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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

Uniqueness of solutions for a fractional thermostat model

J. CABALLERO, J. HARJANI and K. SADARANGANI

ABSTRACT. In this paper, we present a sufficient condition for the uniqueness of solutions to a nonlocal fractional boundary value problem which can be considered as the fractional version to the thermostat model. As application of our result, we study the eigenvalues problem associated and, moreover, we get a Lyapunov-type inequality.

1. INTRODUCTION

Fractional differential equations arise from a variety of applications including in various fields of science and engineering (see [7, 8] and the references therein).

Two basic concepts in the fractional calculus are the following ones.

Definition 1.1. Suppose $\alpha \geq 0$ and $f\colon [a,b] \to \mathbb{R}$. The Riemann-Liouville fractional integral of order α is defined as

$$\left(I_{a^+}^{\alpha} f\right)(t) = \begin{cases} f(t) & \text{if } \alpha = 0 \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s & \text{if } \alpha > 0 \end{cases}$$

Definition 1.2. For $\alpha \geq 0$ and $f\colon [a,b] \to \mathbb{R}$. The Caputo fractional derivative of order α is given by

$$\left({}^cD_{a^+}^{\alpha} f\right)(t) = \begin{cases} f(t) & \text{if } \alpha = 0 \\ I_{a^+}^{n-\alpha}(D^n f)(t) & \text{if } \alpha > 0 \text{ and } n = [\alpha]+1 \end{cases}$$

In order to motivate the topic treated in the paper we present the following facts.

In Theorem 7.7 of [6], the following result is presented.

Theorem 1.1. *Under the assumption that $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is Lipschitzian with respect to the second variable, i.e., there exists $k > 0$ such that*

$$|f(t,x) - f(t,y)| \leq k|x-y|,$$

for any $t \in [a,b]$ and $x, y \in \mathbb{R}$, if $b - a < \frac{2\sqrt{2}}{k}$ then the boundary value problem

$$\begin{cases} x''(t) = -f(t,x(t)), & a < t < b, \\ x(a) = A,\ x(b) = B, \end{cases} \tag{1.1}$$

where $A, B \in \mathbb{R}$, has a unique continuous solution.

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Corresponding author: Jackie Harjani; jharjani@dma.ulpgc.es

In [4], the author treated a similar question for the following fractional boundary value problem

$$\begin{cases} D^{\alpha}_{a+}x(t) = -f(t,x(t)), & a < t < b, \\ x(a) = 0,\ x(b) = B, \end{cases} \tag{1.2}$$

where $1 < \alpha \leq 2$, $B \in \mathbb{R}$ and D^{α}_{a+} denotes the standard fractional Riemann-Liouville derivative.

The main result in [4] is the following.

Theorem 1.2. *Suppose that $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies*

$$|f(t,x) - f(t,y)| \leq k|x-y|\,,$$

for any $t \in [a,b]$ and $x, y \in \mathbb{R}$, where $k > 0$.

If $b - a < \Gamma(\alpha)^{1/\alpha} \dfrac{\alpha^{\frac{\alpha+1}{\alpha}}}{k^{1/\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}}$ then the boundary value problem (1.2) *has a unique continuous solution.*

Recently, in [1, 3] the following fractional boundary value problem

$$\begin{cases} {}^{c}D^{\alpha}_{a+}x(t) = -f(t,x(t)), & a < t < b, \\ x(a) = A,\ x(b) = B, \end{cases} \tag{1.3}$$

where $1 < \alpha < 2$, $A, B \in \mathbb{R}$ and ${}^{c}D^{\alpha}_{a+}$ denotes the Caputo fractional derivative, was considered, and a similar result to Theorem 1.2 was obtained.

