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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

On Caristi's fixed point theorem in metric spaces with a graph

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ABSTRACT. We generalize the Caristi's fixed point theorem for single valued as well as multivalued mappings defined on a metric space endowed with a graph and w-distance. Particularly, we modify the concept of the (OSC)-property due to Alfuraidan and Khamsi (Alfuraidan M. R. and Khamsi, M. A., *Caristi fixed point theorem in metric spaces with graph*, Abstr. Appl. Anal., (2014) Art. ID 303484, 5.) which enable us to reformulated their stated graph theory version theorem (Theorem 3.2 in Alfuraidan M. R. and Khamsi, M. A., *Caristi fixed point theorem in metric spaces with graph*, Abstr. Appl. Anal., (2014) Art. ID 303484, 5.) to the case of w-distance. Consequently, we extend and improve some recent works concerning extension of Banach Contraction Theorem to w-distance with graph e.g. (Jachymski, J., *The contraction principle for mappings on a metric space with graph*, Proc. Amer. Math. Soc., **136** (2008), No. 4, 1359–1373; Nieto, J. J., Pouso, R. L. and Rodriguez-Lopez R., *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc., **135** (2007), 2505–2517 and Petrusel, A. and Rus, I., *Fixed point theorems in ordered $L-$spaces endowed with graph*, Proc. Amer, Math. Soc., **134** (2006), 411–418).

1. INTRODUCTION

Caristi's [8] fixed point theorem is considered as one of the most beautiful extensions of Banach [5] contraction theorem. The proofs given for Caristi's result vary and use different techniques (e.g. [6, 7, 19]). The original proof is based on iterates and use of transfinite induction. It is worth mentioning that because of Caristi's result close connection with Ekeland's [10] variational principle, many authors refer to it as the Caristi-Ekeland fixed point result. For some important contributions on this topic and related results, we refer the reader to see [4, 9, 12–14, 18, 20, 21, 27, 28, 31].

In 1996, Kada et. al.[16] introduced the notion of w-distance and used it primarily to generalize Caristi's fixed point theorem, Ekeland's variational principle applied to nonconvex minimization problems, see Takahashi [29] for details.

Proving existence results of fixed points for monotone, order preserving mappings has been a relatively new development in metric fixed point theory. This direction was initiated by Ran and Reurings [25] (see also [30]) while investigating some applications of matrix equations. Subsequently, Nieto [22, 23] and others [11, 24] modified and improved Ran and Reurings results. In 2008, Jachymski [15] using the language of graph theory (which subsumes the partial ordering) introduced the concept of G-contraction on a metric space endowed with a graph and obtained some fixed point results which unified most of the previous results concerning partial ordering e. g. ([11, 22–25]). Recently, Alfuraidan and Khamsi [3] proved the analogue of Caristi's fixed point theorem for monotone, order preserving mapping satisfying the (OSC)-property and also stated a graph theory version theorem (Theorem 3.2 in [3]).

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In this article, we modify the concept of the (OSC)-property (graph theory version) due to Alfuraidan and Khamsi [3] and extend their theorem to the case of w-distance. We also generalize some recent work on Caristi's fixed point theorems to metric spaces endowed with a graph and a w-distance.

2. Preliminaries

First, we start by recalling the concept of w-distance which was initiated by Kada et. al. in [16].

Definition 2.1. Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called w-distance on X if it satisfies the following conditions:

(P_1) $p(x, z) \leq p(x, y) + p(y, z)$;

(P_2) $p(x, \cdot) : X \times X \to [0, \infty)$ is lower semicontinuous, i.e, if $y_n \to y$ in X, then $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$ holds;

(P_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$;

for all $x, y, z \in X$

In general, we do not have $p(x, y) = p(y, x)$ for all $x, y \in X$ and $p(x, x)$ need not be zero. The paper [16] contains many examples of w-distances. For our purpose, we mention the following examples.

Example 2.1. Let (X, d) be a metric space with a usual metric. Then $p = d$ is a w-distance on X.

Example 2.2. Let (X, d) be a metric space with usual metric. If the function $p : X \times X \to [0, \infty)$ is defined by $p(x, y) = c$ where c is a positive real number and $x, y \in X$, then p is a w-distance but p is not a metric.

