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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

On the geometry of b-distances and the fixed points of mappings

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ABSTRACT. In the present article we study the geometrical configuration of b-perturbation of the norm of the normed space. The notion of b-metric is important in the solving of the fixed point problem.

1. PRELIMINARIES

By a space we understand a topological T_0-space. We use the terminology from [20, 22, 34]. The problem of fixed points is one of the most investigated one and consists in finding conditions under which for a given mapping $\varphi : X \longrightarrow X$ the set of fixed points $Fix(\varphi)$ = $\{x \in X : \varphi(x) = x\}$ of φ is non-empty. Till now, there were founded various conditions that use distinct structures on the space X: metrical structures; ordering structures; linear structures etc., see for example [34].

Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a mapping such that for all $x, y \in X$, we have

(i_m) $d(x, y) \geq 0$; (ii_m) $d(x, y) + d(y, x) = 0 \Rightarrow x = y$.

Then (X, d) is called a *dislocated distance space* and d is called *a dislocated distance* on X. A dislocated distance d on a set X is called a *distance* if we have

(iii_m) $d(x, x) = 0$ for any $x \in X$.

A dislocated distance $d : X \times X \longrightarrow \mathbb{R}$ on a set X is called *a dislocated b-quasi-metric* if there exists a positive number k, named a b-constant, such that for all $x, y, z \in X$ we have

(iv_m) $d(x, z) \leq k(d(x, y) + d(y, z))$.

A dislocated distance $d : X \times X \longrightarrow \mathbb{R}$ on a set X is called symmetric if for all $x, y \in X$ we have:

(v_m) $d(x, y) = d(y, x)$.

A symmetric dislocated b-quasi-metric d on X is called *a dislocated b-metric*.

A dislocated b-quasi-metric d generates the following dislocated b-metrics

$$d_s(x, y) = d(x, y) + d(y, x), \qquad d_m(x, y) = \max\{d(x, y), d(y, x)\}.$$

Any dislocated b-quasi-metric (dislocated b-metric) with the b-constant $k = 1$ is called a *dislocated quasi-metric* (*dislocated metric*).

A distance d which is a dislocated b-quasi-metric (dislocated b-metric) is called a *b-quasi-metric* (*b-metric*). A b-quasi-metric (b-metric) with the b-constant $k = 1$ is called a *quasi-metric* (*metric*).

Let d be a dislocated distance on X, $B(x, d, r) = \{y \in X : d(x, y) < r\} \cup \{x\}$ be the open d-ball and $B[x, d, r] = \{y \in X : d(x, y) \leq r\} \cup \{x\}$ be the closed d-ball with the center x and

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radius $r > 0$. The set $U \subset X$ is called d-open if for any $x \in U$ there exists $r > 0$ such that $B(x, d, r) \subset U$. The family $T(d)$ of all d-open subsets is the topology of X generated by d. A distance space is a sequential space, i.e., a set $B \subseteq X$ is closed if and only if together with any sequence it contains all its limits [20].

If d is a dislocated distance on X and Y is a subset of X, then $\rho = d|Y$ is the restriction of the distance d to Y: $\rho(x, y) = d(x, y)$, for all $x, y \in Y$.

Let (X, d) be a dislocated distance space, $\{x_n : n \in \mathbb{N} = \{1, 2, ...\}\}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n : n \in \mathbb{N}\}$:

1) is *convergent* in X if for any d-open subset $U \in T(d)$, for which $x \in U$, there exists a number $n \in \mathbb{N} = \{1, 2, 3, ...\}$ such that $\{x_m : m \in \mathbb{N}, m \geq n\} \subset U$. We denote this by $x_n \to x$ or $x = \lim_{n\to\infty} x_n$;
2) is *d-convergent to x* if and only if $\lim_{n\to\infty} d(x, x_n) = 0$. We denote this by $x = d$-$\lim_{n\to\infty} x_n$.
3) is *Cauchy* if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

We say that a dislocated distance space (X, d) is *complete* if every Cauchy sequence in X d-converges to some point in X.

Any d-convergent sequence is convergent. The reverse assertion is not true, in general.

Example 1.1. Let $X = \{a, b\} \cup \mathbb{N}$, where $a \neq b$ and $\{a, b\} \cap \mathbb{N} = \emptyset$. We put $d(x, x) = 0$, for each $x \in X$, $d(a, b) = 1$, $d(b, a) = 0$ and $d(n, b) = d(b, n) = 1$, $d(n, a) = d(a, n) = 2^{-n}$, $d(n, m) = |2^{-n} - 2^{-m}|$, for each $n, m \in \mathbb{N}$. Then $(X, T(d))$ is a compact T_0-space. The distance $d_Y = d|Y$ is a metric on the subspace $Y = \{a\} \cup \mathbb{N}$ and $(Y, T(d))$ is a compact metrizable subspace of $(X, T(d))$. If $Z = \{b\} \cup \mathbb{N}$ and $d_Z = d|Z$, then:

(a) Z as the subspace of $(X, T(d))$ is a compact Hausdorff space homeomorphic to $(Y, T(d_Y))$;
(b) $(Z, T(d_Z))$ is a countable discrete space;
(c) (Z, d_Z) is a symmetric space;
(d) the sequence $\{n : n \in \mathbb{N}\}$ is convergent to the points a, b in the space $(X, T(d))$;
(e) the sequence $\{n : n \in \mathbb{N}\}$ is d-convergent to the point $a : \lim_{n\to\infty} d(a, n) = 0$;
(f) the sequence $\{n : n \in \mathbb{N}\}$ is not d-convergent to the point $b : \lim_{n\to\infty} d(b, n) = 1$.

Example 1.2. Let a, c be two non-negative numbers, $c > 0$ and X be a set with the cardinality $|X| \geq 2$. We put $d_{(a,c)}(x, x) = a$ for each $x \in X$ and $d_{(a,c)}(x, y) = c$ for all distinct points $x, y \in X$. By construction:

(a) $d_{(a,c)}$ is a complete dislocated b-metric on X with the b-constant $k \geq 1$ for which $2kc \geq a$;
(b) if $a = 0$, then $d_{(a,c)}$ is a complete metric on X;
(c) if $a > 0$ and $2c \geq a$, then $d_{(a,c)}$ is a complete dislocated metric on X;
(d) the topology $T(d_{(a,c)})$ is discrete.

Example 1.3. Let $(X, \|\cdot\|)$ be a normed space. For any $x, y \in X$ and any number $\lambda > 1$ we denote:

$$d(x, y) = \|x - y\|, \qquad d_0(x, y) = \max\{\|x\|, \|y\|\}, \qquad d_\lambda(x, y) = \lambda\|x\| + \|y\|.$$

Then, by construction:

(a) d is an invariant metric on X;
(b) d_0 is a complete dislocated metric on X;
(c) d_λ is a complete dislocated quasi-metric on X.

This article was motivated by the following questions put by Selma Gülyaz-Özyurt to the author (personal communication):

1. *Is it true that every closed ball in a dislocated b-metric spaces is a closed set?*

2. *I thought that "an open ball" is an "open set", but I am not sure about the "closed" case. If I am not bothering you, could I ask whether I am right on the "open case"?*

According to the constructions and examples in Sections 3 – 5, the answers to these questions for b-metric spaces and dislocated b-metric spaces are in the negative.

2. On a topology of a dislocated metric space

Let ρ be a dislocated distance on X. The topology $T(\rho)$ is not dislocated in the sense of the works [24, 23]. In the dislocated topology the neighborhoods of points may to be empty, or may not include the points.

The set $MK(X, \rho) = \{x \in X : \rho(x, x) = 0\}$ is the metric kernel of (X, ρ). If ρ is a dislocated b-metric, then for any $x \in X \setminus MK(X, \rho)$ we have $\{x\} \in T(\rho)$, i.e., the point x is isolated in the space $(X, T(\rho))$. Therefore $MK(X, d)$ is a closed subspace and $X \setminus MK(X, d)$ is a discrete subspace of the space $(X, T(d))$.

A dislocated distance space (X, ρ) is a *dislocated H-distance space* if for any two distinct points $a, b \in X$ there exists $r > 0$ such that $B(a, \rho, r) \cap B(b, \rho, r) = \emptyset$. A dislocated b-distance space (X, ρ) is a *dislocated H^*-distance space* if any convergent Cauchy sequence in X converges to a unique point in X.

Two dislocated distances ρ_1 and ρ_2 on X are called:

- *topologically equivalent* if $T(\rho_1) = T(\rho_2)$;
- *uniformly equivalent* if for any $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that $\rho_1(x, y) < \varepsilon$ provided $\rho_2(x, y) < \delta$ and $\rho_2(x, y) < \varepsilon$ provided $\rho_1(x, y) < \delta$.

