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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

A strong convergence theorem for maximal monotone operators in Banach spaces with applications

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ABSTRACT. An algorithm is constructed to approximate a zero of a maximal monotone operator in a uniformly convex and uniformly smooth real Banach space. The sequence of the algorithm is proved to converge strongly to a zero of the maximal monotone map. In the case where the Banach space is a real Hilbert space, our theorem complements the celebrated proximal point algorithm of Martinet and Rockafellar. Furthermore, our convergence theorem is applied to approximate a solution of a Hammerstein integral equation in our general setting. Finally, numerical experiments are presented to illustrate the convergence of our algorithm.

1. INTRODUCTION

Throughout this paper, we assume H is a real Hilbert space, and E is an arbitrary real Banach space with dual space E^*, unless otherwise stated. A mapping $A : E \to 2^{E^*}$ is called *monotone* if for each $x, y \in E$, the following inequality holds:

$$\langle \eta - \nu, x - y \rangle \geq 0, \ \forall \, \eta \in Ax, \ \nu \in Ay. \tag{1.1}$$

A is said to be *maximal monotone* if, in addition, its graph is not included in the graph of any other monotone mapping. Monotone operators were first introduced by Minty [43] to aid in the abstract study of electrical networks and later studied by Browder [3] and his school in the setting of partial differential equations. Later, Zarantonello [59], Minty [45], Kačurovskii [39] and a host of other authors studied this class of mappings in Hilbert spaces. Interest in such mappings stems mainly from their usefulness in several areas of mathematics such as optimization, evolution equations, differential equations, Hammerstein equations, variational inequality problems, calculus of variation, and so on.

As an example of applications of monotone mappings, consider the following: Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *subdifferential* of f, $\partial f : E \to 2^{E^*}$ is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle, \ \forall \, y \in E\}.$$

It is well known that $\partial f : E \to 2^{E^*}$ is a monotone operator on E, and that $0 \in \partial f(x^*)$ *if and only if x^* is a minimizer of f*. Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Ax$, in this case, is equivalent to solving for a minimizer of f.

In general, a fundamental problem in the study of monotone operators in Banach spaces is the following:

$$\text{Find } u \in E \text{ such that } 0 \in Au. \tag{1.2}$$

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This problem has been investigated in Hilbert spaces by numerous researchers.

The *proximal point algorithm* introduced by Martinet [44] and studied extensively by Rockafellar [54] and numerous authors is concerned with an iterative method for approximating a solution of the inclusion $0 \in Au$, where A is a *maximal monotone operator*. Specifically, given $x_n \in H$, the proximal point algorithm generates the next iterate x_{n+1} by solving the following equation:

$$x_{n+1} = \big(I + \lambda_n A\big)^{-1} x_n + e_n, \tag{1.3}$$

where $\lambda_n > 0$ is a regularizing parameter. Rockafellar [54] proved that if the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is bounded from above, then the resulting sequence $\{x_n\}_{n=1}^{\infty}$ of proximal point iterates converges *weakly* to a solution of (1.2), when $E = H$, provided that a solution exists (see also Bruck and Reich [12]). Rockafellar [54] then posed the following question.

Question 1. *Does the proximal point algorithm always converge strongly?*

The question was resolved in the negative by Güler [38] who produced a proper closed convex function g in the infinite dimensional Hilbert space l_2 for which the proximal point algorithm converges *weakly but not strongly*, (see also Bauschke *et al.* [2]). This naturally raised the following questions.

Question 2. *Can the proximal point algorithm be modified to guarantee strong convergence?*

Question 3. *Can another iterative algorithm be developed to approximate a solution of* (1.2), *assuming existence, such that the sequence of the algorithm converges strongly to a solution of* (1.2)*?*

In connection with Question 3, Bruck [11] considered an iteration process of the Mann-type and proved that the sequence of the process converges strongly to a solution of (1.2) in a real Hilbert space where A is a maximal monotone map, provided the initial vector is chosen in a neighborhood of a solution of (1.2). Chidume [34] extended this result to L_p spaces, $p \geq 2$ (see also Reich [50, 51, 52]). These results of Bruck [11] and Chidume [34] are not easy to use in any possible application because the neighborhood of a solution in which the initial vector must be chosen is not known precisely.