Example 2.3. Let X be a normed linear space with norm $\|.\|$. If the function $p : X \times X \to [0, \infty)$ is defined by

$$p(x, y) = \|y\|$$

for every $x, y \in X$, then p is a w-distance but p is not a metric.

Throughout if (X, d) is a metric space, we will consider on $X \times X$ the following distance

$$d_1((x, y), (u, v)) = d(x, u) + d(y, v),$$

for any $(x, y), (u, v) \in X \times X$. We have $\{(x_n, y_n)\} \in X \times X$ converges to (x, y) if and only if $\{x_n\}$ converges to x and $\{y_n\}$ converges to y. Obviously a subset Y of $X \times X$ is closed if whenever $\{(x_n, y_n)\}$ is in Y and converges to (x, y), we have $(x, y) \in Y$. Throughout, we denote the diagonal of $X \times X$ by Δ, i.e. $\Delta = \{(x, x); x \in X\}$. We will use the notation 2^X to designate the family of all nonempty subsets of X and $CB(X)$ the family of bounded closed nonempty subsets of X.

Lemma 2.1. [16] *Let (X, d) be a metric space with metric d and let p be a w-distance on X. If $\{x_n\}$ is a sequence in X and α_n is a sequence in $[0, \infty)$ converging to zero, then $\{x_n\}$ is a Cauchy sequence if $p(x_n, x_m) < \alpha_n$ for each $n, m \in \mathbb{N}$ and $m > n$.*

Definition 2.2. Let (X, d) be a metric space. Let $T : X \to 2^X$ and $F : X \to X$ be two mappings. Then we define their graphs as follow

$$Gr(T) = \{(x, y) : y \in Tx\}, \text{ and } Gr(F) = \{(x, y) : y = Fx)\}.$$

Let $G = (V(G), E(G))$ be a directed graph(digraph), such that the set of vertices $V(G)$ coincides with X and the set of edges $E(G) \subset X \times X$ contains all loops (i.e., $\triangle \subset E(G)$). Also assume that the graph G has no parallel edges that is, we do not allow it to get two or more edges that are incident to the same two vertices.
If x, y are in $V(G)$, then the path from x to y of length n is a sequence $\{x_i\}_{i=0}^{n}$ of $n + 1$ vertices such that

$$x_0 = x, x_n = y, \text{ and } (x_i, x_{i+1}) \in E(G) \text{ for any } i = 0, 1, \ldots, n.$$

A directed cycle from x to y is a closed directed path of length $n > 1$, i.e., $x = y$ and digraph that has no directed cycle is called acyclic digraph.

A graph G is said to be transitive if $(x, y) \in E(G)$ and $(y, z) \in E(G)$, then $(x, z) \in E(G)$ for all $x, y, z \in X$.

Since we are dealing with set-valued mappings, the following definition will be needed.

Definition 2.3. For a multivalued mapping $T : X \to 2^X$, let

(a) $Fix(T) = \{x \in X : x \in Tx\}$
(b) $X(T, G) = \{x \in X : (x, u) \in E(G) \text{ for some } u \in Tx\}$

Since the proof of the main result of our work relies on a transfinite induction ([17]), the following result will be needed.

Proposition 2.1. [17] *Suppose $\{x_\alpha\}_{\alpha \in \Gamma} \subset \mathbb{R}$ is bounded and either nonincreasing or nondecreasing. Then there exists β such that $x_\alpha = x_\beta$ for all $\alpha \geq \beta$.*

3. Main results

The single valued version of the Caristi's fixed point theorem for graph-monotone mapping was first discovered in [3]. Next, we extend these results to multivalued mapping when w-distance are involved.

Theorem 3.1. *Let (X, d, G) be a complete metric space endowed with a directed graph G and p is w-distance on X. Assume that $T : X \to 2^X$ has the following properties:*

(i) $X(T, G)$ is nonempty;
(ii) for any $x, y \in X$ with $(x, y) \in E(G)$, for any $u \in Tx$ there exists $v \in Ty$ such that $(u, v) \in E(G)$;
(iii) there exists a lower semicontinuous $\phi : X \to [0, \infty)$ such that

$$p(x, y) \leq \phi(x) - \phi(y)$$

for any $x \in X$ with $(x, y) \in E(G)$ and for some $y \in Tx$
(iv) $Gr(T)$ is closed;

then T has a fixed point.