In this paper, motivated by the above mentioned results, we study the uniqueness of solutions for the following fractional boundary value problem

$$\begin{cases} {}^{c}D^{\alpha}_{a+}x(t) = -f(t,x(t)), & a < t < b, \\ x'(a) = 0, \quad \beta\, {}^{c}D^{\alpha-1}_{a^+}x(b) + x(\eta) = 0, \end{cases} \tag{1.4}$$

where $1 < \alpha \leq 2$, $\beta > 0$ and $a \leq \eta \leq b$.

The boundary value problem (1.4) is the fractional version of the nonlocal boundary value problem

$$\begin{cases} x''(t) = -f(t,x(t)), & 0 < t < 1, \\ x'(0) = 0,\ \beta\, x'(1) + x(\eta) = 0, \end{cases} \tag{1.5}$$

with $0 \leq \eta \leq 1$. This problem has been treated as particular case, with $\eta = 0$ in [5] and it models a thermostat insulated at $t = 0$ with a controller dissipating heat at $t = 1$ depending on the temperature detected by a sensor at $t = \eta$.

2. Background

We start this section by presenting a result which transforms the problem (1.4) into an integral equation and it appears in [2].

Lemma 2.1. *Let $h \in \mathcal{C}[a,b]$. Then the unique solution $x \in \mathcal{C}^2[a,b]$ to the fractional boundary value problem*

$$\begin{cases} {}^{c}D^{\alpha}_{a+}x(t) = -h(t), & a < t < b, \\ x'(a) = 0,\ \beta\, {}^{c}D^{\alpha-1}_{a^+}x(b) + x(\eta) = 0, \end{cases} \tag{2.6}$$

where $1 < \alpha \le 2$, $\beta > 0$ *and* $a \le \eta \le b$, *is given by*

$$x(t) = \int_a^b G(t,s)h(s)\,\mathrm{d}s\,,$$

where $G(t,s)$ *is the Green's function defined by*

$$G(t,s) = \beta + H_\eta(s) - H_t(s)\,,$$

where, for $r \in [a,b]$, $H_r\colon [a,b] \to \mathbb{R}$ *is the function given by*

$$H_r(s) = \begin{cases} \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)}, & a \le s \le r \le b, \\ 0, & a \le r \le s \le b, . \end{cases}$$

Remark 2.1. It is clear that $G(t,s)$ is a continuous function on $[a,b] \times [a,b]$. Moreover, it is proved in [2] that

$$|G(t,s)| \le \max\left\{\beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}, \left|\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right|\right\}.$$

Remark 2.2. Suppose that $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $x \in \mathcal{C}^2[a,b]$. By Lemma 2.1, we have x is solution to the boundary value problem (1.4) if and only if x is a fixed point of the operator T defined on $\mathcal{C}[a,b]$ by

$$(Tx)(t) = \int_a^b G(t,s)f(s,x(s))\,\mathrm{d}s.$$

3. Main result

Our starting point in this section is the following result which gives us a sufficient condition for the uniqueness of solutions to the equation $Tx = x$.

Theorem 3.3. *Suppose that* $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous and there exists a constant* $L > 0$ *such that*

$$|f(t,x) - f(t,y)| \le L|x-y|\,, \tag{3.7}$$

for any $t \in [a,b]$ *and* $x, y \in \mathbb{R}$. *If the condition*

$$\frac{L}{b-a}\max\left\{\beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}, \left|\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right|\right\} < 1 \tag{3.8}$$

holds then the equation $Tx = x$ *has a unique continuous solution on* $[a,b]$.

Proof. Consider the metric space $\mathcal{C}[a,b]$ of real and continuous functions defined on $[a,b]$ with the metric

$$d(f,g) = \max\left\{|f(t) - g(t)| : t \in [a,b]\right\}\,.$$

It is well known that $(\mathcal{C}[a,b], d)$ is a complete metric space.

Next, we define, for $x \in \mathcal{C}[a,b]$, Tx as

$$(Tx)(t) = \int_a^b G(t,s)f(s,x(s))\,\mathrm{d}s, \quad t \in [a,b]\,.$$

Since G and g are continuous functions, it is clear that T applies $\mathcal{C}[a,b]$ into itself.