Proposition 2.1. *Let ρ_1 and ρ_2 be two uniformly equivalent dislocated distances on X. Then:*

1) $MK(X, \rho_1) = MK(X, \rho_2)$.
2) $T(X, \rho_1) = T(X, \rho_2)$.
3) *If $x \in X$, then the spaces (X, ρ_1) and (X, ρ_2) share the same ρ_1-convergent and ρ_2-convergent to the point x sequences.*
4) *The spaces (X, ρ_1) and (X, ρ_2) share the same Cauchy sequences.*
5) *The space (X, ρ_1) is complete if and only if the space (X, ρ_2) is complete.*

Proof. It is obvious. □

Proposition 2.2. *Let d be a dislocated distance on X. We put $\rho(x, x) = 0$ for any $x \in X$ and $\rho(x, y) = d(x, y)$ for any distinct points $x, y \in X$. The function ρ is called a location of d. Are true:*

1) *ρ is a distance on X.*
2) $T(X, d) = T(X, \rho)$.
3) *If $x \in X$, then the spaces (X, d) and (X, ρ) are the same d-convergent and ρ-convergent to the point x sequences.*
4) *The spaces $(X, d) = (X, \rho)$ are the same Cauchy sequences.*
5) *The space (X, d) is complete if and only if the space (X, ρ) is complete.*
6) *If d is a dislocated b-quasi-metric on X, then ρ is a b-quasi-metric on X with the same b-constant.*
7) *d is a symmetric if and only if ρ is symmetric.*
8) *If d is a dislocated b-metric on X, then ρ is a b-metric on X with the same b-constant.*

Proof. It is obvious. □

Remark 2.1. Let G be a group with the binary operation $+$. A dislocated distance d on G is called *invariant dislocated distance* if $d(x + z, y + z) = d(z + x, z + y) = d(-x, -y)$ for all $x, y, z \in G$. In this case $d(x, x) = d(y, y)$ for all $x, y \in G$. Hence for an invariant dislocated distance d we have $MK(G, d) = \emptyset$ or d is a distance on G.

3. Perturbation of the norms

Let E be a linear space of linear dimension $\dim E \geq 2$. A *b-norm* on E is a function $\nu : E \longrightarrow \mathbb{R}$ with the following properties:

(i_n) $\nu(x) \geq 0$ for each $x \in E$;
(ii_n) $\nu(x) = 0$ if and only if $x = 0$;
(iii_n) $\nu(tx) = |t| \cdot \nu(x)$ for all $x \in X$ and $t \in \mathbb{R}$;
(iv) there exists a constant $k \geq 1$, named a b-constant, such that $\nu(x+y) \leq k(\nu(x) + \nu(y))$.

A b-norm with the b-constant $k = 1$ is called a *norm*.

Any b-norm ν with the b-constant k generates the invariant b-metric $\rho_\nu(x,y) = \nu(x-y)$ with the same b-constant k. We put $T(\nu) = T(\rho_\nu)$.

Let L be a non-empty subset of the linear normed space (E, ν) and $L \subset S(\nu, 1) = \{x \in E : \nu(x) = 1\}$. If $x \in E$ and $tx \in L$ for some $t > 0$, then we denote $x \parallel L$. By definition, we consider that $0 \parallel L$. If $x \neq 0$ and $tx \notin L$ for any $t > 0$, then we denote $x \nparallel L$.

Two b-norms ν_1 and ν_2 are called:

- *topologically equivalent* if $T(\nu_1) = T(\nu_2)$;
- *uniformly equivalent* if for any $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that $\nu_1(x) < \varepsilon$ provided $\nu_2(x) < \delta$ and $\nu_2(x) < \varepsilon$ provided $\nu_1(x) < \delta$.

Proposition 3.3. *Let ν and μ be two b-norms on a linear space E. Then the following assertions are equivalent:*

1) *The b-norms ν and μ are topologically equivalent.*
2) *The b-norms ν and μ are uniformly equivalent.*
3) *The b-metrics ρ_ν and ρ_μ are topologically equivalent.*
4) *The b-metrics ρ_ν and ρ_μ are uniformly equivalent.*

Proof. It is obvious. □

Theorem 3.1. *Let (E, ν) be a linear normed space of linear dimension $\dim E \geq 2$, $F \subset S(\nu, 1)$ be a non-empty set, $F = -F$, the set $\Phi = S(\nu, 1) \setminus F$ is non-empty, $\lambda > 0$, $\nu_\lambda(0) = 0$, $\nu_\lambda(x) = \nu(\lambda x)$ provided $x \parallel F$ and $\nu_\lambda(x) = \nu(x)$ provided $x \parallel \Phi$. Then:*

(1) *ν_λ is a b-norm on E.*
(2) *The b-norms ν and ν_λ are uniformly equivalent.*
(3) *If $\lambda > 1$, then $k = \lambda$ is the minimal b-constant of ν_λ.*
(4) *If $\lambda < 1$, then $k = \lambda^{-1}$ is the minimal b-constant of ν_λ.*

Proof. By construction, we have:

- $\nu_\lambda(x) \geq 0$ for each $x \in E$;
- $\nu_\lambda(x) = 0$ if and only if $x = 0$;
- if $x \in E$ and $x \parallel F$, then $\nu(x) = \lambda^{-1}\nu_\lambda(x)$;
- if $x \in E$, $x \parallel F$ and $t \in \mathbb{R}$, then $\nu_\lambda(tx) = \nu(t\lambda x) = |t| \cdot \nu(\lambda x) = |t| \cdot \nu_\lambda(x)$;
- if $x, y \in E$ and $x + y = 0$, then $\nu_\lambda(x+y) \leq \nu_\lambda(x) + \nu_\lambda(y)$;
- if $x, y \in E$ and $0 \in \{x, y\}$, then $\nu_\lambda(x+y) = \nu_\lambda(x) + \nu_\lambda(y)$;
- if $x \in E$, $x \parallel \Phi$ and $t \in \mathbb{R}$, then $\nu_\lambda(tx) = \nu(tx) = |t| \cdot \nu(x) = |t| \cdot \nu_\lambda(x)$.

Assume that $\lambda > 1$. Fix $x, y \in E$. We can suppose that $0 \notin \{x, y\}$ and $x + y \neq 0$. We have the following cases:

Case 1. $x \parallel F$, $y \parallel F$ and $(x+y) \parallel F$.

In this case $\nu_\lambda(x+y) = \nu(\lambda(x+y)) = \lambda\nu(x+y) \leq \lambda(\nu(x) + \nu(y)) = \lambda\nu(x) + \lambda\nu(y) = (\nu_\lambda(x) + \nu_\lambda(y)) < \lambda(\nu_\lambda(x) + \nu_\lambda(y))$.

Case 2. $x \parallel F$, $y \parallel F$ and $(x+y) \parallel \Phi$.

In this case $\nu_\lambda(x+y) = \nu(x+y) \le \nu(x)+\nu(y) = \lambda^{-1}(\nu_\lambda(x)+\lambda\nu_\lambda(y)) < \lambda(\nu_\lambda(x)+\nu_\lambda(y))$.

Case 3. $x \parallel F$, $y \parallel \Phi$ and $(x+y) \parallel F$.

In this case $\nu_\lambda(x+y) = \nu(\lambda(x+y)) = \lambda\nu(x+y) \le \lambda(\nu(x)+\nu(y)) = \lambda\nu(x)+\lambda\nu(y) = \nu_\lambda(x)+\lambda\nu_\lambda(y) < \lambda(\nu_\lambda(x)+\nu_\lambda(y))$.

Case 4. $x \parallel F$, $y \parallel \Phi$ and $(x+y) \parallel \Phi$.

In this case $\nu_\lambda(x+y) = \nu(x+y) \le \nu(x)+\nu(y) = \lambda^{-1}\nu(\lambda x)+\nu_\lambda(y) < \nu_\lambda(x)+\nu_\lambda(y)) < \lambda(\nu_\lambda(x)+\nu_\lambda(y))$.

Case 5. $x \parallel \Phi$, $y \parallel \Phi$ and $(x+y) \parallel F$.

In this case $\nu_\lambda(x+y) = \nu(\lambda(x+y)) = \lambda\nu(x+y) \le \lambda(\nu(x)+\nu(y)) = \lambda(\nu_\lambda(x)+\nu_\lambda(y))$.

Case 6. $x \parallel \Phi$, $y \parallel \Phi$ and $(x+y) \parallel \Phi$.

In this case $\nu_\lambda(x+y) = \nu(x+y) \le \nu(x)+\nu(y)) = \nu_\lambda(x)+\nu_\lambda(y)) < \lambda(\nu_\lambda(x)+\nu_\lambda(y))$.

Hence ν_λ is a b-norm with the b-constant λ. Since $\nu(x) \le \lambda\nu_\lambda(x) \le \lambda\nu(x)$, the b-norms ν and ν_λ are uniformly equivalent.

Now we fix $a \in F$ and $b \in \Phi$. Since $-a \in F$ and $-b \in \Phi$, we have the following cases:

Case 7. $(a+b) \parallel F$.

We put $x = a+b$ and $y = -b$. Then $(x+y) = a$ and $a \parallel F$. In this case, for any $t \in \mathbb{R}$ we have $\nu_\lambda(tx+y) = \nu(\lambda(tx+y)) = \lambda\nu(tx+y) \le \lambda(\nu(tx)+\nu(y)) = |t|\cdot\lambda\nu(x)+\lambda\nu_\lambda(b) \le \lambda(\nu_\lambda(tx)+\lambda\nu_\lambda(y)$. If $1 < k < \lambda$, then $2\varepsilon = \lambda\nu(y) > 0$ and there exists $t > 0$ such that $0 < kt\lambda\nu(x) < \varepsilon$. Hence $k(\nu_\lambda(tx)+\nu_\lambda(y)) < \lambda\nu_\lambda(y)$ and λ is the minimal b-constant of ν_λ.

Case 8. $(a-b) \parallel F$.

We put $x = a-b$ and $y = b$. Then $(x+y) = a$ and $a \parallel F$. As in the Case 7 we can established that λ is the minimal b-constant of ν_λ.

Case 9. $(a+b) \parallel \Phi$ and $(a-b) \parallel \Phi$.

