. Then $T$ has a unique fixed point in $M$.*

Now we give an illustrative example.

**Example 2.1.** Let $M = \{\frac{(-1)^n}{2^n} : n = 1, 2, \cdots\} \cup \{0,1\}$ with the usual metric defined by $\rho(x,y) = |x-y|$ for $x, y \in X$. Define a mapping $T : X \to \mathcal{P}_C(M)$ as

$$Tx = \begin{cases} \{\zeta\} & , \quad \zeta \in \{0,1\} \\ & , \\ \{\frac{(-1)^{n+1}}{2^{n+1}}, \frac{(-1)^n}{2^{n+2}}\} & , \quad \zeta = \frac{(-1)^n}{2^n}, n \geq 1 \end{cases}.$$

In this case since

$$f(\zeta) = \rho(\zeta, T\zeta) = \begin{cases} 0 & , \quad \zeta \in \{0,1\} \\ \frac{3}{2^{n+2}} & , \quad \zeta = \frac{(-1)^n}{2^n}, n \geq 1 \end{cases},$$

then $f$ is lower semicontinuous. Now we claim that the inequality (2.3) is satisfied with $\varphi(t) = \frac{2}{27}$ and $b = \frac{1}{2}$. To see this we have to consider the following cases:

Case 1. Let $\zeta \in \{0,1\}$, then for $\eta = \zeta \in I^x_{\frac{1}{2}}$ we have $\rho(\eta, T\eta) = \rho(\zeta, T\zeta) = 0$ and so (2.3) holds.

Case 2. Let $\zeta = \frac{(-1)^n}{2^n}$ for $n \geq 1$, then for $\eta = \frac{(-1)^{n+1}}{2^{n+1}} \in I^{\zeta}_{\frac{1}{2}}$ we have

$$\rho(\eta, T\eta) = \frac{3}{2^{n+3}}$$

and

$$\rho(\zeta, \eta) + |\rho(\zeta, T\zeta) - \rho(\eta, T\eta)| = \frac{3}{2^{n+1}} + \frac{3}{2^{n+4}} = \frac{27}{2^{n+4}}.$$

Thus we have

$$\rho(\eta, T\eta) = \frac{3}{2^{n+3}} = \frac{2}{27}\frac{27}{2^{n+4}} = \frac{2}{27}\left[\rho(\zeta, \eta) + |\rho(\zeta, T\zeta) - \rho(\eta, T\eta)|\right].$$

Therefore all conditions of Theorem 2.3 (and also Theorem 2.4) are satisfied and so $T$ has a fixed point in $X$.

## 3. Application to second order two point boundary value problem

In this section by considering the Corollary 2.3 we present an existence and uniqueness theorem for the second order two point boundary value problem as follows:

$$\begin{cases} -\frac{d^2u}{dt^2} = f(t, u(t)), \ t \in [0, 1] \\ u(0) = u(1) = 0 \end{cases}, \tag{3.8}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. By considering some certain conditions on the function $f$, many existence results provided for this problem in the literature (see [2, 14]). Here, we will consider a different type condition on $f$ which is more general, and so we provide a new existence and uniqueness result for this problem.

By considering the Green's function defined as

$$G(t, s) = \begin{cases} t(1-s) & , \quad 0 \leq t \leq s \leq 1 \\ \\ s(1-t) & , \quad 0 \leq s \leq t \leq 1 \end{cases}$$

we can see that the problem (3.8) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t, s)f(s, u(s))ds, \ t \in [0, 1]. \tag{3.9}$$

Therefore, $u \in C^2[0, 1]$ is a solution of (3.8) if and only if it is a solution of (3.9). Hence, the existence of solution of (3.8) can be considered as the existence of fixed point of the operator $T$ defined on $X = C[0, 1]$ by

$$Tu(t) = \int_0^1 G(t, s)f(s, u(s))ds, \tag{3.10}$$

where $X = C[0, 1]$ is the space of all continuous real vallued functions defined on $[0, 1]$. It is clear that $\int_0^1 G(t, s)ds = \frac{t(1-t)}{2}$ and thus $\sup_{t\in[0,1]} \int_0^1 G(t, s)ds = \frac{1}{8}$.

Consider the uniform metric $\rho_\infty$ on $X$, that is,

$$\rho_\infty(u, v) = \|u - v\| = \sup\{|u(t) - v(t)| : t \in [0, 1]\},$$

then, it is well known that the space $(X, \rho_\infty)$ is complete.