In connection with Question 2, Solodov and Svaiter [57] proposed a modification of the proximal point algorithm which guarantees strong convergence in a real Hilbert space. Their algorithm is as follows.

Choose any $x_0 \in H$ and $\sigma \in [0, 1)$. At iteration k, having x_k, choose $\mu_k > 0$, and find (y_k, v_k), an inexact solution of $0 \in Tx + \mu_k(x - x_k)$, with tolerance σ. Define

$$C_k = \{z \in H : \big\langle z - y_k, v_k \big\rangle \leq 0\}, \quad Q_k = \{z \in H : \big\langle z - x_k, x_0 - x_k \big\rangle \leq 0\}.$$

$$\text{Take} \qquad x_{k+1} = P_{C_k \cap Q_k} x_0,\ k \geq 1.$$

The authors themselves noted ([57], p. 195) that ". . . at each iteration, there are two sub-problems to be solved. . . " : (i) find an inexact solution of the proximal point algorithm, and (ii) find the projection of x_0 onto $C_k \cap Q_k$. They also acknowledged that these two subproblems constitute a serious drawback in using their algorithm.

Kamimura and Takahashi [40] extended this work of Solodov and Svaiter [57] to the framework of Banach spaces that are both uniformly convex and uniformly smooth. Reich and Sabach [53] extended this result to reflexive Banach spaces.

Xu [58] noted that "...Solodov and Svaiter's algorithm, though strongly convergent, does need more computing time due to the projection in the second subproblem...". He then proposed and studied the following algorithm:

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\big(I + c_n A\big)^{-1} x_n + e_n,\ n \geq 0. \tag{1.4}$$

He proved that the sequence $\{x_n\}$ generated by algorithm (1.4) converges strongly to a solution of $0 \in Au$ provided that the sequences $\{\alpha_n\}$ and $\{c_n\}$ of real numbers and the sequence $\{e_n\}$ of errors are chosen appropriately. We note here, however, that the occurrence of errors is random and so the sequence $\{e_n\}$ cannot actually be chosen.

Lehdili and Moudafi [41] considered the technique of the proximal map and the Tikhonov regularization to introduce the so-called Prox-Tikhonov method which generates the sequence $\{x_n\}$ by the algorithm:

$$x_{n+1} = J_{\lambda_n}^{A_n} x_n,\ n \geq 0, \tag{1.5}$$

where $A_n := \mu_n A + A,\ \mu_n > 0$ and $J_{\lambda_n}^{A_n} := \big(I + \lambda_n A_n\big)^{-1}$. Using the notion of variational distance, Lehdili and Moudafi [41] proved strong convergence theorems for this algorithm and its perturbed version, under appropriate conditions on the sequences $\{\lambda_n\}$ and $\{\mu_n\}$.

Some recent results connected with the proximal point algorithm can be found in the following papers: Matsushita and Xu [42], Bot and Csetnek [13].

In response to Question 3, Chidume *et al.* [16] recently proved the following theorem, where J is the *normalized duality map* on E.

Theorem 1.1 (Chidume *et al.* [16]). *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $A : E \to 2^{E^*}$ be a maximal monotone and bounded mapping with $A^{-1}(0) \neq \emptyset$. For arbitrary $u_1 \in E$, define a sequence $\{u_n\}$ iteratively by:*

$$u_{n+1} = J^{-1}\big(Ju_n - \lambda_n \eta_n - \lambda_n \theta_n (Jx_n - Ju_1)\big),\ \eta_n \in Ax_n,\ n \geq 1, \tag{1.6}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying certain conditions. Then, the sequence $\{u_n\}$ converges strongly to a solution of $0 \in Au$.

Remark 1.1. This result of Chidume *et al.* [16] answers Question 3 for the restricted class of maximal monotone operators that are *bounded*. Hence, the following question is of interest.

Question 4. *Can the requirement that A be bounded imposed in Theorem 1.1 be dispensed with?*

It is our purpose in this paper to first give an affirmative answer to this question. Secondly, we apply the convergence theorem proved to approximate a solution of a Hammerstein integral equation. Finally, a numerical example is presented to illustrate the convergence of the sequence of our algorithm.