We put $x = a+b$ and $y = a-b$. Then $(x+y) = 2a$ and $2a \parallel F$.

In this case $\nu_\lambda(x+y) = \nu(\lambda(x+y)) = \lambda\nu(x+y) \le \lambda(\nu(x)+\nu(y)) = \lambda(\nu_\lambda(x)+\nu_\lambda(y))$. Hence λ is the minimal b-constant of ν_λ. In the case $\lambda > 1$ the assertions of the theorem are proved.

Assume now that $0 < \lambda < 1$. Fix $x, y \in E$. We can suppose that $0 \notin \{x, y\}$ and $x+y \ne 0$. We have the following cases:

Case 10. $x \parallel F$, $y \parallel F$ and $(x+y) \parallel F$.

In this case $\nu_\lambda(x+y) = \nu(\lambda(x+y)) = \lambda\nu(x+y) \le \lambda(\nu(x)+\nu(y)) = \lambda\nu(x)+\lambda\nu(y) = \nu_\lambda(x)+\nu_\lambda(y) < \lambda^{-1}(\nu_\lambda(x)+\nu_\lambda(y))$.

Case 11. $x \parallel F$, $y \parallel F$ and $(x+y) \parallel \Phi$.

In this case $\nu_\lambda(x+y) = \nu(x+y) \le \nu(x)+\nu(y) = \lambda^{-1}(\nu(\lambda x)+\nu(\lambda y)) = \lambda^{-1}(\nu_\lambda(x)+\nu_\lambda(y))$.

Case 12. $x \parallel F$, $y \parallel \Phi$ and $(x+y) \parallel F$.

In this case $\nu_\lambda(x+y) = \nu(\lambda(x+y)) = \lambda\nu(x+y) \le \lambda(\nu(x)+\nu(y)) = \lambda\nu(x)+\lambda\nu(y) = \nu_\lambda(x)+\lambda\nu_\lambda(y) < \lambda^{-1}(\nu_\lambda(x)+\nu_\lambda(y))$.

Case 13. $x \parallel F$, $y \parallel \Phi$ and $(x+y) \parallel \Phi$.

In this case $\nu_\lambda(x+y) = \nu(x+y) \le \nu(x)+\nu(y) = \lambda^{-1}\nu(\lambda x)+\nu_\lambda(y) = \lambda^{-1}\nu_\lambda(x)+\nu_\lambda(y)) < \lambda^{-1}(\nu_\lambda(x)+\nu_\lambda(y))$.

Case 14. $x \parallel \Phi$, $y \parallel \Phi$ and $(x+y) \parallel F$.

In this case $\nu_\lambda(x+y) = \nu(\lambda(x+y)) = \lambda\nu(x+y) \le \lambda(\nu(x)+\nu(y)) = \lambda(\nu_\lambda(x)+\nu_\lambda(y)) < \lambda^{-1}(\nu_\lambda(x)+\nu_\lambda(y))$.

Case 15. $x \parallel \Phi$, $y \parallel \Phi$ and $(x+y) \parallel \Phi$.

In this case $\nu_\lambda(x+y) = \nu(x+y) \leq \nu(x)+\nu(y)) = \nu_\lambda(x)+\nu_\lambda(y)) < \lambda^{-1}(\nu_\lambda(x)+\nu_\lambda(y))$.

Hence ν_λ is a b-norm with the b-constant λ^{-1}. Since $\nu_\lambda(x) \leq \nu(x) \leq \lambda^{-1}\nu_\lambda(x)$, the b-norms ν and ν_λ are uniformly equivalent. As for $\lambda > 1$ we established that $k = \lambda^{-1}$ is the minimal b-constant of ν_λ. The proof is complete. □

Remark 3.2. Let (E,ν) be a linear normed space of linear dimension $\dim E \geq 2$, $F \subset S(\nu,1)$ be a non-empty set, $F = -F$, the set $\Phi = S(\nu,1) \setminus F$ be non-empty, $\lambda > 0$ and $r > 0$.

1) If $\lambda > 1$ and the set F is closed in the topology $T(\nu)$, then:
   - the "open" ν_λ-ball $B(0,\nu_\lambda,r)$ is open in the topology $T(\nu)$;
   - the "closed" ν_λ-ball $B[0,\nu_\lambda,r]$ is not closed in the topology $T(\nu)$.
2) If $\lambda < 1$ and the set F is closed in the topology $T(\nu)$, then:
   - the "open" ν_λ-ball $B(0,\nu_\lambda,r)$ is not open in the topology $T(\nu)$;
   - the "closed" ν_λ-ball $B[0,\nu_\lambda,r]$ is closed in the topology $T(\nu)$.
3) If $\lambda \neq 1$ and the sets F and Φ are dense in $S(\nu,1)$ in the topology $T(\nu)$, then:
   - the "open" ν_λ-ball $B(0,\nu_\lambda,r)$ is not open in the topology $T(\nu)$;
   - the "closed" ν_λ-ball $B[0,\nu_\lambda,r]$ is not closed in the topology $T(\nu)$.

4. Perturbation of the distances

Let (E,ν) be a linear normed space. A distance d on E is called a *homogeneous distance* if $d(x,x+t(y-x)) = t\cdot d(x,y)$ for all $x,y \in E$ and $t > 0$.

Theorem 4.2. *Let (E,ν) be a linear normed space of linear dimension $\dim E \geq 2$. Let $L \subset S(\nu,1)$ be a non-empty subset and the set $M = S(\nu,1) \setminus L$ is non-empty, too. For a distance ρ and a number $\lambda > 0$, we put $\rho_{(L,\lambda)}(x,y) = \lambda\cdot\rho(x,y)$, provided $(y-x) \parallel L$ and $\rho_{(L,\lambda)}(x,y) = \rho(x,y)$, provided $(y-x) \parallel M$. Then*

(1) $\rho_{(L,\lambda)}$ is a distance and $MK(E,\rho_{(L,\lambda)}) = MK(E,\rho)$.

(2) The distances ρ and $\rho_{(L,\lambda)}$ are uniformly equivalent.

(3) If $d = \rho_{(L,\lambda)}$, then $\rho = d_{(L,\lambda^{-1})}$.

(4) If ρ is a b-quasi-metric ρ with the b-constant $k \geq 1$, then $\rho_{(L,\lambda)}$ is a b-quasi-metric ρ with the b-constant $\leq \max\{\lambda k, \lambda^{-1}k\}$. If ρ is a quasi-metric and $T(\rho) = T(\nu)$, then this b-constant is minimal.

(5) If $L = -L$ and ρ is a symmetric distance, then $\rho_{(L,\lambda)}$ is a symmetric distance, too.

(6) If ρ is an invariant distance, then $\rho_{(L,\lambda)}$ is an invariant distance, too.

(7) If ρ is a homogeneous distance, then $\rho_{(L,\lambda)}$ is a homogeneous distance, $\rho_{(L,\lambda)}(x,y) = \rho(x,x+\lambda(y-x))$, provided $(y-x) \parallel L$ and $\rho_{(L,\lambda)}(x,y) = \rho(x,y)$, provided $(y-x) \parallel M$.

Proof. If $\lambda > 1$, then $\lambda^{-1}\rho(x,y) \leq \rho(x,y) \leq \lambda\rho(x,y)$ for all $x,y \in E$. If $0 < \lambda \leq 1$, then $\lambda\rho(x,y) \leq \rho(x,y) \leq \lambda^{-1}\rho(x,y)$ for all $x,y \in E$. Hence ρ and $\rho_{(L,\lambda)}$ are uniformly equivalent distances. Assertions 1 and 2 are proved. Assertion 3 is obvious.

Assume that ρ is a b-quasi-metric ρ with the b-constant $k \geq 1$. If $\lambda > 1$, then $\rho_{(L,\lambda)}(x,z) \leq \lambda\cdot\rho(x,z) \leq \lambda k(\rho(x,y)+\rho(y,z)) \leq \lambda k(\rho_{(L,\lambda)}(x,y)+\rho_{(L,\lambda)}(y,z))$. If $\lambda < 1$, then $\rho_{(L,\lambda)}(x,z) \leq \rho(x,z) \leq k(\rho(x,y)+\rho(y,z)) \leq \lambda^{-1}k(\rho_{(L,\lambda)}(x,y)+\rho_{(L,\lambda)}(y,z))$. Assertion 4 is proved. Assertions 5, 6 and 7 are obvious. The proof is complete. □

Remark 4.3. Let (E,ν) be a linear normed space of linear dimension $\dim E \geq 2$. Let $L \subset S(\nu,1)$ be a non-empty subset and the set $M = S(\nu,1) \setminus L$ be non-empty too. Fix on E an invariant homogeneous metric ρ and a number $\lambda > 0$. Assume that $T(\rho) = T(\nu)$.

1) If $\lambda > 1$ and the set L is closed in the topology $T(\nu)$, then:
   - the "open" $\rho_{(L,\lambda)}$-ball $B(0,\rho_{(L,\lambda)},r)$ is open in the topology $T(\rho)$;

- the "closed" $\rho_{(L,\lambda)}$-ball $B[0, \rho_{(L,\lambda)}, r]$ is not closed in the topology $T(\rho)$.