Now, we estimate $d(Tx,Ty)$ for $x, y \in \mathcal{C}[a,b]$. To do this, we take $t \in [a,b]$ and we have

$$\begin{aligned}|(Tx)(t)-(Ty)(t)| &= \left|\int_a^b G(t,s)(f(s,x(s))-f(s,y(s)))\,\mathrm{d}s\right| \\ &\le \int_a^b |G(t,s)|\,|f(s,x(s))-f(s,y(s))|\,\mathrm{d}s \\ &\le L\int_a^b |G(t,s)|\,|x(s)-y(s)|\,\mathrm{d}s \\ &\le L\,\sup\{|x(s)-y(s)|\colon t\in[a,b]\}\int_a^b |G(t,s)|\,\mathrm{d}s \\ &\le L\,d(x,y)\frac{1}{b-a}\,\max\left\{\beta+\frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)},\left|\beta-\frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right|\right\},\end{aligned}$$

where we have used assumption (3.7) and Remark 2.1.

By (3.8), applying the Banach contraction principle, the equation $Tx = x$ has a unique solution in $\mathcal{C}[a,b]$. □

The following result gives us uniqueness of solutions to the boundary value problem (1.4).

Theorem 3.4. *Under assumptions of Theorem 3.3 and if*

$$\frac{L}{b-a}\,\max\left\{\beta+\frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)},\left|\beta-\frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right|\right\}<1$$

then the boundary value problem (1.4) *has a unique solution.*

Proof. By Remark 2.2, the solutions to the boundary value problem (1.4) are fixed points of the operator T and by Theorem 3.3, we get the desired result. □

Next, we present an example illustrating our results.

Example 3.1. Consider the following fractional boundary value problem

$$\begin{cases} {}^cD_{0^+}^{3/2}x(t) = -\arctan\left(\dfrac{t}{3}x(t)\right)+1, & 0<t<1, \\ x'(0)=0, \quad \dfrac{1}{2}\,{}^cD_{0^+}^{1/2}x(1)+x(1/2)=0, & \end{cases} \tag{3.9}$$

The problem (3.9) is a particular case to the problem (1.4) with $\alpha = 3/2$, $a = 0$, $b = 1$, $\beta = \eta = 1/2$ and $f(t,x) = \arctan(\frac{t}{3}x(t)) - 1$.

Since, for any $t \in [0,1]$ and $x, y \in \mathbb{R}$ we have

$$|f(t,x)-f(t,y)| = \left|\arctan\left(\frac{t}{3}x\right)-\arctan\left(\frac{t}{3}y\right)\right| \le \frac{t}{3}|x-y| \le \frac{1}{3}|x-y|,$$

where we have used the fact that $|\arctan x - \arctan y| \le |x-y|$, condition (3.7) of Theorem 3.3 is satisfied with $L = 1/3$.

Moreover, in this case the inequality appearing in 3.8 has the expression

$$\frac{1}{3}\,\max\left\{\frac{1}{2}+\frac{\frac{1}{2}^{\frac{1}{2}}}{\Gamma(3/2)},\left|\frac{1}{2}-\frac{\frac{1}{2}^{\frac{1}{2}}}{\Gamma(3/2)}\right|\right\} = \frac{1}{3}\left(\frac{1}{2}+\frac{\sqrt{2}}{\frac{1}{2}\sqrt{\pi}}\right) = \frac{1}{3}\left(\frac{1}{2}+2\sqrt{\frac{2}{\pi}}\right) \approx 0.699 < 1.$$

Therefore, by Theorem 3.4 the problem (3.9) has a unique continuous solution on $[0.1]$.

Moreover, since the trivial solution is not solution to the problem (3.9), this unique solution is not trivial.

4. Application

In this section, we present an application of our result to the eigenvalues problem.