2. Preliminaries

In the sequel, we shall need the following definitions and results. Let E be a smooth real Banach space with dual E^*. The Lyapounov functional $\phi : E \times E \to \mathbb{R}$ introduced in Alber and Ryazantseva [1], is defined by:

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \text{for } x, y \in E, \tag{2.7}$$

where J is the *normalized duality mapping* from E into E^*. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2 \text{ for } x, y \in E. \tag{2.8}$$

Define a mapping $V : E \times E^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \text{ for } x \in E,\ x^* \in E^*. \tag{2.9}$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}(x^*)),\ \forall x \in E,\ x^* \in E^*. \tag{2.10}$$

We shall use the following lemmas in the sequel where $Int\ D(A)$ denotes the interior of the domain of A.

Lemma 2.1 (Pascali and Sburian [48] Lemma 3.6, Ch. III). *Let X be a real normed space and $A : X \to 2^{X^*}$ be a monotone mapping with $0 \in IntD(A)$. Then, A is quasi-bounded, i.e., for any $M > 0$, there is exists $C > 0$ such that:*

$$(i)\ (y, v) \in G(A); \quad (ii)\ \langle v, y \rangle \leq M\|y\|; \text{ and } (iii)\ \|y\| \leq M, \text{ imply } \|v\| \leq C.$$

Lemma 2.2 (Alber and Ryazantseva [1]). *Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*) \tag{2.11}$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Lemma 2.3 (Alber and Ryazantseva [1], p.50). *Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Let $W : X \times X \to \mathbb{R}^{\not{K}}$ be defined by $W(x,y) = \frac{1}{2}\phi(y,x)$. Then,*

$$W(x,y) - W(z,y) \geq \langle Jx - Jz, z - y \rangle,$$

$$\text{i.e.,} \qquad \phi(y,x) - \phi(y,z) \geq 2\langle Jx - Jz, z - y \rangle,$$

$$\text{and also} \qquad W(x,y) \leq \langle Jx - Jy, x - y \rangle,$$

for all $x,\ y,\ z \in X$.

Lemma 2.4 (Alber and Ryazantseva [1], p.45). *Let X be a uniformly convex Banach space. Then, for any $R > 0$ and any $x,\ y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$, the following inequality holds: $\langle Jx - Jy, x - y \rangle \geq (2L)^{-1}\delta_X(c_2^{-1}\|x - y\|)$, where $c_2 = 2max\{1, R\}$, $1 < L < 1.7$.*

Define

$$K := 4RLsup\{\|Jx - Jy\| : \|x\| \leq R,\ \|y\| \leq R\} + 1. \tag{2.12}$$

Lemma 2.5 (Alber and Ryazantseva [1], p.46). *Let X be a uniformly smooth and strictly convex Banach space. Then for any $R > 0$ and any $x,\ y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds: $\langle Jx - Jy, x - y \rangle \geq (2L)^{-1}\delta_{X^*}(c_2^{-1}\|Jx - Jy\|)$, where $c_2 = 2max\{1, R\}$, $1 < L < 1.7$, and δ_X is the modulus of convexity of X.*

3. Main Results

In Theorem 3.2 below, λ_n and θ_n are real sequences in $(0,1)$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$; $\quad (ii)$ $\lambda_n M_0^* \leq \gamma_0\theta_n$; $\delta_E^{-1}(\lambda_n M_0^*) \leq \gamma_0\theta_n$,

$$(iii)\ \lim_{n\to\infty} \frac{\delta_E^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n} = \frac{\delta_{E^*}^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n} = 0, \qquad (iv)\ \frac{1}{2}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right) \in (0,1),$$

for some constants $M_0^* > 0$, and $\gamma_0 > 0$; where $\delta_E : (0,\infty) \to (0,\infty)$ is the modulus of convexity of E and $K > 0$ is as defined in 2.12.

Example 3.1. *The prototypes for λ_n and θ_n in Theorem 3.2 below are the following: $\lambda_n = \frac{1}{(n+1)^a}$, $\quad \theta_n = \frac{1}{(n+1)^b}$, where $0 < b < \min\{\frac{a}{r}, \frac{1}{K}\}$, $a + b < 1/r$, where $K > 0$ is as defined in 2.12, $r = max\{p,q\}$, $p,q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

The verification that λ_n and θ_n defined above satisfy conditions $(i) - (iv)$ is given in Chidume and Idu [18]. We now prove the following theorem.