2) If $\lambda < 1$ and the set M is closed in the topology $T(\nu)$, then:
   - the "open" $\rho_{(L,\lambda)}$-ball $B(0, \rho_{(L,\lambda)}, r)$ is not open in the topology $T(\rho)$;
   - the "closed" $\rho_{(L,\lambda)}$-ball $B[0, \rho_{(L,\lambda)}, r]$ is closed in the topology $T(\rho)$.
3) If $\lambda \neq 1$ and the sets L and M are dense in $S(\nu, 1)$ in the topology $T(\rho)$, then:
   - the "open" $\rho_{(L,\lambda)}$-ball $B(0, \rho_{(L,\lambda)}, r)$ is not open in the topology $T(\rho)$;
   - the "closed" $\rho_{(L,\lambda)}$-ball $B[0, \rho_{(L,\lambda)}, r]$ is not closed in the topology $T(\rho)$.
4) When switching to $B(0, \rho_{(L,\lambda)})$-balls in case $\lambda > 1$ the ρ-balls are drilling in the L-directions, and in the case $\lambda > 1$ are added needles in the L-directions.
5) If the sets L and M are dense in $S(\nu, 1)$ in the topology $T(\rho)$, then the "open" $\rho_{(L,\lambda)}$-ball $B(0, \rho_{(L,\lambda)}, r)$ and the "closed" $\rho_{(L,\lambda)}$-ball $B[0, \rho_{(L,\lambda)}, r]$ are the form of a hedgehog with needles in the directions M for $\lambda > 1$ and in directions L for $\lambda < 1$.

Example 4.4. Let $E = \{(x, y) : x, y \textit{ are real numbers}\}$ be the Euclidean plane and $T(\rho)$ be the topology on E, where $\rho((x, y), (u, v)) = ((x - u)^2 + (y - v)^2)^{1:2}$ is the Euclidean invariant metric generate by the norm $\nu((x, y)) = (x^2 + y^2)^{1:2}$. Fix positive number $\lambda \neq 1$ and non-empty subset L of the circle $S = \{(\cos t, \sin t) : 0^\circ \leq t \leq 360^\circ\}$ of the radius 1 for which $M = S \setminus L$ is non-empty too. Let $m(x, y)$ be the magnitude of the angle in degrees between the vectors $(1, 0)$ and (x, y). If $0^\circ \leq t \leq 360^\circ$, then $m(\cos t, \sin t) = t$. The set L generates the b-quasi-metric $\rho_{(L,\lambda)}$. If $-L = L$, then $\rho_{(L,\lambda)}$ is an invariant homogeneous b-metric with the b-constant $k \in \{\lambda, \lambda^{-1}\}$ and $k \geq 1$. The structure of $\rho_{(L,\lambda)}$-balls:

$B((a, b), \rho_{(L,\lambda)}, r)$, the "open" $\rho_{(L,\lambda)}$-ball of the radius $r > 0$,

$B[(a, b), \rho_{(L,\lambda)}, r]$, the "closed" $\rho_{(L,\lambda)}$-ball of the radius $r > 0$,

depends of the set L and the number λ.

It is interesting the form of $\rho_{(L,\lambda)}$-balls for the following cases:

Case 1. $\lambda = 2$ *and* $L_1 = \{(1, 0), (0, 1)\}$.

In this case:

- $B((0, 0), \rho_{(L_1,2)}, 1) = B((0, 0), \nu, 1) \setminus \{t \cdot (x, y) : (x, y) \in L_1.2^{-1} \leq t \leq 1\}$ is an open and a not closed set and $B[(0, 0), \rho_{(L_1,2)}, 1] = B[(0, 0), \nu, 1] \setminus \{t \cdot (x, y) : (x, y) \in L_1.2^{-1} < t \leq 1\}$ is a not open and a not closed set;
- the sets $B((0, 0), \rho_{(L_1,2)}, 1)$ and $B[(0, 0), \rho_{(L_1,2)}, 1]$ are dense in $B[(0, 0), \nu, 1]$;
- $B((0, 0), \nu, 2^{-1}) \subset B((0, 0), \rho_{(L_1,2)}, 1) \subset B[(0, 0), \rho_{(L_1,2)}, 1]$;
- $\rho_{(L_1,2)}$ is a b-quasi-metric with the b-constant $k = 2$ and $\rho_{(L_1,2)}$ is not a b-metric.

Case 2. $\lambda = 2^{-1}$ *and* $L_1 = \{(1, 0), (0, 1)\}$.

In this case:

- $B((0, 0), \rho_{(L_1,2^{-1})}, 1) = B((0, 0), \nu, 1) \cup \{t \cdot (x, y) : (x, y) \in L_1, 1 \leq t < 2\}$ is a not open and a not closed set and $B[(0, 0), \rho_{(L_1,2^{-1})}, 1] = B[(0, 0), \nu, 1] \setminus \{t \cdot (x, y) : (x, y) \in L_1, 1 \leq t \leq 2\}$ is a not open and a closed set;
- $B((0, 0), \nu, 1) \subset B((0, 0), \rho_{(L_1,2^{-1})}, 1) \subset B[(0, 0), \rho_{(L_1,2^{-1})}, 1]$;
- $\rho_{(L_1,2^{-1})}$ is a b-quasi-metric with the b-constant $k = 2$ and $\rho_{(L_1,2^{-1})}$ is not a b-metric.

Case 3. $\lambda = 2$ *and* $L_2 = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$.

In this case:

- $B((0, 0), \rho_{(L_2,2)}, 1) = B((0, 0), \nu, 1) \setminus \{t \cdot (x, y) : (x, y) \in L_1.2^{-1} \leq t \leq 1\}$ is an open and a not closed set and $B[(0, 0), \rho_{(L_2,2)}, 1] = B[(0, 0), \nu, 1] \setminus \{t \cdot (x, y) : (x, y) \in L_1.2^{-1} < t \leq 1\}$ is a not open and a not closed set;
- the sets $B((0, 0), \rho_{(L_2,2)}, 1)$ and $B[(0, 0), \rho_{(L_2,2)}, 1]$ are dense in $B[(0, 0), \nu, 1]$;
- $B((0, 0), \nu, 2^{-1}) \subset B((0, 0), \rho_{(L_2,2)}, 1) \subset B[(0, 0), \rho_{(L_2,2)}, 1]$;
- $\rho_{(L_2,2)}$ is a b-metric with the b-constant $k = 2$.

Case 4. $\lambda = 2^{-1}$ *and* $L_2 = \{(1,0),(0,1),(-1,0),(0,-1)\}$.

In this case:

- $B((0,0),\rho_{(L_2,2^{-1})},1) = B((0,0),\nu,1) \cup \{t\cdot(x,y) : (x,y) \in L_2, 1 \leq t < 2\}$ is a not open and a not closed set and $B[(0,0),\rho_{(L_2,2^{-1})},1] = B[(0,0),\nu,1] \setminus \{t\cdot(x,y) : (x,y) \in L_2, 1 \leq t \leq 2\}$ is a not open and a closed set;
- $B((0,0),\nu,1) \subset B((0,0),\rho_{(L_2,2^{-1})},1) \subset B[(0,0),\rho_{(L_2,2^{-1})},1]$;
- $\rho_{(L_2,2^{-1})}$ is a b-metric with the b-constant $k = 2$.

Case 5. $\lambda = 2$ *and* $L_3 = \{(\cos t, \sin t) : 0° < t \leq 90°\}$.

In this case:

- $B((0,0),\rho_{(L_3,2)},1) = B((0,0),\nu,1) \setminus \{t\cdot(x,y) : (x,y) \in L_3.2^{-1} \leq t \leq 1\}$ is a not open and a not closed set and $B[(0,0),\rho_{(L_3,2)},1] = B[(0,0),\nu,1] \setminus \{t\cdot(x,y) : (x,y) \in L_3.2^{-1} < t \leq 1\}$ is a not open and a not closed set;
- $B((0,0),\nu,2^{-1}) \subset B((0,0),\rho_{(L_3,2)},1) \subset B[(0,0),\rho_{(L_3,2)},1]$;
- $\rho_{(L_3,2)}$ is a b-quasi-metric with the b-constant $k = 2$ and $\rho_{(L_1,2)}$ is not a b-metric.

Case 6. $\lambda = 2^{-1}$ *and* $L_3 = \{(\cos t, \sin t) : 0° < t \leq 90°\}$.

In this case:

- $B((0,0),\rho_{(L_3,2^{-1})},1) = B((0,0),\nu,1) \cup \{t\cdot(x,y) : (x,y) \in L_3, 1 \leq t < 2\}$ is a not open and a not closed set and $B[(0,0),\rho_{(L_3,2^{-1})},1] = B[(0,0),\nu,1] \setminus \{t\cdot(x,y) : (x,y) \in L_3, 1 \leq t \leq 2\}$ is a not open and a closed set;
- $\rho_{(L_1,2^{-1})}$ is a b-quasi-metric with the b-constant $k = 2$ and $\rho_{(L_1,2^{-1})}$ is not a b-metric.

Case 7. $\lambda = 2$ *and* $L_4 = \{(\cos t, \sin t) : 0° \leq t \leq 360°, t$ *is rational*$\}$.

In this case:

- $B((0,0),\rho_{(L_4,2)},1) = B((0,0),\nu,1) \setminus \{t\cdot(x,y) : (x,y) \in L_4.2^{-1} \leq t \leq 1\}$ is an open and a not closed set and $B[(0,0),\rho_{(L_4,2)},1] = B[(0,0),\nu,1] \setminus \{t\cdot(x,y) : (x,y) \in L_4.2^{-1} < t \leq 1\}$ is a not open and a not closed set;
- the sets $B((0,0),\rho_{(L_4,2)},1)$ and $B[(0,0),\rho_{(L_4,2)},1]$ are dense in $B[(0,0),\nu,1]$;
- $B((0,0),\nu,2^{-1}) \subset B((0,0),\rho_{(L_4,2)},1) \subset B[(0,0),\rho_{(L_4,2)},1]$;
- $\rho_{(L_2,2)}$ is a b-metric with the b-constant $k = 2$.