Proof. By assumption (i), pick $x_0 \in X(T, G)$ then there exists $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$. From the assumption (ii), there exists $x_2 \in Tx_1$ such that $(x_1, x_2) \in E(G)$. If we continue this process, we construct a sequence $\{x_n\}_{n \geq 0}$ such that $x_{n+1} \in Tx_n$ with $(x_n, x_{n+1}) \in E(G)$. Furthermore, by using (iii), we have

$$p(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1}),$$

for any $n \in \mathbb{N}$. Since p takes positive values, we conclude that $\phi(x_n)$ is decreasing sequence of positive numbers. Therefore, $\lim\limits_{n \to \infty} \phi(x_n) = \inf\limits_{n \geq 1} \phi(x_n) = \phi_0$. Next, we claim that $\{x_n\}$ is a Cauchy sequence. Indeed, we have

$$\begin{aligned} p(x_n, x_m) &\le \sum_{k=0}^{m-n-1} [p(x_{n+k}, x_{n+k+1})] \\ &\le \sum_{k=0}^{m-n-1} [\phi(x_{n+k}) - \phi(x_{n+k+1})] \\ &= \phi(x_n) - \phi(x_m), \end{aligned}$$

for any $n \le m$. Fix $\varepsilon > 0$. From the definition of the w-distance, there exists $\delta > 0$ such that $\max\{p(x,y); p(x,z)\} \le \delta$ implies $d(y,z) \le \varepsilon$, for any $x, y, z \in X$. Moreover, there exists $N_0 \ge 1$ such that $\phi_0 \le \phi(x_n) \le \phi_0 + \varepsilon$, for each $n \ge N_0$, which implies

$$p(x_n, x_m) \le (\phi_0 + \delta) - \phi_0 = \delta,$$

for $m \ge n \ge N_0$. Hence $\max\{p(x_{N_0}, x_n); p(x_{N_0}, x_m)\} \le \delta$, for any $n, m \ge N_0$. Therefore, we must have $d(x_n, x_m) \le \varepsilon$, for any $n, m \ge N_0$, i.e., $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, then there exists $\bar{x} \in X$ such that

$$\lim_{n\to\infty} x_n = \bar{x}.$$

Using the fact $x_{n+1} \in Tx_n$, we conclude that $(x_n, x_{n+1}) \in Gr(T)$, for any $n \in \mathbb{N}$. Since $Gr(T)$, is closed, we deduce that $(\bar{x}, \bar{x}) \in Gr(T)$. Therefore $\bar{x}$ is a fixed point of T. □

At this moment, we really do not know weather the assumption that $Gr(T)$ is closed in Theorem 3.1 can be relaxed or not. However, in the case of single valued mapping it can be done (see Theorem 3.2). In order to do this, we give first the following definition:

Definition 3.4. Let (X, d, G) be a complete metric space endowed with directed graph G. We say that G satisfies the (OSC)-property if and only if for any convergent sequence $\{x_n\}$, with $(x_{n+1}, x_n) \in E(G)$ for any $n \in \mathbb{N}$, which converges to x^*, then we have

(i) $(x^*, x_n) \in E(G)$;
(ii) If $(y, x_n) \in E(G)$, then $(y, x^*) \in E(G)$.

Remark 3.1. The above definition is a modification of the original definition of the (OSC)-property (graph theory version) due to Alfuraidan and Khamsi [3] which enable us to reformulated their stated graph theory version theorem (Theorem 3.2 in [3]) to the case of w-distance.

Now we prove the following:

Theorem 3.2. *Let (X, d, G) be a complete metric space endowed with a directed graph G and p is a w-distance on X. Assume G is transitive. Let $T : X \to X$ be mapping which satisfies the following properties:*

(i) there exists x_0 in X such that $(Tx_0, x_0) \in E(G)$;
(ii) T is G-preserving or G-monotone, i.e., $(Tx, Ty) \in E(G)$ whenever $(x, y) \in E(G)$ for any $x, y \in X$;
(iii) there exists a lower semicontinuous $\phi : X \to [0, \infty)$ such that

$$p(x, Tx) \le \phi(x) - \phi(Tx)$$

for any $x \in X$ with $(Tx, x) \in E(G)$;
(iv) G satisfies the (OSC)-property,

then T has a fixed point.