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $A : E \to 2^{E^*}$ be a maximal monotone mapping. Assume $A^{-1}(0) \neq \emptyset$. For arbitrary $u \in E$, define a sequence $\{x_n\}$ iteratively by: $x_1 \in E$,*

$$x_{n+1} = J^{-1}\Big(Jx_n - \lambda_n\eta_n - \lambda_n\theta_n(Jx_n - Ju)\Big),\ n \geq 1, \tag{3.13}$$

where $\eta_n \in Ax_n, n \geq 1$. Then, the sequence $\{x_n\}$ converges strongly to a zero of A.

Proof. First, we show that $\{x_n\}$ is bounded. Since $A^{-1}(0) \neq \emptyset$, let $x^* \in A^{-1}(0)$. Then, there exists $r > 0$ such that $max\{\phi(x^*,u), \phi(x^*,x_1)\} \leq \dfrac{r}{8}$ and $0 \in B := \{x \in E : \phi(x^*,x) < r\}$. Observe that for any $x \in B$, we have that $\|x\| \leq \sqrt{r} + \|x^*\|$. Since A is locally bounded at 0, there exist $h_0 > 0$, $m_0 > 0$ such that $||\eta_x|| \leq m_0,\ \eta_x \in Ax,\ \forall\, x \in B_{h_0}(0) \subset B$. Let $y \in B$ be arbitrary. By monotonicity of A, we have that:

$$\begin{aligned} \langle \eta_y, y\rangle &\geq \langle \eta_x, y - x\rangle + \langle \eta_y, x\rangle,\ \eta_y \in Ay \\ \langle \eta_y, -y\rangle &\leq \langle \eta_x, x - y\rangle + \langle \eta_y, -x\rangle \end{aligned}$$

Set $v = -y$, we have that:

$$\begin{aligned} \langle \eta_y, v\rangle &\leq \langle \eta_x, x + v\rangle + \langle \eta_y, -x\rangle \\ &\leq ||\eta_x||\,(||x|| + ||v||) + ||\eta_y||\,||x|| \\ \sup_{||v|| \leq \sqrt{r} + \|x^*\|} |\langle \eta_y, v\rangle| &\leq m_0\left(h_0 + ||x^*|| + \sqrt{r}\right) + ||\eta_y||\,h_0 \\ (\sqrt{r} + ||x^*||)||\eta_y|| = \sup_{||v||\leq 1} |\langle \eta_y, v\rangle| &\leq m_0\left(h_0 + ||x^*|| + \sqrt{r}\right) + ||\eta_y||\,h_0 \\ ||\eta_y|| &\leq \frac{m_0\left(h_0 + ||x^*|| + \sqrt{r}\right)}{||x^*|| + \sqrt{r} - h_0} := M^*,\ \forall\, y \in B. \end{aligned}$$

Define $M_0^* := \max\{M^*,\ ||x^*|| + \sqrt{r}\}$. This implies that

$$\langle \eta_y, y\rangle \leq M_0^*||y|| \ \text{ and } \ ||y|| \leq M_0^*.$$

By Lemma 2.1 there exists $c > 0$ such that $\|\eta_y\| \leq c,\ \eta_y \in Ay$. Since E is uniformly smooth, J is uniformly continuous on bounded sets and so maps bounded sets to bounded sets.

Now, define

$M_0 := \sup\{\|\eta + \theta(Jx - Ju)\| : \theta \in (0,1),\, x \in B,\, \eta \in Ax\} + 1.$

$M_1 := \sup\{\|Jx - Ju\| : x \in B\} + 1.$

$M_2 := \sup\Big\{\Big\|J^{-1}\Big(Jx - \lambda\eta - \lambda_n\theta_n(Jx - Ju)\Big) - x\Big\| : \lambda, \theta \in (0,1),\, x \in B,\, \eta \in Tx\Big\} + 1.$

Let $M := \max\{M_2M_0, c_2M_0, c_2M_1\}$, and $\gamma_0 := \min\Big\{1, \dfrac{r}{16M}\Big\}$, where c_2 is the constant in Lemma 2.4.