Case 8. $\lambda = 2^{-1}$ *and* $L_4 = \{(\cos t, \sin t) : 0° \leq t \leq 360°, t$ *is rational*$\}$.

In this case:

- $B((0,0),\rho_{(L_4,2^{-1})},1) = B((0,0),\nu,1) \cup \{t\cdot(x,y) : (x,y) \in L_4, 1 \leq t < 2\}$ is a not open and a not closed set and $B[(0,0),\rho_{(L_4,2^{-1})},1] = B[(0,0),\nu,1] \setminus \{t\cdot(x,y) : (x,y) \in L_4, 1 \leq t \leq 2\}$ is a not open and a closed set;
- $B((0,0),\nu,1) \subset B((0,0),\rho_{(L_4,2^{-1})},1) \subset B[(0,0),\rho_{(L_4,2^{-1})},1]$;
- $\rho_{(L_2,2^{-1})}$ is a b-metric with the b-constant $k = 2$.

The next figures represent the shape of the balls for the Cases 1 – 7.

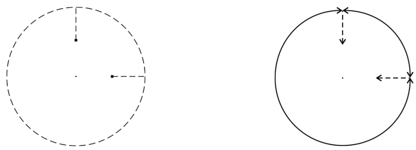
5. Partial perturbation of the distances

The idea is to change the distance between two different points.

Let (X,ρ) be a dislocated distance space, $a,b \in MK(X,\rho)$, $\rho(b,a) > 0$, $\rho(a,b) > 0$ and a be a non-isolated point of the space (X,ρ), i.e., there exists a sequence $\{a_n \in X : n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} \rho(a,a_n) = 0$ and $\rho(a,a_n) > 0$, $0 < 2\mu < \rho(b,a_n) \leq \lambda$, for all $n \in \mathbb{N}$ and a fixed $\lambda \geq \max\{1,\rho(a,b),\rho(b,a)\}$.

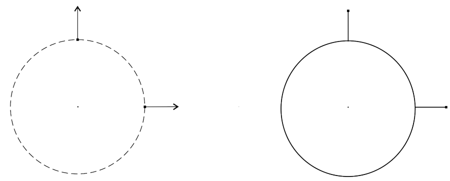
We put $d^\star(x,y) = d(x,y) = \rho(x,y)$ if $b \notin \{x,y\}$, $d(b,a) = d(b,a) = \lambda + 1$, $d^\star(b,a) = d^\star(b,a) = \mu$, $d^\star(b,x) = d(b,x) = \rho(b,x)$ and $d^\star(x,b) = d(x,b) = \rho(x,b)$ for $x \neq a$.

By construction, the following properties of d hold.



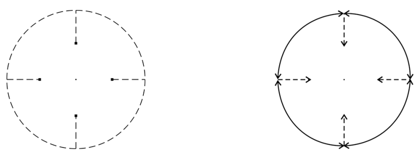
(a) $B(O, \rho_{(L,\lambda)}, 1)$ **(b)** $B[O, \rho_{(L,\lambda)}, 1]$

Figure 1. $\lambda = 2, L = L_1$



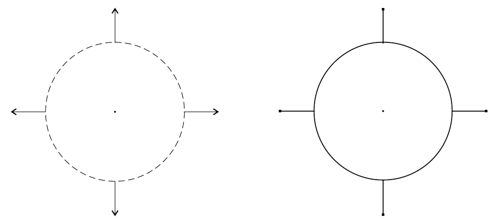
(a) $B(O, \rho_{(L,\lambda)}, 1)$ **(b)** $B[O, \rho_{(L,\lambda)}, 1]$

Figure 2. $\lambda = 2^{-1}, L = L_1$



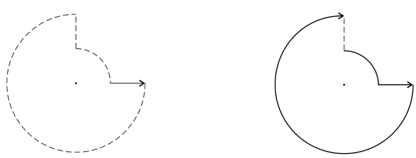
(a) $B(O, \rho_{(L,\lambda)}, 1)$ **(b)** $B[O, \rho_{(L,\lambda)}, 1]$

Figure 3. $\lambda = 2, L = L_2$



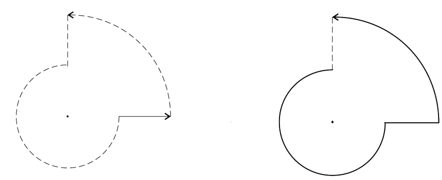
(a) $B(O, \rho_{(L,\lambda)}, 1)$ **(b)** $B[O, \rho_{(L,\lambda)}, 1]$

Figure 4. $\lambda = 2^{-1}, L = L_2$



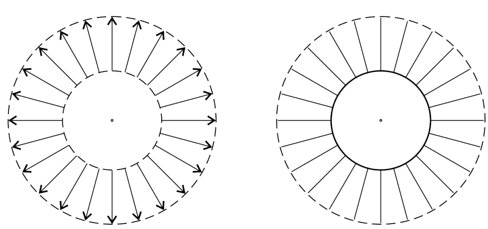
(a) $B(O, \rho_{(L,\lambda)}, 1)$ **(b)** $B[O, \rho_{(L,\lambda)}, 1]$

Figure 5. $\lambda = 2, L = L_3$



(a) $B(O, \rho_{(L,\lambda)}, 1)$ **(b)** $B[O, \rho_{(L,\lambda)}, 1]$

Figure 6. $\lambda = 2^{-1}, L = L_3$



(a) $B(O, \rho_{(L,\lambda)}, 1)$ **(b)** $B[O, \rho_{(L,\lambda)}, 1]$

Figure 7. $\lambda = 2, L = L_4$

Property 1. *d and $d^\star$ are dislocated distances and $MK(X, d) = MK(X, d^\star) = MK(X, \rho)$. If ρ is a distance, then d and $d^\star$ are distances, too.*

Property 2. *$\rho(x, y) \le d(x, y)$ for all $x, y \in X$ and $d, d^\star, \rho$ are uniformly equivalent.*

Property 3. *If $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$, then $d(x, y) = d(y, x)$, for all $x, y \in X$ too.*

Property 4. *If (X, ρ) is a dislocated b-quasi-metric space, then:*
- *(X, d) is a dislocated b-quasi-metric space;*
- *$(X, d^\star)$ is a dislocated b-quasi-metric space.*

Property 5. *If (X, ρ) is a dislocated b-metric space, then:*
- *(X, d) is a dislocated b-metric space;*
- *$(X, d^\star)$ is a dislocated b-metric space.*

Property 6. *If $\lambda r < 1 = \lambda$, then the "closed" d-ball $B[b, d, r]$ is not closed in the space $(X, T(d))$.*

Property 7. *If $\mu < r \leq 2\mu$, then the "open" $d^\star$-ball $B(b, d^\star, r)$ is not open in the space $(X, T(d^\star))$.*

6. Spaces with H-distances and the fixed point problem

In the last fifty years, many metrical fixed point results have been obtained. Initially, these results were demonstrated for complete metric spaces, and then expanded for spaces with special distances.

Example 6.5. Let $X = [0, 1] \cup \{s\}$, where $s \notin [0, 1]$, and $A = \{2^{-n} : n \in \mathbb{N}\}$. Consider on X the symmetric d, where $d(x, y) = |x - y|$ if $0, s \notin \{x, y\}$, $d(0, 2^{-n}) = d(0, s) = 1$, $d(0, x) = x$ if $x \in [0, 1] \setminus A$, $d(s, 2^{-n}) = 2^{-n}$ and $d(s, x) = 1$ if $x \in [0, 1] \setminus A$. The space $(X, \mathcal{T}(d))$ is not Hausdorff: if $U, V \in \mathcal{T}(d)$, $0 \in U$ and $s \in V$, then $U \cap V \neq \emptyset$. Since $B(0, d, 1) \cap B(s, d, 1) = \emptyset$, d is a H-distance.

Example 6.6. Let $X = \{a, b\} \cup \mathbb{N}$, where $a \neq b$ and $\{a, b\} \cap \mathbb{N} = \emptyset$. We put $\rho(x, x) = 0$ for each $x \in X$, $\rho(a, b) = 1$, $\rho(b, a) = 0$ and $\rho(n, b) = \rho(b, n) = 1$, $\rho(n, a) = \rho(a, n) = 2^{-n}$, $\rho(n, m) = |n - m|$ for each $n, m \in \mathbb{N}$. Then $(X, T(\rho))$ is a compact T_0-space. If d is the distance on X from Example 1.1, then $T(\rho) = T(d)$. Hence we have:
- the sequence $\{n : n \in \mathbb{N}\}$ is convergent to the points a, b in $(X, T(\rho))$;
- the sequence $\{n : n \in \mathbb{N}\}$ is convergent and is not a Cauchy sequence in (X, ρ);
- the sequence $\{n : n \in \mathbb{N}\}$ is ρ-convergent to the point $a : \lim_{n\to\infty} \rho(a, n) = 0$;
- the sequence $\{n : n \in \mathbb{N}\}$ is not ρ-convergent to the point $b : \lim_{n\to\infty} \rho(b, n) = 1$;
- ρ is a H^*-distance and is not a H-distance.