**Theorem 3.6.** *Suppose the following conditions hold:*

*(i) there exists a continuous function $p : [0, 1] \to [0, \infty)$ satisfying*

$$|f(t, u(t)) - f(t, v(t))| \leq p(t)\left\{|u(t) - v(t)| + |\|u - Tu\| - \|v - Tv\||\right\},$$

*(ii) there exists $c < 1$ such that $\int_0^1 G(t, s)p(s)ds \leq c$.*

*Then the second order two point boundary value problem given by (3.8) has a unique solution.*

**Remark 3.2.** Note that if $\max_{s\in[0,1]} p(s) \leq 8c$ for $c < 1$, we have $\int_0^1 G(t,s)p(s)ds \leq c$.

*Proof of Theorem 3.6.* Consider the complete metric space $(X, \rho_\infty)$ and the operator $T$ on $X$ given by (3.10). Then for any $u, v \in X$ and $t \in [0,1]$ we have

$$\begin{aligned}
|Tu(t) - Tv(t)| &= \left|\int_0^1 G(t,s)f(s,u(s))ds - \int_0^1 G(t,s)f(s,v(s))ds\right| \\
&\leq \int_0^1 G(t,s)\left|f(s,u(s)) - f(s,v(s))\right| ds \\
&\leq \int_0^1 G(t,s)p(s)\left\{|u(s)-v(s)| + |\|u-Tu\| - \|v-Tv\||\right\} ds \\
&\leq \left\{\|u-v\| + |\|u-Tu\| - \|v-Tv\||\right\} \int_0^1 G(t,s)p(s)ds \\
&\leq c\left\{\|u-v\| + |\|u-Tu\| - \|v-Tv\||\right\}.
\end{aligned}$$

Hence we have

$$\|Tu - Tv\| \leq c\left\{\|u-v\| + |\|u-Tu\| - \|v-Tv\||\right\}$$

or equivalently

$$\rho_\infty(Tu, Tv) \leq c\left\{\rho_\infty(u,v) + |\rho_\infty(u,Tu) - \rho_\infty(v,Tv)|\right\}.$$

Therefore the contractive condition of Corollary 2.3 is satisfied for $\varphi(t) = c$ and $\theta(t) = \frac{2c}{1+c}$. Hence the operator $T$ has a unique fixed point, and thus the second order two point boundary value problem given by (3.8) has a unique solution. □

## 4. Conclusions

We introduced the class of set-valued nonlinear $P$-contraction, which includes the classes of set-valued contractions in the sense of Hançer [7], Feng-Liu [4] and Klim- Wardowski [9]. We obtained fixed point results when such mappings are closed (compact) set-valued. We presented some corollaries for single valued mappings. We gave an illustrative example. Finally, we presented an existence and uniqueness theorem for a kind of second order two point boundary value problem.

**Acknowledgement.** The authors are grateful to the referees because their suggestions contributed to improve the paper.

## References

[1] Altun, I., Durmaz, G. and Olgun, M., *P-contractive mappings on metric spaces*, J. Nonlinear Funct. Anal., **2018** (2018), Article ID 43, pp. 1–7

[2] Aydi, H. and Karapinar, E., *Fixed point results for generalized $\alpha$-$\psi$-contractions in metric-like spaces and applications*, Electron J Differ Equ. **133** (2015), 1–15

[3] Berinde, V. and Păcurar, M., *The role of the Pompeiu-Hausdorff metric in fixed point theory*, Creat. Math. Inform., **22** (2013), No. 2, 35–42

[4] Feng, Y. and Liu, S., *Fixed point theorems for multivalued contractive mappings and multivalued Caristi type mappings*, J. Math. Anal. Appl., **317** (2006), 103–112

[5] Fulga, A. and Proca, A., *A new generalization of Wardowski fixed point theorem in complete metric spaces*, Advances in the Theory of Nonlinear Analysis and its Applications, **1** (2017), No. 1, 57–63

[6] Fulga, A. and Proca, A., *Fixed points for $\varphi_E$-Geraghty contractions*, J. Nonlinear Sci. Appl., **10** (2017), No. 9, 5125–5131

[7] Hançer, H. A., *On multivalued $P$-contractive mappings that belongs to class of weakly Picard operators*, Fixed Point Theory (Cluj). Accepted.

[8] Istrăţescu, V. I., *Fixed Point Theory an Introduction*, Dordrecht D. Reidel Publishing Company, 1981

[9] Klim, D. and Wardowski, D., *Fixed point theorems for set-valued contractions in complete metric spaces*, J. Math. Anal. Appl., **334** (2007), 132–139

[10] Mizoguchi, N. and Takahashi, W., *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl., **141** (1989), 177–188
[11] Nadler, S. B., *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488
[12] Popescu, O., *Fixed point theorem in metric spaces*, Bull. of Transilvania Univ., **1** (2008), 479–482
[13] Reich, S., *Fixed points of contractive functions*, Boll. Un. Mat. Ital., **4** (1972), No. 5, 26–42
[14] Saipara, P., Khammahawong, K. and Kumam, P., *Fixed-point theorem for a generalized almost Hardy-Rogers-type $F$ contraction on metric-like spaces*, Math. Methods Appl. Sci., **42** (2019), No. 17, 5898–5919

NONLINEAR ANALYSIS RESEARCH GROUP
TON DUC THANG UNIVERSITY
HO CHI MINH CITY, VIETNAM

FACULTY OF MATHEMATICS AND STATISTICS
TON DUC THANG UNIVERSITY
HO CHI MINH CITY, VIETNAM
*Email address*: ishak.altun@tdtu.edu.vn

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND ARTS
KIRIKKALE UNIVERSITY
71450 YAHSIHAN, KIRIKKALE, TURKEY
*Email address*: haticeaslanhancer@gmail.com
*Email address*: ali.erduran1@yahoo.com