Proof. Using the same approach used in the proof of Theorem 3.1, we construct a sequence $\{x_n\}$ in X such that

(1) $x_{n+1} = Tx_n$,
(2) $(x_{n+1}, x_n) \in E(G)$,
(3) $p(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1})$,

for any $n \in \mathbb{N}$. As in the proof of Theorem 3.1, we see that $\{x_n\}$ is convergent. Set $x_\omega = \lim\limits_{n\to\infty} x_n$. Next, we claim that $(Tx_\omega, x_\omega) \in E(G)$. Indeed, the (OSC) property and the fact that $(x_{n+1}, x_n) \in E(G)$, for any any $n \in \mathbb{N}$, will imply $(x_\omega, x_n) \in E(G)$, for any $n \in \mathbb{N}$. Since T is G-preserving, then $(Tx_\omega, Tx_n) \in E(G)$, i.e., $(Tx_\omega, x_{n+1}) \in E(G)$, for any $n \in \mathbb{N}$. Again using the (OSC)-property, we conclude that $(Tx_\omega, x_\omega) \in E(G)$ holds as claimed. Next, we construct a transfinite orbit $\{x_\alpha\}_{\alpha\in\Gamma}$, where Γ is the the smallest uncountable ordinal. We will do this by a transfinite induction. Indeed, assume that $\{x_\beta\}_{\beta<\alpha}$ is constructed and satisfies the following properties:

i) If $\beta + 1 < \alpha$, then $x_{\beta+1} = Tx_\beta$.
ii) If $\beta < \gamma < \alpha$, then $(x_\gamma, x_\beta) \in E(G)$.
iii) If $\gamma < \alpha$ with γ is an ordinal limit and $(x, x_\beta) \in E(G)$ for any $\beta < \gamma$, then $(x, x_\gamma) \in E(G)$
iv) If $\beta < \gamma < \alpha$, then $p(x_\beta, x_\gamma) \leq \phi(x_\beta) - \phi(x_\gamma)$.

Now we construct x_α. We have two cases to consider whether α has predecessor or is a limit ordinal. First assume $\alpha - 1$ exists. In this case, we set $x_\alpha = Tx_{\alpha-1}$. Let us prove that $(x_\alpha, x_{\alpha-1}) \in E(G)$.

(1) If $\alpha-2$ exists, then $(x_{\alpha-1}, x_{\alpha-2}) \in E(G)$. Since T is G- preserving, then $(x_\alpha, x_{\alpha-1}) \in E(G)$ holds.
(2) Otherwise assume $\alpha - 2$ does not exists, i.e., $\alpha - 1$ is an ordinal limit. Then for any γ with $\gamma < \alpha - 1$, we have $(x_{\alpha-1}, x_\gamma) \in E(G)$ which implies $(Tx_{\alpha-1}, Tx_\gamma) = (x_\alpha, x_{\gamma+1}) \in E(G)$, since T is G-preserving. And since $(x_{\gamma+1}, x_\gamma) \in E(G)$, then we can apply the transitivity of G to obtain that $(x_\alpha, x_\gamma) \in E(G)$, for any $\gamma < \alpha - 1$. Using the transfinite induction assumptions, we get $(x_\alpha, x_{\alpha-1}) \in E(G)$.

Using the properties satisfied by T, we get

$$p(x_{\alpha-1}, x_\alpha) = p(x_{\alpha-1}, Tx_{\alpha-1}) \leq \phi(x_{\alpha-1}) - \phi(x_\alpha). \tag{3.1}$$

Fix $\beta < \alpha$ and let us prove next that

$$p(x_\beta, x_\alpha) \leq \phi(x_\beta) - \phi(x_\alpha).$$

This inequality is satisfied if $\beta = \alpha - 1$. Assume that $\beta < \alpha - 1$. The transfinite induction assumptions imply

$$p(x_\beta, x_{\alpha-1}) \leq \phi(x_\beta) - \phi(x_{\alpha-1}). \tag{3.2}$$

Putting the inequalities (3.1) and (3.2) together, we get

$$\begin{array}{rcl} p(x_\beta, x_\alpha) & \leq & p(x_\beta, x_{\alpha-1}) + p(x_{\alpha-1}, x_\alpha) \\ & \leq & \phi(x_\beta) - \phi(x_{\alpha-1}) + \phi(x_{\alpha-1}) - \phi(x_\alpha) \\ & = & \phi(x_\beta) - \phi(x_\alpha). \end{array}$$