Let (X, d) be a dislocated distance space and $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$. Consider the conditions:

(F) there exists a function $\delta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that from $x, y, z \in X$, $d(x, y) \leq \delta(\varepsilon)$ and $d(y, z) \leq \delta(\varepsilon)$ it follows $d(x, z) \leq \varepsilon$.

(AU) for any point $x \in X$ there exists a function $\delta_x : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that from $y, z \in X$, $d(x, y) \leq \delta_x(\varepsilon)$ and $d(y, z) \leq \delta_x(\varepsilon)$ it follows $d(x, z) \leq \varepsilon$.

The distance d with the condition (F) is called an *F-distance*. The distance d with the condition (AU) is called an *AU-distance*. Any b-quasi-metric is an F-distance and any F-distance is an AU-distance. Any b-metric is a H-distance. Topological properties of spaces with distances were studied by many authors (see [1, 21, 10, 3, 27, 28, 29, 30, 31, 32]).

The notion of b-distance is due to P. Alexandroff and P. Urysohn [1], M. Fréchet [10], S. Czerwik [19], I. A. Bakhtin [4], V. Berinde [5] (see [34]). That notion is very important for the solving fixed point problem [4, 19, 34, 22, 6, 7, 11, 12, 13, 14, 15, 16, 17, 18, 23, 25].

Fix a dislocated distance space (X, d) and a mapping $\varphi : X \longrightarrow X$. For any point $x \in X$ we put $\varphi^0(x) = x$, $\varphi^1(x) = \varphi(x), ..., \varphi^n(x) = \varphi(\varphi^{n-1}(x))$,... The sequence $O(\varphi, x) = \{x_n = \varphi^n(x) : n \in \mathbb{N} = \{1, 2, ...\}$ is called the orbit of φ with respect to the point x or the Picard sequence of the point x. We say that the mapping φ:

1) is *contractive* if $d(\varphi(x), \varphi(y)) < d(x, y)$ provided $d(x, y) > 0$;
2) is a *contraction* if there exists $\lambda \in [0, 1)$ such that $d(\varphi(x), \varphi(y)) \leq \lambda d(x, y)$ for all $x, y \in X$;
3) is a *Lipschitz mapping* if there exists $\lambda > 0$ such that $d(\varphi(x), \varphi(y)) \leq \lambda d(x, y)$ for all $x, y \in X$.

Any contraction or any contractive mapping is Lipschitz mapping and every Lipschitz mapping is continuous.

The following statements, which were discussed in [14, 16] for b-quasi-metrics, are powerful tools for fixed point problems.

Proposition 6.4. *Let (X, d) be a dislocated distance space, $\varphi : X \longrightarrow X$ be a contractive mapping. Then $Fix(\varphi) \subset MK(X, d)$ and $Fix(\varphi^n) = Fix(\varphi)$, for each $n \in \mathbb{N}$.*

Proof. If $a \notin MK(X, d)$ and $a \in Fix(\varphi)$, then $0 < d(a, a) = d(\varphi(a), \varphi(a)) < d(a, b)$, a contradiction. Let $a \in Fix(\varphi^n)$, $n > 1$ and $b = \varphi(a) \neq a$. Since φ^n is a contractive mapping, too, we have $0 < d(a, b) + d(b, a) = d(\varphi^n(a), \varphi^n(b)) < d(a, b) + d(b, a)$, a contradiction. □

Proposition 6.5. *Let (X, d) be a dislocated distance space, $\varphi : X \longrightarrow X$ be a contractive mapping. Then the mapping φ is continuous and the set of fixed points $Fix(\varphi)$ of the mapping φ is empty or a singleton.*

Proof. Since $\varphi(B(x, d, r)) \subset B(\varphi(x), d, r)$ for all $x \in X$ and $r > 0$, the mapping φ is continuous.

Let $a, b \in Fix(\varphi)$ be two distinct points. Then $d(a, b) + d(b, a) > 0$. Then $d(a, b) + d(b, a) = d(\varphi(a), \varphi(b)) + d(\varphi(b), \varphi A(b)) < d(a, b) + d(b, a)$, a contradiction. □

Proposition 6.6. *Let (X, d) be a dislocated H-distance space, $\varphi : X \longrightarrow X$ be a continuous mapping and for some point $x \in X$ the Picard sequence $O(\varphi, x)$ is convergent. Then the set of fixed points $Fix(\varphi)$ of the mapping φ is non-empty.*

Proof. Let $\{x_n = \varphi^n(x) \in X : n \in \mathbb{N}\}$ be the Picard sequence of the given point $x \in X$ which is a convergent to a point $a \in X$. Then, since the mapping φ is continuous and $\lim_{n\to\infty} x_n = a$, we have $\varphi(a) = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} x_{n+1} = a$ and $\varphi(a) = a$. □

Proposition 6.7. *Let (X, d) be a dislocated b-quasi-metric space with the b-constant $\lambda \geq 1$, $k > 0$, $k\lambda < 1$, $\varphi : X \longrightarrow X$ be a mapping and $d(\varphi(x), \varphi(y)) \leq k \cdot d(x, y)$ for all $x, y \in X$. Then the orbit $O(\varphi, a)$ is a Cauchy sequence for any point $a \in X$.*

Proof. Let $\Sigma\{(k\lambda)^n : n \in \mathbb{N}\} = c$. Fix $a \in X$. We put $s = d(a, \varphi(a)) + d(\varphi(a), a)$. We put $a_n = \varphi^n(a)$. For all $n, m \in \mathbb{N}$ we have $d(a_n, a_{n+m}) + d(a_{n+m}, a_n) \leq s \cdot k^{n-1} \cdot \Sigma\{(k\lambda)^i : 1 \leq i \leq m\} \leq s \cdot c \cdot k^{n-1}$. The proof is complete. □

Proposition 6.8. *Let (X, d) be a dislocated F-distance space, $\varphi : X \longrightarrow X$ be a contraction mapping, $m \in \mathbb{N}$, $a \in X$ and the orbit $O(\varphi^m, a)$ is a Cauchy sequence. Then the orbit $O(\varphi, a)$ is a Cauchy sequence, too.*

Proof. There exists $\lambda \in [0, 1)$ such that $d(\varphi(x), \varphi(y)) \leq \lambda d(x, y)$ for all $x, y \in X$. Fix a function $\delta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that from $x, y, z \in X$, $d(x, y) \leq \delta(\varepsilon)$ and $d(y, z) \leq \delta(\varepsilon)$ it follows $d(x, z) \leq \varepsilon$. We put $q = d(a_1, a_2) + d(a_2, a_1)$. Then $d(a_n, a_{n+1}) + d(a_{n+1}, a_n) \leq \lambda^{n-1}q$. Let $\{a_n = \varphi^n(x) \in X : n \in \mathbb{N}\}$ be the Picard sequence of the given point $a \in X$. As contractive mappings, the mappings φ^i, $i \in \mathbb{N}$, are continuous and the image of a Cauchy sequence is a Cauchy sequence. Hence $O(\varphi^{m+1}, a) = \varphi(O(\varphi^m, a))$ is a Cauchy sequence. Fix $\varepsilon > 0$. There exists $n_\varepsilon \in \mathbb{N}_m = \{n \cdot m : n \in \mathbb{N}\}$ such that:

- $d(a_{n_\varepsilon}, a_{n_\varepsilon+1}) < \delta(\varepsilon)$;
- if $n_1, n_2 \in \mathbb{N}_m$ and $n_1 \geq n_\varepsilon$, $n_2 \geq n_\varepsilon$, then $d(a_{n_1}, a_{n_2}) < \delta(\varepsilon)$.

In this case $d(a_{n_1}, a_{n_2+1}) < \varepsilon$, $d(a_{n_1+1}, a_{n_2})$ and $d(a_{n_1+1}, a_{n_2+1}) < \varepsilon$ and $O(\varphi^{m+1}, a) \cup \varphi(O(\varphi^m, a))$ is a Cauchy sequence. By induction, we establish that $O(\varphi^m, a) \cup \varphi(O(\varphi^m), a) \cup \varphi^2(O(\varphi^m, a)) \cup ... \cup \varphi^i(O(\varphi^m, a))$ is a Cauchy sequence for any $i \in \mathbb{N}$. Hence $O(\varphi, a) = O(\varphi^m, a) \cup \varphi(O(\varphi^m, a)) \cup \varphi^2(O(\varphi^m, a)) \cup ... \cup \varphi^m(O(\varphi^m, a))$ is a Cauchy sequence. □

Corollary 6.1. *Let (X, d) be a dislocated H-distance space, $0 \leq k < 1$, $\varphi : X \longrightarrow X$ be a mapping, $d(\varphi(x), \varphi(y)) \leq kd(x, y)$ for all $x, y \in X$ and for some point $a \in X$ the Picard sequence $O(\varphi, a)$ is convergent. Then:*

1) There exists a unique fixed point $b \in X$ of the mapping φ and $Fix(\varphi) = \{b\}$.

2) If $x \in X$ and $O(\varphi, x) = \{x_n : n \in \mathbb{N}\}$, then $\lim_{n\to\infty} x_n = b$ and $lim_{n\to\infty} d(b, x_n) = 0$.

Proof. Let $\{a_n = \varphi^n(a) \in X : n \in \mathbb{N}\}$ be the Picard sequence of the given point $a \in X$ which is a convergent to a point $b \in X$, i.e. $\lim_{n\to\infty} a_n = b$. Since φ is continuous as contraction and $a_{n+1} = \varphi(a_n)$ for any $n \in \mathbb{N}$, we have $b = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \varphi(a_n) = \varphi(b)$. Since d is a H-distance, $b = \varphi(b)$. Hence $b \in Fix(\varphi)$.

