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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

A probabilistic Meir-Keeler type fixed point theorem which characterizes metric completeness

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ABSTRACT. A probabilistic version of the Meir-Keeler type fixed point theorem, which characterizes completeness of the metric space is established. In addition to it, a fixed point theorem for non-expansive mappings satisfying $(\epsilon - \delta)$ type condition in Menger probabilistic metric space (Menger PM-space) is proved. As a byproduct we find an affirmative answer to the open question on the existence of contractive mappings which admit discontinuity at the fixed point (see Rhoades, B. E., *Contractive definitions and continuity*, Contemporary Mathematics **72** (1988), 233–245, p. 242) in the setting of Menger PM-space.

1. INTRODUCTION

In a seminal work, Menger [10] introduced the notion of a statistical metric space as an extension of a metric space (X, d), in which the distance $d(x, y)$ $(x, y \in X)$ was replaced by a distributive function $F_{x,y} : X \times X \to \mathbb{R}$, where $F_{x,y}(t)$ represents the probability that the distance between x and y is less than t. Following Menger, Schweizer and Sklar [21, 22] gave detailed study of various properties, namely, topology, convergence of sequences, continuity of mappings, completeness, etc., of these spaces. In 1972, Sehgal and Bharucha–Reid [23] proved the first fixed point result in probabilistic metric space, which was the probabilistic metric version of the classical Banach contraction mapping principle. Since then the study of fixed point theorems in probabilistic metric space (PM-space) has emerged as an active area of research.

In 1971, Ćirić [5] introduced the notion of orbital continuity. If f is a self-mapping of a metric space (X, d) then the set $O_f(x) = \{f^n x \mid n = 0, 1, 2, \ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i}x$ implies $fu = \lim_i ff^{m_i}x$. Every continuous self-mapping is orbitally continuous but not conversely. In a recent work Pant and Pant [13] introduced the notion of $k-$continuity. A self-mapping f of a metric space X is called k-continuous, $k = 1, 2, 3, \ldots$, if $f^k x_n \to ft$ whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $f^{k-1}x_n \to t$. It may be observed that 1-continuity is equivalent to continuity and continuity implies 2-continuity, 2-continuity implies 3-continuity and so on but not conversely.

More recently Pant et al. [14] introduced the notion of weak orbital continuity.

Definition 1.1. A self-mapping f of a metric space (X, d) is called weakly orbitally continuous [14] if the set $\{y \in X : lim_i f^{m_i}y = u$ implies $lim_i ff^{m_i}y = fu\}$ is nonempty, whenever the set $\{x \in X : lim_i f^{m_i}x = u\}$ is nonempty.

Definition 1.2. Let (X, d) be a metric space and $f : X \to X$. A mapping $H : X \to \mathbb{R}$ is said to be f-orbitally lower semi-continuous [7] at a point $z \in X$ if $\{x_n\}$ is a sequence in $O_f(x)$ for some $x \in X$, $\lim_{n\to+\infty} x_n = z$ implies $H(z) \le \liminf_{n\to+\infty} H(x_n)$.

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Example 1.1. Let $X = [0, 2]$ and d be the usual metric. Define $f : X \to X$ by

$$fx = \frac{(1+x)}{2} \quad \text{if } 0 \leq x < 1, \quad fx = 0 \quad \text{if } 1 \leq x < 2, \quad f2 = 2.$$

Then [14]:

(i) f is not orbitally continuous. Since $f^n 0 \to 1$ and $f(f^n 0) \to 1 \neq f1$.

(ii) f is weakly orbitally continuous. If we take $x = 2$ then $f^n 2 \to 2$ and $f(f^n 2) \to 2 = f2$.

(iii) f is not $k-$continuous. If we consider the sequence $\{f^n 0\}$, then for any integer ≥ 1, we have $f^{k-1}(f^n 0) \to 1$ and $f^k(f^n 0) \to 1 \neq f1$.