Claim: $\phi(x^*, x_n) \le r$, for all $n \ge 1$.

We proceed by induction. By construction, $\phi(x^*, x_1) \le r$. Suppose $\phi(x^*, x_n) \le r$, for some $n \ge 1$. We show $\phi(x^*, x_{n+1}) \le r$. Suppose this is not the case, then $\phi(x^*, x_{n+1}) > r$. From Lemma 2.4 and the recurrence relation (3.13), we have that

$$
\begin{aligned}
(2L)^{-1}\delta_E(c_2^{-1}\|x_{n+1} - x_n\|) &\le \langle Jx_{n+1} - Jx_n, x_{n+1} - x_n\rangle \\
&\le \|Jx_{n+1} - Jx_n\|\|x_{n+1} - x_n\| \\
&\le \lambda_n M_0\|x_{n+1} - x_n\|. \qquad (3.14)
\end{aligned}
$$

We hence obtain that

$$\|x_{n+1} - x_n\| \le c_2\delta_E^{-1}(\lambda_n M_0^*), \text{ for some } M_0^* > 0. \qquad (3.15)$$

Using inequality (2.11) with $y^* = \lambda_n\eta_n + \lambda_n\theta_n(Jx_n - Ju)$, we obtain, using also inequality (3.15) and the monotonicity of A that:

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &= V(x^*, Jx_n - \lambda_n\eta_n - \lambda_n\theta_n(Jx_n - Ju)) \\
&\le V(x^*, Jx_n) - 2\lambda_n\langle x_{n+1} - x_n, \eta_n + \theta_n(Jx_n - Ju)\rangle \\
&\quad -2\lambda_n\langle x_n - x^*, \eta_n + \theta_n(Jx_n - Ju)\rangle \\
&\le \phi(x^*, x_n) - 2\lambda_n\langle x_n - x^*, \eta_n + \theta_n(Jx_n - Ju)\rangle \\
&\quad +2\lambda_n\|x_{n+1} - x_n\|\|\eta_n + \theta_n(Jx_n - Ju)\| \\
&\le \phi(x^*, x_n) - 2\lambda_n\theta_n\langle x_n - x^*, Jx_n - Ju\rangle + 2\lambda_n M_0 c_2\delta_E^{-1}(\lambda_n M_0^*) \\
&\le \phi(x^*, x_n) - 2\lambda_n\theta_n\langle x_n - x_{n+1}, Jx_n - Ju\rangle \\
&\quad -2\lambda_n\theta_n\langle x_{n+1} - x^*, Jx_n - Jx_{n+1}\rangle \\
&\quad -2\lambda_n\theta_n\langle x_{n+1} - x^*, Jx_{n+1} - Ju\rangle + 2\lambda_n M_0 c_2\delta_E^{-1}(\lambda_n M_0^*).
\end{aligned}
$$

We have from Lemma 2.3 that,

$$-2\lambda_n\theta_n\langle x_{n+1} - x^*, Jx_{n+1} - Ju\rangle \le \lambda_n\theta_n\phi(x^*, u) - \lambda_n\theta_n\phi(x^*, x_{n+1}).$$

Substituting this in inequality (3.16), we obtain that:

$$\begin{aligned} r &< \phi(x^*,x_{n+1}) \\ &\leq \phi(x^*,x_n)-\lambda_n\theta_n\phi(x^*,x_{n+1})+\lambda_n\theta_n\phi(x^*,u)+2\lambda_n\theta_n M_1c_2\delta_E^{-1}(\lambda_nM_0^*) \\ &\quad +2\lambda_n\theta_nM_2(\lambda_nM_0)+2\lambda_nM_0c_2\delta_E^{-1}(\lambda_nM_0^*) \\ &\leq \phi(x^*,x_n)-\lambda_n\theta_n\phi(x^*,x_{n+1})+\lambda_n\theta_n\phi(x^*,u)+2\lambda_n\theta_n\gamma_0M_1c_2 \\ &\quad +2\lambda_n\theta_n\gamma_0M_2M_0+2\lambda_n\theta_n\gamma_0M_0c_2 \\ &\leq \phi(x^*,x_n)-\lambda_n\theta_n\phi(x^*,x_{n+1})+4\lambda_n\theta_n\frac{r}{8} \\ &\leq r-\lambda_n\theta_nr+\frac{\lambda_n\theta_nr}{2}=r-\frac{\lambda_n\theta_nr}{2}<r. \end{aligned}$$