Otherwise, assume $\alpha-1$ does not exist, i.e., α is an ordinal limit. Set $\phi_0 = \inf\limits_{\beta<\alpha} \phi(x_\beta)$. Using the decreasing behavior of the family of positive numbers $\{\phi(x_\beta)\}_{\beta<\alpha}$, we can construct an increasing sequence of ordinals $\{\beta_n\}$ such that $\lim\limits_{n\to\infty} \phi(x_{\beta_n}) = \phi_0$. For any $n \in \mathbb{N}$, we have $\beta_n < \beta_{n+1} < \alpha$ which implies

$$p(x_{\beta_n}, x_{\beta_{n+1}}) \leq \phi(x_{\beta_n}) - \phi(x_{\beta_{n+1}}).$$

Using the same argument in the proof of Theorem 3.1, we will show that $\{x_{\beta_n}\}$ is a Cauchy sequence in X. Since (X, d) is complete, the sequence $\{x_{\beta_n}\}$ converges. Set $\bar{x} = \lim_{n\to\infty} x_{\beta_n}$. Since ϕ is lower semicontinuous, then

$$\phi(\bar{x}) \leq \liminf_{n\to\infty} \ \phi(x_{\beta_n}) = \phi_0,$$

which implies $\phi(\bar{x}) \leq \phi_0 \leq \phi(x_{\beta_{n+h}}) \leq \phi(x_{\beta_n})$, for any $n, h \in \mathbb{N}$. Hence

$$p(x_{\beta_n}, x_{\beta_{n+h}}) \leq \phi(x_{\beta_n}) - \phi(x_{\beta_{n+h}}) \leq \phi(x_{\beta_n}) - \phi(\bar{x}),$$

for any $n, h \in \mathbb{N}$. Since p is lower semicontinuous in the second variable, we get

$$p(x_{\beta_n}, \bar{x}) \leq \liminf_{h\to\infty} \ p(x_{\beta_n}, x_{\beta_{n+h}}) \leq \phi(x_{\beta_n}) - \phi(\bar{x}).$$

Set $x_\alpha = \bar{x}$. Let $\beta < \alpha$. Assume there exists $n_0 \in \mathbb{N}$ such that $\beta \leq \beta_{n_0} < \alpha$, then we have

$$\begin{array}{rcl} p(x_\beta, x_\alpha) & \leq & p(x_\beta, x_{\beta_{n_0}}) + p(x_{\beta_{n_0}}, x_\alpha) \\ & = & p(x_\beta, x_{\beta_{n_0}}) + p(x_{\beta_{n_0}}, \bar{x}) \\ & \leq & \phi(x_\beta) - \phi(x_{\beta_{n_0}}) + \phi(x_{\beta_{n_0}}) - \phi(\bar{x}) \\ & = & \phi(x_\beta) - \phi(\bar{x}) \\ & = & \phi(x_\beta) - \phi(x_\alpha). \end{array}$$

Since $\lim_{n\to\infty} x_{\beta_n} = \bar{x} = x_\alpha$ and $(x_{\beta n+1}, x_{\beta_n}) \in E(G)$, then the (OSC)-property implies $(x_\alpha, x_{\beta_n}) \in E(G)$, for any $n \in \mathbb{N}$. Since $\beta \leq \beta_{n_0}$, then $(x_{\beta n_0}, x_\beta) \in E(G)$. Since G is transitive, we get $(x_\alpha, x_\beta) \in E(G)$. Otherwise, assume $\beta_n < \beta$, for any $n \in \mathbb{N}$. Then we have

$$p(x_{\beta_n}, x_\beta) \leq \phi(x_{\beta_n}) - \phi(x_\beta),$$

for any $n \in \mathbb{N}$. Since $\phi_0 \leq \phi(x_\beta) \leq \phi(x_{\beta_n})$, for any $n \in \mathbb{N}$, and $\lim_{n\to\infty} \phi(x_{\beta_n}) = \phi_0$, we conclude that $\phi(x_\beta) = \phi_0$. So, we have that