Example 1.2. Let $X = [0, +\infty)$ equipped with usual metric and let $f : X \to X$ be defined by

$$fx = 1 \text{ if } 0 \leq x \leq 1, \quad fx = \frac{x}{5} \text{ if } x > 1.$$

Then f is orbitally continuous. Let $k \geq 1$ be any integer. Consider the sequence $\{x_n\}$ given by $x_n = 5^{k-1} + \frac{1}{n}$. Then $f^{k-1}x_n = 1 + \frac{1}{n5^{k-1}}, f^k x_n = \frac{1}{5} + \frac{1}{n5^k}$. This implies $f^{k-1}x_n \to 1, f^k x_n \to \frac{1}{5} \neq f1$ as $n \to +\infty$. Hence f is not $k-$continuous.

The above examples show that orbital continuity implies weak orbital continuity but the converse need not be true. Also, every k-continuous mapping is orbitally continuous, but the converse is not true. In [12] the author has shown that the f-orbital lower semicontinuity of $x \to d(x, fx)$ is weaker than orbital continuity.

The question of continuity of contractive definitions at their fixed point was studied by Rhoades [20] (see also, Hicks and Rhoades [6]). All the contractive definitions studied by them forced the mappings to be continuous at the fixed point. Rhoades [20] also listed the question of the existence of a contractive condition that intromits discontinuity at the fixed point as an open problem. Pant [15] gave the first affirmative answer to this problem in the setting of metric space. Various other distinct answers to this problem and their possible applications to neural networks having discontinuities in activation functions can be found in (Bisht and Pant [3], Bisht and Rakočević [4], Pant and Pant [13], Pant et al. [14, 16, 17, 18], Tas and Özgür [26]).

In general, a self-mapping f satisfying some contractive condition on a complete Menger PM-space X ensures existence of a Cauchy sequence of successive iterates $\{f^n x\}_{n\in\mathbb{N}}$ for each x in X, which converges to some point, say $z \in X$, and the limiting point z of the sequence of iterates is known as a fixed point of f. However, there may exist contractive definitions that ensure the existence of the Cauchy sequence of iterates, which converges to some limit point, but the limit point may not be a fixed point. Therefore, to ensure the existence of a fixed point under such contractive definitions, one needs to assume some additional hypotheses. In this paper, we assume the notions of weak orbital continuity and lower semicontinuity which may imply discontinuity at the fixed point but characterize completeness of the space.

2. Preliminaries

First we recall some standard definitions and notations used in probabilistic metric spaces.

Let D^+ be the set of all distribution functions $F : \mathbb{R} \to [0,1]$ such that F is a non-decreasing, left-continuous mapping satisfying $F(0) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$. The space D^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for D^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.3. [22] A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \leq T(c,d)$, whenever $a \leq c$ and $b \leq d$, and $a,b,c,d \in [0,1]$.

Some of the simple examples of t-norm are $T(a,b) = \max\{a+b-1,0\}$, $T(a,b) = \min\{a,b\}$, $T(a,b) = ab$ and

$$T(a,b) = \begin{cases} \frac{ab}{a+b-ab}, & ab \neq 0, \\ 0, & ab = 0. \end{cases}$$

The t-norms are defined recursively by $T^1 = T$ and

$$T^n(x_1,\ldots,x_{n+1}) = T(T^{n-1}(x_1,\ldots,x_n),x_{n+1}),$$

for $n \geq 2$ and $x_i \in [0,1]$ for all $i \in \{1,\ldots,n+1\}$.

Definition 2.4. A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X,\mathcal{F},T)$ where X is a non-void set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X\times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x,y), then the following conditions hold:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t+s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x,y,z \in X$ and $s,t \geq 0$.