This is a contradiction. Hence, $\{x_n\}_{n=1}^{\infty}$ is bounded. The rest of the argument to establish that $\{x_n\}$ converges strongly to a zero of A now follows exactly as in the proof of Theorem 3.2 in [16]. □

4. Application to Hammerstein integral equations

In this section, we apply Theorem 3.2 to approximate a solution of a Hammerstein integral equation.

Definition 4.1. Let $\Omega\subset\mathbb{R}^n$ be bounded. Let $k:\Omega\times\Omega\to\mathbb{R}$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form

$$u(x)+\int_\Omega k(x,y)f(y,u(y))dy=w(x), \tag{4.16}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions.

By some transformation, equation (4.16) can be written as

$$u+KFu=0, \tag{4.17}$$

where, without loss of generality, we have taken $w(x)=0,\ \forall\, x\in E$ (see e.g., Pascali and Sburian [48]).

Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts posses Green's functions can, as a rule, be transformed into the form (4.16) (see e.g., Pascali and Sburian [48], Chapter IV, p. 164).

Several existence and uniqueness theorems have been proved for equations of the Hammerstein type (see e.g., Brezis and Browder [5, 6, 7], Browder [8], Browder *et al.* [9], Browder and Gupta [10], Cydotchepanovich [14], and De Figueiredo and Gupta [37]). In general, these equations are nonlinear and there is no known method to find closed form solutions for them. Consequently, methods of approximating solutions of such equations are of interest (see e.g., Brezis and Browder [5], Zegeye [60], Ofoedu and Onyi [49], Minjibir and Mohammed [47], Chidume and Djitte [21, 26, 27, 29, 30], Chidume and Ofeodu [28], Chidume *et al.* [23, 24] Chidume and Shehu [19, 20, 22, 25], Chidume and Bello [17], Chidume and Zegeye [31, 32, 33], Djitte and Sene [35, 36], Mindy *et al.* [46], Sow *et al.* [55, 56] and also Chapter 13 of [15]).

Here, we shall apply Theorem 3.2 to approximate a solution of problem (4.17). The following lemmas and remark would be needed in what follows.

Lemma 4.6 (Browder, [4]). *Let X be a strictly convex reflexive Banach space with a strictly convex conjugate space X^*, T_1 a maximal monotone mapping from X to X^*, T_2 a hemicontinuous monotone mapping of all of X into X^* which carries bounded subsets of X into bounded subsets of X^*. Then, the mapping $T = T_1 + T_2$ is a maximal monotone mapping of X into X^*.*

Lemma 4.7. *Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^* and $X = E \times E^*$. Let $F : E \to E^*$ and $K : E^* \to E$ be monotone mappings. Let $A : X \to X^*$ be defined by $A[u, v] = [Fu - v, Kv + u]$. Then, A is maximal monotone.*

Proof. Define $S, T : E \times E^* \to E^* \times E$ as

$$S[u, v] = [Fu, Kv]; \quad T[u, v] = [-v, u].$$

Define $A := S + T$. It is shown in [18] that S is maximal monotone and that T is monotone and hemicontinuous. Hence, by Lemma 4.6, A is maximal monotone. □

Lemma 4.8 (Chidume and Idu [18]). *Let $q > 1$ and let X, Y be real uniformly convex and uniformly smooth spaces. Let $E = X \times Y$ with the norm $\|z\|_E = (\|u\|_X^q + \|v\|_Y^q)^{\frac{1}{q}}$, for arbitrary $z = [u, v] \in E$. Let $E^* = X^* \times Y^*$ denote the dual space of E. For arbitrary $x = [x_1, x_2] \in E$, define the mapping $j_q^E : E \to E^*$ by*

$$j_q^E(x) = j_q^E[x_1, x_2] := [j_q^X(x_1), j_q^Y(x_2)],$$

so that for arbitrary $z_1 = [u_1, v_1]$, $z_2 = [u_2, v_2]$ in E, the duality pairing $\langle \cdot, \cdot \rangle$ is given by

$$\langle z_1, j_q^E \rangle := \langle u_1, j_q^X(u_2) \rangle + \langle v_1, j_q^Y(v_2) \rangle.$$

Then,

(a.) E is uniformly smooth and uniformly convex,
(b.) j_q^E is single-valued duality mapping on E.