$$p(x_{\beta_n}, x_\beta) \leq \phi(x_{\beta_n}) - \phi_0.$$

for any $n \in \mathbb{N}$. This will imply that $\lim_{n\to\infty} p(x_{\beta_n}, x_\beta) = 0$. Since $\lim_{n\to\infty} p(x_{\beta_n}, \bar{x}) = 0$, the w-distance properties imply $x_\beta = \bar{x} = x_\alpha$. Next, for $\beta < \gamma < \alpha$, we have $\beta_n < \gamma < \alpha$ for any $n \in \mathbb{N}$. A similar argument will show that $x_\gamma = x_\alpha$, i.e., $x_\gamma = x_\alpha$, for any $\beta < \gamma < \alpha$. The assumptions of the transfinite induction are satisfied when we include α. Therefore by transfinite induction, we built $\{x_\alpha\}_{\alpha\in\Gamma}$ such that:

i) $x_{\beta+1} = Tx_\beta$;
ii) if $\beta < \alpha$, then $(x_\alpha, x_\beta) \in E(G)$;
iii) if $\gamma < \alpha$ with γ is an ordinal limit and $(x, x_\beta) \in E(G)$ for any $\beta < \alpha$, then $(x, x_\alpha) \in E(G)$;
iv) if $\beta < \alpha$, then $p(x_\beta, x_\alpha) \leq \phi(x_\beta) - \phi(x_\alpha)$.

Hence $\{\phi(x_\alpha)\}_{\alpha\in\Gamma}$ is an decreasing sequence or positive numbers in $\mathbb{R}$. Proposition 2.1 implies the existence of $\alpha_0 \in \Gamma$ such that $\phi(x_\alpha) = \phi(x_{\alpha_0})$, for any $\alpha > \alpha_0$. Fix $\lambda > \alpha > \alpha_0$. Then we have

$$\begin{array}{rcl} p(x_{\alpha_0}, x_\alpha) & \leq & \phi(x_{\alpha_0}) - \phi(x_\alpha) = 0, \\ p(x_{\alpha_0}, x_\lambda) & \leq & \phi(x_{\alpha_0}) - \phi(x_\lambda) = 0. \end{array}$$

Using the w-distance properties, we easily show that $d(x_\alpha, x_\lambda) < \varepsilon$, for any $\varepsilon > 0$. Therefore, we must have $x_\alpha = x_\lambda$. In particular, we have $x_{\alpha_0+1} = x_{\alpha_0+2} = Tx_{\alpha_0+1}$, i.e., x_{α_0+1} is a fixed point of T. □

Next, we include following examples to illustrate Theorem 3.2.

Example 3.4. Let $X = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N} \cup \{0\}\}$ and G be the directed graph defined by $V(G) = X$ and $E(G) = \Delta \cup \{(\frac{1}{2^n}, \frac{1}{2^{n+1}}), (0, \frac{1}{2^n}), (\frac{1}{2^n}, 0), (\frac{1}{2}, \frac{1}{2^n}), (\frac{1}{2^n}, \frac{1}{2}) : n \in \mathbb{N} \cup \{0\}\}$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x = 0, \frac{1}{2}; \\ 1, & \text{if } x = 1; \\ \frac{1}{2^{n-1}}, & \text{if } x = \frac{1}{\text{ffl}2^n} \text{ when } n \geq 2. \end{cases}$$

Let $d(x, y) = |x - y| = p(x, y)$, then (X, d) is complete metric space and p is w - distance on X. From the setting, we can see that G is transitive and there exist $x_0 = 0$ such that $(Tx_0, x_0) \in E(G)$ and T is also G - preserving. If we define

$$\phi(x) = \begin{cases} \frac{17}{6}, & \text{if } x = 0; \\ \frac{1}{x}, & \text{if } x \neq 0, \end{cases}$$

then it can be shown that

$$p(x, Tx) \leq \phi(x) - \phi(Tx)$$

for any $x \in X$ with $(Tx, x) \in E(G)$. Moreover, the sequence $\{x_n\}$ which constructed by mapping T satisfy the (OSC)-property. Therefore, T satisfies all conditions of Theorem 3.2 and hence, T admit fixed points $x = \frac{1}{2}$ and $x = 1$.

Remark 3.2. In the above example 3.4, we note that **1.** since T admit more than one fixed points and hence T is not a Banach contraction. **2.** T is not G-continuous, then the Theorem 3.1 given in [26] cannot be applied on this example.