Remark 2.1. [23] Every metric space is a PM-space. Let (X,d) be a metric space and $T(a,b) = \min\{a,b\}$ is a continuous t-norm. Define $F_{x,y}(t) = \varepsilon_0(t - d(x,y))$ for all $x,y \in X$ and $t > 0$. The triple $(X,\mathcal{F},T)$ is a PM-space induced by the metric d.

Definition 2.5. Let $(X,\mathcal{F},T)$ be a Menger PM-space.

(1) A sequence $\{x_n\}_{n=1,2,\ldots}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists positive integer N such that $F_{x_n,x}(\varepsilon) > 1-\lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}_{n=1,2,\ldots}$ in X is called Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1-\lambda$ whenever $n,m \geq N$.

(3) A Menger PM-space is said to be complete if every Cauchy sequence in X is convergent to a point in X.

The following lemma was given in [21, 22].

Lemma 2.1. [22] *Let $(X,\mathcal{F},T)$ be a Menger PM-space. Then the function $\mathcal{F}$ is lower semi-continuous for every fixed $t > 0$, i.e., for every fixed $t > 0$ and every two convergent sequences $\{x_n\},\{y_n\} \subseteq X$ such that $x_n \to x, y_n \to y$ it follows that*

$$\liminf_{n\to+\infty} F_{x_n,y_n}(t) = F_{x,y}(t).$$

3. Main results

3.1. **Fixed points of Meir-Keeler type mappings in Menger PM-space.** The Meir-Keeler [9] (see also, [8, 11, 27]) type contractive condition employed in the next proposition ensures the convergence of sequence of iterates but does not ensure the existence of a fixed point.

Proposition 3.1. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let f be self-mapping of X satisfying the condition*

(i) for every $\epsilon \in (0,1)$ there exists $\delta \in (0, \epsilon]$ such that

$$\epsilon - \delta \leq \min\left\{F_{x,fx}(t), F_{y,fy}(t)\right\} < \epsilon \ \textit{implies} \ F_{fx,fy}(t) > \epsilon,$$

for all $x, y \in X$. Then for any x in X the sequence of iterates $\{f^n x\}_{n=1,2,\dots}$ is a Cauchy sequence and there exists a point z in X such that $\lim\limits_{n\to+\infty} f^n x = z$ for each x in X.

Proof. It is obvious that f satisfies the following contractive condition:

$$F_{fx,fy}(t) > \min\left\{F_{x,fx}(t), F_{y,fy}(t)\right\}. \tag{3.1}$$

Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}, n = 1, 2, \dots$. If $x_p = x_{p+1}$ for some $p \in \mathbb{N}$, then x_p is a fixed point of f. Suppose $x_n \neq x_{n+1}$ for all $n \geq 0$. Then using (3.1) we have

$$\begin{aligned} F_{x_n,x_{n+1}}(t) &= F_{fx_{n-1},fx_n}(t) > \min\left\{F_{x_{n-1},fx_{n-1}}(t), F_{x_n,f_{x_n}}(t)\right\} = \\ &= \min\left\{F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)\right\} = \\ &= F_{x_{n-1},x_n}(t). \end{aligned}$$

Thus $\{F_{x_n,x_{n+1}}(t)\}$ is a strictly increasing sequence of positive real numbers in $[0, 1]$ and, hence, tends to a limit $r \leq 1$. Suppose $r < 1$. Then there exists a positive integer N with $n \geq N$ such that

$$r - \delta(r) < F_{x_n,x_{n+1}}(t) < r. \tag{3.2}$$

This further implies

$$r - \delta(r) < \min\left\{F_{x_n,x_{n+1}}(t), F_{x_{n+1},x_{n+2}}(t)\right\} < r,$$

that is,

$$r - \delta(r) < \min\left\{F_{x_n,fx_n}(t), F_{x_{n+1},fx_{n+1}}(t)\right\} < r.$$