Remark 4.2. We remark that for A defined in Lemma 4.7, $[u^*, v^*]$ is a zero of A if and only if u^* solves (4.17), where $v^* = Fu^*$. Also, Lemma 4.8 holds for the normalized duality map, $J_2^E = J$.

We now prove the following theorem.

Theorem 4.3. *Let X be a uniformly smooth and uniformly convex real Banach space with dual space X^*. Let $F : X \to X^*$, and $K : X^* \to X$ be maximal monotone mappings. Let $E := X \times X^*$ and $A : E \to E^*$ be define by $A[u, v] := [Fu - v, Kv + u]$. For arbitrary z_1, $w_1 \in E$, define the sequence $\{w_n\}$ in E by*

$$w_{n+1} = J^{-1}\left[Jw_n - \lambda_n A w_n - \lambda_n \theta_n (Jw_n - Jz_1)\right], \; n \geq 1.$$

Assume that the equation $u + KFu = 0$ has a solution. Then, the sequence $\{w_n\}_{n=1}^\infty$ converge strongly to a solution of $u + KFu = 0$.

Proof. By Lemma 4.8, E is uniformly convex and uniformly smooth, and by Lemma 4.7, A is maximal monotone. Hence, the conclusion follows from Theorem 3.2 and Remark 4.2. □

Theorem 4.3 can also be stated as follows.

Theorem 4.4. *Let X be a uniformly smooth and uniformly convex real Banach space with dual space X^*. Let $F : X \to X^*$, and $K : X^* \to X$ be maximal monotone mappings. For (x_1, y_1), $(u_1, v_1) \in X \times X^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^* respectively, by*

$$u_{n+1} = J^{-1}\left[Ju_n - \lambda_n (Fu_n - v_n) - \lambda_n \theta_n (Ju_n - Jx_1)\right], \; n \geq 1, \tag{4.18}$$

(4.19) $$v_{n+1} = J\left[J^{-1}v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(J^{-1}v_n - J^{-1}y_1)\right],\ n \geq 1.$$

Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

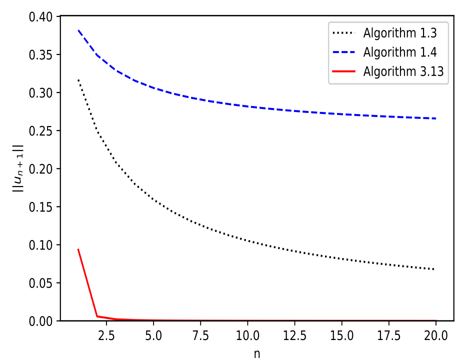
5. Numerical illustrations

In this section, we demonstrate numerically the convergence of the sequence generated by our algorithm and some important algorithms.

Example 5.2. Let $E = L_2([0,1])$ and $Ax(t) = tx(t)$. Then, A is monotone and $0 \in A^{-1}0$. Taking $\lambda_n = c_n = \frac{1}{n+1}$ in algorithms (1.3) and (1.4), respectively, and $\lambda_n = \frac{1}{(n+1)^{\frac{1}{2}}}$ and $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$ in algorithm (3.13), setting maximum number of iteration $n = 20$ and tolerance 10^{-5}, we obtain the following iterates and graph.

n	Alg. (1.3) $\|x_{n+1}\|$	Alg. (1.4) $\|x_{n+1}\|$	Alg. (3.13) $\|x_{n+1}\|$
1	0.3819	0.3169	0.0935
3	0.329	0.2496	2.05e-3
5	0.3059	0.1798	6.19e-4
7	0.293	0.1309	2.75e-4
10	0.2816	0.1052	1.1e-4
15	0.2716	0.0813	3.54e-5
20	0.2659	0.0677	1.48e-5