Consider the eigenvalues problem associated to the boundary value problem (1.4), that is,

$$\begin{cases} {}^cD^{\alpha}_{a+}x(t) + \lambda\, x(t) = 0, & a < t < b, \\ x'(a) = 0,\ \beta\, {}^cD^{\alpha-1}_{a^+}x(b) + x(\eta) = 0, \end{cases} \tag{4.10}$$

where $1 < \alpha \le 2$, $\beta > 0$ and $a \le \eta \le b$.

The real values of λ for which there exists a nontrivial solution to the problem (4.10) are called eigenvalues associated to the problem (4.10) and the corresponding solutions are called eigenfunctions.

The boundary value problem (4.10) is a particular case of Problem (1.4) with $f(t,x) = \lambda\, x$ which is a continuous function on $[a,b] \times \mathbb{R}$.

Moreover,

$$|f(t,x) - f(t,y)| \le |\lambda|\,|x-y|\,,$$

and the constant L appearing in (3.7) is $L = |\lambda|$.

If the inequality (3.8) is satisfied, that is

$$\frac{|\lambda|}{b-a}\max\left\{\beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}\,, \left|\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right|\right\} < 1$$

then, by Theorem 3.4, the problem (4.10) has a unique solution. Since the trivial solution $x(t) = 0$ for $t \in [0,1]$ satisfies (4.10) and belongs to $\mathcal{C}^2[a,b]$, it will be the unique solution and, consequently, λ is not an eigenvalue to the problem (4.10).

Summarizing, if

$$|\lambda| < \frac{b-a}{\max\left\{\beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}\,, \left|\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right|\right\}}$$

then λ is not an eigenvalue to the problem (4.10).

Therefore, we have the following result

Theorem 4.5. *If λ is an eigenvalue to the problem* (4.10) *then*

$$|\lambda| \ge \frac{b-a}{\max\left\{\beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}\,, \left|\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right|\right\}}\,.$$

Finally, we present an Lyapunov-type inequality.

Consider the following fractional boundary value problem

$$\begin{cases} {}^cD^{\alpha}_{a+}x(t) + q(t)\cdot x(t) = 0, & a < t < b, \\ x'(a) = 0,\ \beta\, {}^cD^{\alpha-1}_{a^+}x(b) + x(\eta) = 0, \end{cases} \tag{4.11}$$

where $1 < \alpha \le 2$, $\beta > 0$ and $a \le \eta \le b$ and $q\colon [a,b] \to \mathbb{R}$ is a continuous function.

The boundary value problem (4.11) is a particular case of the problem (1.4) with $f(t,x) = q(t)\,x$. The continuity of q gives us the continuity of f on $[a,b] \times \mathbb{R}$. Moreover,

$$|f(t,x) - f(t,y)| = |q(t)||x-y| \le \|q\|_\infty |x-y|\,,$$

where $\|q\|_\infty = \sup\{|q(t)| : t \in [a,b]\}$, whose existence is guaranteed by the continuity of q on $[a,b]$.

In this case, the constant L appearing in (3.7) is $L = \|q\|_\infty$.

Suppose that the inequality (3.8) holds, that is

$$\frac{\|q\|_\infty}{b-a} \max\left\{\beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}, \left|\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right|\right\} < 1 .$$

By, Theorem 3.4, the boundary value problem (4.11) has a unique solution and, as the trivial solution $x(t) = 0$ for $t \in [a,b]$ satisfies (4.11), it will be the unique solution. From this, it follows the next Lyapunov-type result.

Theorem 4.6. *If the boundary value problem* (4.11) *has a nontrivial solution then*

$$\|q\|_\infty \geq \frac{b-a}{\max\left\{\beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}, \left|\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}\right|\right\}} .$$

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Department of Mathematics
University of Las Palmas de Gran Canaria
Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain
Email address: josefa.caballero@ulpgc.es
Email address: jharjani@dma.ulpgc.es
Email address: ksadaran@dma.ulpgc.es