By virtue of (i), this yields $F_{fx_n,fx_{n+1}}(t) = F_{x_{n+1},x_{n+2}}(t) > r$. This contradicts (3.2). Hence $\liminf\limits_{n\to+\infty} F_{x_n,x_{n+1}}(t) = 1$. Further, if q is any positive integer then for each $t > 0$, we have

$$\begin{aligned} F_{x_n,x_{n+q}}(t) &= F_{fx_{n-1},fx_{n+q-1}}(t) > \\ &> \min\left\{F_{x_{n-1},fx_{n-1}}(t), F_{x_{n+q-1},fx_{n+q-1}}(t)\right\} = \\ &= \min\left\{F_{x_{n-1},x_n}(t), F_{x_{n+q-1},x_{n+q}}(t)\right\}. \end{aligned}$$

Since $\liminf\limits_{n\to+\infty} F_{x_n,x_{n+1}}(t) = 1$, making limit as $n \to +\infty$, the above inequality yields

$$\liminf_{n\to+\infty} F_{x_n,x_{n+q}}(t) = 1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} f^n x_0 = z$. Moreover, if y_0 is any other point in X and $y_n = fy_{n-1} = f^n y_0$, then (3.1) yields

$$\begin{aligned} F_{x_n,y_n}(t) &= F_{fx_{n-1},fy_{n-1}}(t) > \min\left\{F_{x_{n-1},fx_{n-1}}(t), F_{y_{n-1},fy_{n-1}}(t)\right\} = \\ &= \min\left\{F_{x_{n-1},x_n}(t), F_{y_{n-1},y_n}(t)\right\}. \end{aligned}$$

Letting $n \to +\infty$, we get $\liminf_{n\to+\infty} F_{z,y_n}(t) = 1$ for each $t > 0$. Therefore, $\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} f^n y_0 = z$. □

The triple $(X, \mathcal{F}, T_{min})$ is a complete Menger PM-space, for $X \subseteq \mathbb{R}$ (Remark 2.1). Thus, the mapping in the next example validates all the condition of Lemma 3.1, but f is fixed point free [14].

Example 3.3. Let $X = [0, 2]$. Mapping $f : X \mapsto X$ defined by

$$fx = \begin{cases} \dfrac{1+x}{2}, & \text{if } 0 \le x < 1, \\ 0, & \text{if } 1 \le x < 2. \end{cases}$$

Then f satisfies the contractive condition *(i)* with $\delta(\epsilon) = 1 - \epsilon$ for $\epsilon < 1$ and $\delta(\epsilon) = \epsilon$, for $\epsilon \ge 1$, but does not possess a fixed point. It can be easily verified that for each x in X, the sequence of iterates $\{f^n x\}_{n=1,2,\ldots}$ is a Cauchy sequence and $f^n x \to 1$.

In order to ensure the existence of a fixed point under Meir-Keeler type conditions, we need some additional conditions. In the next theorem, we assume f to be weakly orbitally continuous which may not imply continuity at the fixed point.

Theorem 3.1. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let f be a self-mapping of X satisfying the condition*

(i) for every $\epsilon \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that

$$\epsilon - \delta \le \min\left\{F_{x,fx}(t), F_{y,fy}(t)\right\} < \epsilon \ \ implies \ \ F_{fx,fy}(t) > \epsilon,$$

for all $x, y \in X$. Then f has a unique fixed point if and only if f is weakly orbitally continuous.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}$, $n = 1, 2, \ldots$. Then by virtue of Proposition 3.1, it follows that $\{x_n\}_{n=1,2,\ldots}$ is a Cauchy sequence. Since $(X, \mathcal{F}, T)$ is complete, there exists a point z in X such that $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} fx_n = z$.

Suppose that f is weakly orbitally continuous. Since $f^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of f we get, $f^n y_0 \to z$ and $f^{n+1} y_0 \to fz$ for some $y_0 \in X$. This implies that $z = fz$ since $f^{n+1} y_0 \to z$. Therefore z is a fixed point of f.