Fix $x \in X$. Let $\lambda = d(b, x)$ and $x_n = \varphi^n(x)$ for each $n \in \mathbb{N}$. Then $d(b, x_1) \leq kd(b, x) = k\lambda$ and $d(b, x_n) \leq k^n\lambda$ for each $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} d(b, x_n) = \lim_{n\to\infty} k^n\lambda = \lambda \lim_{n\to\infty} k^n = 0$. □

Corollary 6.2. *Let (X, d) be a complete H^*-b-quasi-metric space, $0 \leq k < 1$, $\varphi : X \longrightarrow X$ be a mapping and $d(\varphi(x), \varphi(y)) \leq kd(x, y)$ for all $x, y \in X$. Then:*

1) There exists a unique fixed point $b \in X$ of the mapping φ and $Fix(\varphi) = \{b\}$.

2) If $x \in X$ and $O(\varphi, x) = \{x_n : n \in \mathbb{N}\}$, then $\lim_{n\to\infty} x_n = b$ and $\lim_{n\to\infty} d(b, x_n) = 0$.

3) $Fix(\varphi) \subset \varphi(MK(X, d)) \subset MK(X, d)$.

Proof. Let $\lambda \geq 1$ be the b-constant of d, i. e. $d(x, z) \leq \lambda(d(x, y) + d(y, z))$ for all $x, y, z \in X$. We have $\varphi(MK(X, d)) \subset M(X, d)$.

Case 1. $MK(X, d) \neq \emptyset$.

In this case we can assume that $X = MK(X, d)$ and d be a b-quasi-metric on X.

Fix $a \in X$. Let $\{a_n = \varphi^n(a) \in X : n \in \mathbb{N}\}$ be the Picard sequence of the given point $a \in X$.

We put $d_s(x, y) = d(x, y) + d(y, x)$ for all $x, y \in X$. Then d_s is a b-metric on X with the b-constant λ and $d_s(\varphi(x), \varphi(y)) \leq kd_s(x, y)$ for all $x, y \in X$.

There exists $m \in \mathbb{N}$ such that $k^m\lambda < 1$. If $f = \varphi^m$, then $d_s(f(x), f(y)) \leq k^m d(x, y)$ for all $x, y \in X$. Hence, by virtue of Proposition 6.7, $\{b_n = f^m(a) : n \in \mathbb{N}\}$ is a Cauchy sequence of the b-metric space (X, d_s) ([4], [34], pag. 54). In this case $\{a_n : n \in \mathbb{N}\}$ is a Cauchy sequence of the b-metric space (X, d_s), too (see [14, 16]).

Any Cauchy sequence of the distance space (X, d_s) is a Cauchy sequence of the distance space (X, d). Since (X, d) is a complete distance space, the Picard sequence of the given point $a \in X$ is convergent as a Cauchy sequence of the distance space (X, d). Corollary 6.1 completes the proof.

Case 2. $MK(X, d) = \emptyset$.

We put $\rho(y, y) = 0$ for any $y \in X$ and $\rho(x, y) = d(x, y)$ for distinct points $x, y \in X$. Then ρ is the location of d, $\rho(x, z) \leq \lambda(\rho(x, y) + \rho(y, z))$ and $\rho(\varphi(x), \varphi(y)) \leq k\rho(x, y)$ for all $x, y, z \in X$. Therefore (X, ρ) is a b-quasi-metric space and φ is a ρ-contraction, too.

Fix a point $a \in X$. Then $a \notin Fix(\varphi)$. There exists $m \in \mathbb{N}$ such that $0 \leq k^m \cdot \lambda < 1$. In this case, from Proposition 6.7 (see [4], [34], pag. 54), $O(\varphi^m, x)$ is a Cauchy sequence of the distance spaces (X, d_s) and (X, d) for any point $x \in X$. Fix $a \in X$. By Proposition 6.8, $O(\varphi^m, x)$ is a Cauchy sequence of the space (X, d). Let b be the limit of the sequence $\{a_n = \varphi^n(a) \in X : n \in \mathbb{N}\}$. Then $b \in MK(X, d)$, a contradiction. Hence, this case is impossible. The proof is complete. □

The above assertions allow us to state the following general principle.

Reduction principle. *Assume that for any b-metric space (Z, ρ) and any mapping $\psi : Z \longrightarrow Z$ with properties $\mathcal{Q}$ any Picard orbit $O(\psi, z)$ is a Cauchy sequence and the restriction of ψ on the closure of the orbit is continuous. Then, for any dislocated H^*-b-quasi-metric space (X, ρ) and any mapping $f : X \longrightarrow X$ with properties $\mathcal{Q}$, the following assertions are true:*

1) Any Picard orbit $O(f, x)$ is a Cauchy sequence.
2) If (X, ρ) is a complete space, then the set $Fix(f)$ is non-empty.
3) If any mapping f with properties $\mathcal{Q}$ is contractive, then the set $Fix(f)$ is empty or a singleton.

Note that the above assertions are not true for not H-distances.

Example 6.7. (see [15, 16]). Let $X = \mathbb{N}$, $\rho(x, y) = d(x, x) = 0$ for all $x \in X$, $\rho(n, m) = 2^{-m}$ and $d(n, m) = d(m, n) = 2^{-n-m}$ for all distinct $n, m \in \mathbb{N} = X$. We have $d(x, y) = d(y, x)$ and $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The topology $\mathcal{T}(d)$ generated by d is equal with the topology $\mathcal{T}(\rho)$ generated by ρ. The topology $\mathcal{T}(d) = \mathcal{T}(\rho)$ is a compact T_1-topology on X , $\{n : n \in \mathbb{N}\}$ is a Cauchy sequence convergent to any point $x \in X$. On X consider the continuous mapping $\varphi : X \longrightarrow X$, where $\varphi(n) = n + 1$ for any $n \in \mathbb{N}$. Hence:

- d is a symmetric and complete distance on X;
- the distance ρ is a complete quasi-metric;
- d and ρ are not a H-distances on X;
- since $\lim_{n\to\infty}(d(x, n) + d(n, y)) = 0$ for all $x, y \in X$, d is not an AU-distance on X;
- $\mathcal{T}(\rho) = \mathcal{T}(d) = \{\emptyset\} \cup \{X \setminus F : F$ is a finite subset of $X\}$;
- the balls $B(x, d, r)$ and $B(x, \rho, r)$ are open in the space $(X, T(d))$ and the sets $X \setminus B(x, d, r)$ and $X \setminus B(x, \rho, r)$ are finite for all $x \in X$ and $r > 0$;
- φ is a contraction, $d(\varphi(x), \varphi(y)) = 2^{-2}d(x, y)$ and $\rho(\varphi(x), \varphi(y)) = 2^{-1}\rho(x, y)$ for all $x, y \in X$;
- $Fix(\varphi) = \emptyset$.

7. Partial b-distances and the fixed point problem

A function d on a set $X \times X$ is called a *partial distance* on a set X if, for all $x, y \in X$, we have:

(vi_m) $d(x, x) \leq d(x, y)$;
(vii_m) if $d(x, x) = d(x, y) = d(y, x)$, then $x = y$.

A partial distance d on a set X is called a *partial b-quasi-metric* if there exists a positive number k, named a b-constant, such that for all $x, y, z \in X$ we have:

$(viii_m)$ $d(x, z) \leq k(d(x, y) + d(y, z) - d(y, y))$.

The partial quasi-metric, metric and b-metric are defined as in the case of dislocated distances. Partial metrics have been introduced in S. G. Matthews [26] as part of the study of the denotational semantics of data flow networks (see [8, 2, 33]).

Remark 7.4. Let (X, d) be a partial b-quasi-metric space with the b-constant k. If $d(x, x) \geq 0$ for any $x \in X$, then (X, d) is a dislocated b-quasi-metric space with the b-constant k. If $d(x, x) < 0$ for some $x \in X$, then d is called a *dualistic partial metric* [33]. We will not use this term.

Let (X, d) be a partial distance space.

For any $x \in X$ and $r > 0$ we put $B^p(x, d, r) = \{y \in X : d(x, y) - d(x, x) < r\}$ be the d-p-open ball with the center x and radius $r > 0$. The set $U \subset X$ is called *d-p-open* if, for any $x \in U$, there exists $r > 0$ such that $B^p(x, d, r) \subset U$. The family $T^p(d)$ of all

d-p-open subsets is the partial topology on X generated by d. A partial distance space is a sequential space.

Let $\{x_n : n \in \mathbb{N}\}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n : n \in \mathbb{N}\}$:

1) is *p-convergent* in X if for any d-p-open subset $U \in T^p(d)$, for which $x \in U$, there exists a number $n \in \mathbb{N} = \{1, 2, 3, ...\}$ such that $\{x_m : m \in \mathbb{N}, m \geq n\} \subset U$. We denote this by $x = p\text{-}\lim_{n\to\infty} x_n$;
2) is *d-p-convergent to* x if and only if $\lim_{n\to\infty} d(x, x_n) = d(x, x)$. We denote this by $x = d\text{-}p\text{-}\lim_{n\to\infty} x_n$;
3) is *d-convergent to* x if and only if $\lim_{n\to\infty} d(x, x_n) = 0$. We denote this by $x = d\text{-}\lim_{n\to\infty} x_n$;
4) is *d-Cauchy* if there exists the limit $\lim_{n,m\to\infty} d(x_n, x_m) = 0$;
5) is *d-p-Cauchy* if there exists the limit $\lim_{n,m\to\infty} d(x_n, x_m)$.