Table of values choosing $x_1(t) = t^2$



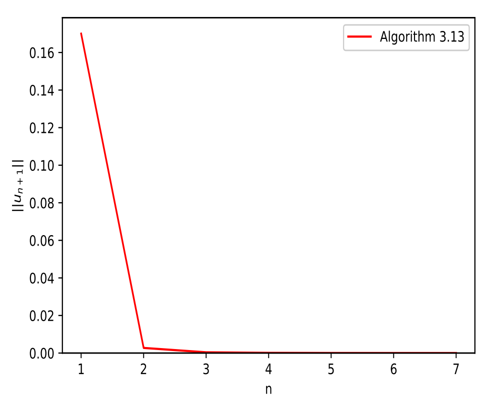
Graph of the first 20 iterates of algorithms (1.3), (1.4) and (3.13) choosing $x_1(t) = t^2$

Next we an example in the Banach space to support our main theorem 3.2.

Example 5.3. Let $E = L_3([0,1])$ and $Ax(t) = x(t)$. Then, A is monotone and $0 \in A^{-1}0$. Taking $\lambda_n = \frac{1}{(n+1)^{\frac{1}{2}}}$ and $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$ in algorithm (3.13), setting maximum number of iteration $n = 20$ and tolerance 10^{-5}, we obtain the following iterates and graph.

n	Algorithm (3.13) $\|x_{n+1}\|$
1	0.17
2	2.27e-3
3	3.99e-4
4	1.01e-4
5	3.35e-5
6	1.31e-5
7	5.69e-6

Table of values choosing $x_1(t) = \sin t$



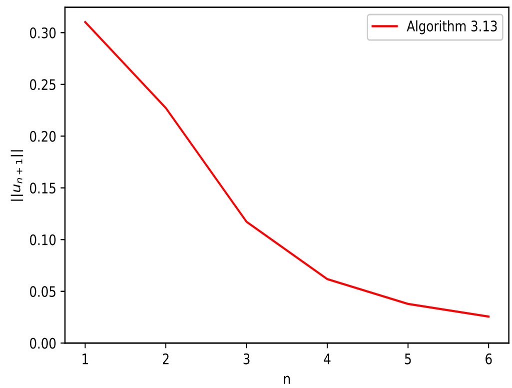
Graph of the first 7 iterates of algorithm (3.13) choosing $x_1(t) = \sin t$

Finally, we a give an example to show that Theorem 4.4 is implementable.

Example 5.4. In Theorem 4.4, set $E = L_5([0,1])$, then $E^* = L_{\frac{5}{4}}([0,1])$. Let $F : E \to E^*$ be defined by $(Fu)(t) := Ju(t)$. Then, it is to see that F maximal monotone. Let $K : E^* \to E$ be defined $(Kv)(t) = tv(t)$. Observe that by definition K is linear. Furthermore, it is easy to see that K maximal monotone and the function $u^*(t) = 0, \forall t \in [0,1]$ is the only solution of the equation $u + KFu = 0$. We take $\lambda_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, as our parameters and fixed. Setting a tolerance of 10^{-6} and maximum number of iterations $n = 6$, we obtain the following iterates and graph

n	Alg. (4.18) $\|u_{n+1}\|$	Alg. (4.19) $\|v_{n+1}\|$
1	0.3102	0.3907
2	0.2271	0.1636
3	0.1171	0.0685
4	0.0618	0.0395
5	0.0378	0.0251
6	0.0255	0.0175

Table of values choosing $u_1(t) = \cos t,\ v_1(t) = \sin t$



Graph of the first 7 iterates of algorithm (4.18) choosing $u_1(t) = \cos t,\ v_1(t) = \sin t$

6. Conclusion

In this paper, an iterative algorithm that complements the proximal point algorithm is constructed. Strong convergence of the sequence generated by the algorithm is proved in a uniformly convex and uniformly smooth real Banach space. The convergence theorem proved is applied to approximate a solution of a Hammerstein integral equation. Finally, a numerical experiment is presented to demonstrate the convergence of the sequence of the proposed algorithm and its convergence is compared with that of two existing algorithms.

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