Conversely, suppose that the mapping f has a fixed point, say z. Then $\{f^n z = z\}$ is a constant sequence such that $\lim_n f^n z = z$ and $\lim_n f^{n+1} z = z = fz$. Hence, f is weak orbitally continuous. Uniqueness of the fixed point follows from (i). □

The following corollary is an easy consequence of Theorem 3.1:

Corollary 3.1. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let f be a self-mapping of X satisfying the condition*

(i) for every $\epsilon \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that

$$\epsilon - \delta \le \min\left\{F_{x,fx}(t), F_{y,fy}(t)\right\} < \epsilon \ \ implies \ \ F_{fx,fy}(t) > \epsilon,$$

for all $x, y \in X$. If f is k-continuous or f^k is continuous or f is orbitally continuous then f has a unique fixed point.

The following example illustrates Theorem 3.1.

Example 3.4. Let $X = [0, 2]$ equipped with the Euclidean metric. Define $f : X \mapsto X$ by

$$fx = \begin{cases} 1, & \text{if } 0 \le x \le 1, \\ x - 1, & \text{if } 1 < x \le 2. \end{cases}$$

Then f satisfies all the conditions of Theorem 3.1 and has a unique fixed point $x = 1$ at which f is discontinuous. The mapping f satisfies condition *(i)* with $\delta(\epsilon) = 1 - \epsilon$, if $\epsilon < 1$, and $\delta(\epsilon) = \epsilon$, for $\epsilon \geq 1$. It is also easy to see that the mapping f is orbitally continuous and, hence, weak orbitally continuous [14].

We now replace the notion of weak orbital continuity assumed in Theorem 3.1 by a weaker continuity notion, i.e., f-orbitally lower semi-continuity.

Theorem 3.2. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let f be a self-mapping of X satisfying condition (i) of Theorem 3.1. Then f has a unique fixed point provided that the function $x \to d(x, fx)$ is lower semi continuous.*

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}, n = 1, 2, \ldots$. Then by virtue of Proposition 3.1, it follows that $\{x_n\}$ is a Cauchy sequence. Since $(X, \mathcal{F}, T)$ is complete, there exists a point z in X such that $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} fx_n = z$. Since $\lim_{n\to+\infty} d(x_n, fx_n) = 0$ and $x \to d(x, fx)$ is lower semi continuous,

$$0 \leq d(z, fz) \leq \lim \inf_{n\to+\infty} d(x_n, fx_n) = 0.$$

This is a contradiction. Hence, f has a fixed point. Uniqueness of the fixed point follows easily.

□

In the next result, we show that Theorem 3.1 characterizes metric completeness of X. Fixed point theorems that characterize completeness of the underlying space are important in mathematical analysis and various workers have analyzed fixed point theorems that characterize metric completeness [1, 2, 17, 19, 24, 25]. However, there is a substantive difference between the next theorem and similar theorems (e. g., Subrahmanyam [24], Suzuki [25]) giving characterization of completeness in terms of fixed point property for contractive type mappings [17]. Subrahmanyam [24] and Suzuki [25] have shown that the contractive condition implies continuity at the fixed point; and completeness of the metric space X is equivalent to the existence of fixed point. In the next theorem, we prove that completeness of the space is equivalent to fixed point property for a large class of mappings including continuous as well as discontinuous mappings.

In what follows we use the notation $a \gg b$ (or $a \ll b$) to show that the positive number a is much greater (smaller) than the positive number b.