A partial distance space (X, d) is complete if any d-Cauchy sequence is d-convergent. Our definition of completeness is more general than the completeness in sense of S. G. Matthews (see [26, 2, 33]): a partial b-quasi-metric space (X, d) is complete if any d-p-Cauchy sequences is d-p-convergent.

Example 7.8. Let (X, ρ) be a b-metric space with the b-constant k. Fix a number $s \neq 0$ and put $d(x, y) = \rho(x, y) + s$. The following properties of d are true:

1) If $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence of the space (X, d), then there exists $n \in \mathbb{N}$ such that $x_{n+m} = x_n$ for all $m \in \mathbb{N}$, i.e. the sequence $\{x_n : n \in \mathbb{N}\}$ is trivial. Hence non-trivial Cauchy sequences of the space (X, ρ) are not Cauchy sequences of the space (X, d).

2) d is a complete partial distance on X.

3) We have $T(\rho) = T^p(d)$. The spaces (X, ρ) and (X, d) are the same convergent sequences to given point $x \in X$.

4) If $k = 1$, then d is a partial metric.

5) If $s > 0$, then d is a complete partial b-metric with the constant k and a complete dislocated b-metric with the constant k. Since $s \neq 0$, the topology $T(d)$ is discrete.

For negative numbers s, the partial distance $d(x, y) = \rho(x, y) + s$ may lose the property of being a b-metric.

Example 7.9. Let (E, ν) be a linear normed space of linear dimension $\dim E \geq 2$. Fix a positive number $\lambda \neq 1$ and a number $s < 0$. Let $L \subset S(\nu, 1)$ be a non-empty subset, $-L = L$ and the set $M = S(\nu, 1) \setminus L$ is non-empty too. As in Theorem 4.2, we put $\rho(x, y) = \nu(x - y)$, $\rho_{(L,\lambda)}(x, y) = \lambda \cdot \rho(x, y)$ provided $(y - x) \parallel L$ and $\rho_{(L,\lambda)}(x, y) = \rho(x, y)$ provided $(y - x) \parallel M$. Then $\rho_{(L,\lambda)}$ is an invariant homogeneous b-metric with b-constant $k = \max\{\lambda, \lambda^{-1}\}$. Since the sets L and M are not empty there exists the distinct points $a_1, a_2, b_1, b_2, c_1, c_2 \in E$ such that $\rho_{(L,\lambda)}(a_1, c_1) > \rho_{(L,\lambda)}(a_1, b_1) + \rho_{(L,\lambda)}(b_1, c_1)$ and $\rho_{(L,\lambda)}(a_2, c_2) = \rho_{(L,\lambda)}(a_2, b_2) + \rho_{(L,\lambda)}(b_2, c_2)$. Now we put $d(x, y) = \rho_{(L,\lambda)}(x, y) + s$ for all $x, y \in E$. Then d is a symmetric partial distance on E and d is not a partial b-metric for any b-constant $k \geq 1$.

Example 7.10. Let a, c be two numbers, X be a set with the cardinality $|X| \geq 2$ and $d_{(a,c)}(x, x)$ be the distance from Example 1.2. If $a \geq c > 0$, then $d_{(a,c)}$ is a dislocated distance and not a partial distance, the topology $T(d_{(a,c)})$ is discrete and for $a = c > 0$ the topology $T^p(d_{(a,c)}) = \{\emptyset, X\}$ is anti-discrete. If $c > a$, then $d_{(a,c)}$ is a partial metric.

Proposition 7.9. *Let (X, d) be a partial b-quasi-metric space. Then:*

1) $d_p(x, y) = d(x, y) - d(x, x)$ is a b-quasi-metric generate by the partial b-quasi-metric d.
2) $T^p(d) = T(d_p)$.
3) If $x \in X$, then the spaces (X, d) and (X, d_p) are the same d-p-convergent and d_p-convergent to the point x sequences.

4) *Any p-Cauchy sequence of the space (X, d) is a Cauchy sequence of the space (X, d_p) too.*

Proof. Is obvious □

Example 7.11. Let (E, ν) be a non-complete linear normed space of linear dimension $\dim E \geq 1$. Fix a point $c \in E \setminus \{0\}$ and the numbers $\lambda \in [0, 1)$ and $s < -1$. We put $g(x) = x + c$ and $\rho(x, y) = s + \min\{0, \nu(x - y)\}$. Then:

1) ρ is an invariant partial metric on E;
2) g is an isometric mappings and $\rho(g(x), g(y)) = \rho(x, y)$ for all $x, y \in E$;
3) $s \leq \rho(g(x), g(y)) = \lambda \cdot \rho(x, y) < 0$ for all $x, y \in E$;
4) $Fix(g) = \emptyset$.
5) partial metric ρ is complete and it is not complete in sense of S. G. Matthews.

Proposition 7.10. *Let (X, d) be a partial b-metric space and $\{a_n \in X : n \in \mathbb{N}\}$ be a Cauchy sequence convergent to the point a. Then:*

(1) $a \in MK(X, \rho) = \{x \in X : d(x, x) = 0\}$.
(2) $\lim_{n\to\infty} d(a_n, a_n) = 0$.

Proof. Is obvious □

Proposition 7.11. *Let (X, d) be a partial b-metric space and $\{a_n \in X : n \in \mathbb{N}\}$ be a sequence convergent to the point a. If $a \in MK(X, \rho)$, then $\{a_n \in X : n \in \mathbb{N}\}$ is a Cauchy sequence.*

Proof. Let $k \geq 1$ be the b-constant of d. Fix $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $2 \cdot d(a, a_{n+m}) < k^{-1} \cdot \varepsilon$ for all $m \in \mathbb{N}$. Then $d(a_{n+m}, a_{n+s}) \leq k(d(a_{n+m}, a) + d(a, a_{n+s}) - 0) < \varepsilon$ for all $m, s \in \mathbb{N}$. The proof is complete. □

Proposition 7.12. *Let (X, d) be a partial b-metric space. Then the distinct points $a, b \in MK(X, d)$ have distinct convergent sequences to them.*

Proof. Let $k \geq 1$ be the b-constant of d. Fix two distinct points $a, b \in MK(X, d)$. In this case $d(a, a) = d(b, b) = 0$ and $d(a, b) > 2k\lambda > 0$ for some $\lambda > 0$. Then $B^p(a, d, \lambda) \cap B^p(b, d, \lambda) = \emptyset$. The proof is complete. □

The following fact is distinct than the Contraction Principle of Matthews, Rus ([26], [34], pag. 55) and is more general than the Contraction Principle from [33].

Theorem 7.3. *Let (X, d) be a complete partial b-metric space, $0 \leq \lambda < 1$, $\varphi : X \longrightarrow X$ be a mapping and $|d(\varphi(x), \varphi(y))| \leq k|d(x, y)|$ for all $x, y \in X$. Then:*

(1) *There exists a unique fixed point $b \in X$ of the mapping φ and $Fix(\varphi) = \{b\}$.*
(2) *If $x \in X$, then the orbit $O(\varphi, x) = \{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence and $\lim_{n\to\infty} x_n = b$, $\lim_{n\to\infty} d(b, x_n) = 0$.*
(3) $b \in MK(X, d)$.
(4) $\varphi(MK(X, d)) \subset MK(X, d)$.

Proof. Let $k \geq 1$ be the b-constant of d, i. e. $d(x, z) \leq k \cdot (d(x, y) + d(y, z) - d(y, y))$ for all $x, y, z \in X$. We have $\varphi(MK(X, d) \subset M(X, d)$.

Case 1. $MK(X, d) \neq \emptyset$.

In this case we can we fix $a \in MK(X, d)$. Since $(MK(X, d), d)$ is a b-metric space, by virtue of Corollary 6.2, the orbit $O(\varphi, a) = \{a_n : n \in \mathbb{N}\}$ is a Cauchy sequence. There exists the limit $\lim_{n\to\infty} a_n = b$. From Proposition 7.10 it follows that $b \in MK(X, d)$. Hence $b \in Fix(\varphi)$. Obviously $Fix(\varphi) \subset MK(X, d)$.

Fix $x \in X$. Then $d(f^n(x), \varphi^n(a)) \leq k^n d(x, a)$. Therefore $\lim_{n\to\infty}(b, \varphi^n(x)) = 0$ and $\lim_{n\to\infty} \varphi^n(x) = b$.

In this case the theorem is proved.

Case 2. $MK(X,d) = \emptyset$.

There exists $m \in \mathbb{N}$ such that $k^m\lambda < 1$. If $f = \varphi^m$, then $|d(f(x), f(y))| \leq k^m|d(x,y)|$ for all $x, y \in X$.

Fix $a \in X$. Let $\{a_n = f^n(a) : n \in \mathbb{N}\}$ be the Picard sequence of the point $a \in X$. We can assume that $|d(a, f(a))| < 1$ and $|d(a,a)| < 1$. Then $|d(a_n, a_{n+1}| < k^{n+m}$ and $|d(a_n, a_n| < k^{n+m}$. We have $a_n \neq a_m$ for distinct $n, m \in \mathbb{N}$. As in the proof of Proposition 6.7 we can proved that the orbit $O(f,a)$ is a Cauchy sequence. There exists the limit $\lim_{n\to\infty} f^n(a) = c$. From Proposition 7.10 it follows that $c \in MK(X,d)$, a contradiction. The proof is complete. □

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