Theorem 3.3. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. If every weakly orbitally continuous self-mappings of X satisfying the condition (i) of Theorem 3.1 has a fixed point, then X is complete.*

Proof. Suppose that every weakly orbitally continuous self-mapping of X satisfying condition (i) of Theorem 3.1 possesses a fixed point. We will prove that X is complete. If possible, suppose X is not complete. Then there exists a Cauchy sequence in X, say $M = \{u_1, u_2, u_3, \ldots\}$, consisting of distinct points which does not converge. Let $x \in X$ be given. Then, since x is not a limit point of the Cauchy sequence M, there exists a least positive integer $N(x)$ such that $x \neq u_{N(x)}$ and for each $m \geq N(x)$ and $t > 0$ we have

$$1 - F_{x,u_{N(x)}}(t) \gg 1 - F_{u_{N(x)},u_m}(t). \tag{3.3}$$

Consider a mapping $f : X \mapsto X$ by $f(x) = u_{N(x)}$. Then, $f(x) \neq x$ for each x and, using (3.3), for any x, y in X and $t > 0$ we get

$$1 - F_{fx,fy}(t) = 1 - F_{u_{N(x)},u_{N(y)}}(t) \ll 1 - F_{x,u_{N(x)}}(t) = 1 - F_{x,fx}(t)$$

if $N(x) \leq N(y)$, or

$$1 - F_{fx,fy}(t) = 1 - F_{u_{N(x)},u_{N(y)}}(t) \ll 1 - F_{y,u_{N(y)}}(t) = 1 - F_{y,fy}(t)$$

if $N(x) > N(y)$.

This implies that

$$F_{fx,fy}(t) > \min\{F_{x,fx}(t), F_{y,fy}(t)\}. \tag{3.4}$$

In other words, given $\epsilon > 0$ we can select $\delta(\epsilon) = \epsilon$ such that

$$\epsilon - \delta \leq \min\{F_{x,fx}(t), F_{y,fy}(t)\} < \epsilon \text{ implies } F_{fx,fy}(t) > \epsilon. \tag{3.5}$$

It is clear from (3.4) and (3.5) that the mapping f satisfies condition (i) of Theorem 3.1. Moreover, f is a fixed point free mapping whose range is contained in the non-convergent Cauchy sequence $M = \{u_n\}_{n\in\mathbb{N}}$. Hence, there exists no sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which $\{fx_n\}_{n\in\mathbb{N}}$ converges, that is, there exists no sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which the condition $fx_n \to t$ implies $f^2x_n \to ft$ is violated. Therefore, f is a 2-continuous mapping, hence, weak orbitally continuous mapping. Thus, we have a self-mapping f of X which satisfies all the conditions of Theorems 3.1 but does not possess a fixed point. This contradicts the hypothesis of the theorem. Hence X is complete. □

3.2. **Fixed points of non-expansive mappings in Menger PM-space.** We now prove a fixed point theorem for a non-expansive mapping satisfying $(\epsilon - \delta)$ condition.

Theorem 3.4. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let f be continuous self-mapping of X satisfying the conditions*

(i) for every $\epsilon \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that

$$\epsilon - \delta < \min\{F_{x,fx}(t), F_{y,fy}(t)\} < \epsilon \ \textit{implies} \ F_{fx,fy}(t) \geq \epsilon,$$

(ii)

$$F_{fx,fy}(t) \geq \min\{F_{x,fx}(t), F_{y,fy}(t)\}.$$

for all $x, y \in X$. Then f has a fixed point.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}, n = 1, 2, \ldots$. Then following the lines of Proposition 3.1, it can be shown that $\{x_n\}$ is a Cauchy sequence. Continuity of f now implies that $fz = z$ and z is a fixed point of f. □

Remark 3.2. Theorem 3.1 provides a new answer to the once open question (see Rhoades [20], p. 242) on the existence of contractive mappings which admit discontinuity at the fixed point.

4. Conclusions and further study

A probabilistic version of the Meir-Keeler type fixed point theorem which characterizes completeness of the metric space has been proved. A Meir–Keeler type solution to the Rhoades problem on the existence of contractive mappings that intromit discontinuity at the fixed point in the setting of Menger PM-space has been given. Using $(\epsilon - \delta)$ type condition, fixed point of non-expansive mapping has been established. New answers of the Rhoades problem using various other families of contractive definitions will be studied in our future work.

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