Example 3.5. Consider the w-distance p on X as defined in example 2.3, where $X = \{0, 1\}$ and G be the directed graph defined by $V(G) = X$ and $E(G) = \Delta \cup \{(0, 1), (1, 0)\}$. Define the mapping $T : X \to X$ by $T(x) = 0$ for all $x \in X$, then it is easy to see that T satisfies all the conditions of Theorem 3.2 and hence, T admit a fixed point $x = 0$.

Remark 3.3. since the w-distance p in the above example 3.5 is not a metric and hence the Theorem 3.2 given in [3] can not be applicable on this example.

Next, we are looking for a condition which imply lower semi continuity of ϕ when defining

$$\phi(x) = \frac{p(x, Tx)}{1 - k}, \quad k \in [0, 1). \tag{3.3}$$

In the case of metric space (X, d), it is well known, as Ran and Reurings did in [25], that if T is continuous then ϕ is continuous. This assumption was further weakened by Alfuraidan et al. in their Theorem 2.1 [2], assuming T is $\preccurlyeq$ - continuous (i.e., $T(x_n)$ is convergent whenever monotone sequence x_n converges in their domain).

At this juncture, we really do not know whether the similar assumptions on T imply lower semicontinuity of ϕ in 3.3. However, in the quest of desirable condition, the idea of "Fatou property" given in [1] inspire us to find the necessary condition for proving the lower semicontinuity of ϕ in 3.3. The following is needed condition for ensuring the lower semicontinuity of ϕ.

Definition 3.5. Let (X, d) be a metric space, p is w - distance on X, $T : X \to X$ is a mapping and $x_n \to x$ where $x_n, x \in X$, then we say that map $x \mapsto p(x, Tx)$ is lower semicontinuous if

$$p(x, Tx) \leq \liminf_{n \to \infty} p(x_n, Tx_n).$$

Remark 3.4. We observed that even if p satisfies Fatou property and T is continuous, we still cannot guarantee the lower semicontinuity of the map $x \mapsto p(x, Tx)$ in Definition 3.5

Remark 3.5. It is easy to see that if T is continuous, then the metric version of map in Definition 3.5 always get satisfied but converse need not be true (see Example 3.7 given below).

Example 3.6. Let $X = [0, 1]$, $p(x, y) = c$ where c is a positive real number. And let $T : X \to X$ such that

$$T(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ \frac{1}{2}, & \text{if } x = 1. \end{cases}$$

Then, we can see that $p(x, Tx) \leq \liminf_{n\to\infty} p(x_n, Tx_n)$ for any $x_n \to x \in X$. However, T is not continuous, in fact T is not $\preccurlyeq$ - continuous (for this, consider $x \preccurlyeq y$ for $x \leq y$, and $x_n = \frac{n}{n+1}$ which converges to 1 for any $n \in \mathbb{N}$).

Example 3.7. Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and $d(x, y) = p(x, y) = |x - y|$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 1, & \text{if } x = \frac{1}{n}; \\ 0, & \text{if } x = 0. \end{cases}$$

Then $p(x, Tx) \leq \liminf_{n\to\infty} p(x_n, Tx_n)$ but T is not $\preccurlyeq$ - continuous.

As a consequence of of Theorem 3.2 with lower semicontinuous of $x \mapsto p(x, Tx)$, we obtain another new result as follows.

Theorem 3.3. *Let (X, d, G) be a complete metric space endowed with a directed graph G with transitivity and p is w-distance on X. If $T : X \to X$ satisfies the following properties:*

(i) there exists x_0 in X such that $(Tx_0, x_0) \in E(G)$;
(ii) T is G - preserving, i.e., for any $x, y \in X$ with $(x, y) \in E(G)$, then $(Tx, Ty) \in E(G)$;
(iii)

$$p(Tx, Ty) \leq kp(x, y)$$

for any $x, y \in X$ with $(x, y) \in E(G)$ and $0 \leq k < 1$;
(iv) G satisfies the (OSC)-property;
(v) $x \mapsto p(x, Tx)$ is lower semicontinuous, i.e, if $x_n \to x$ in X, then $p(x, Tx) \leq \liminf_{n\to\infty} p(x_n, Tx_n)$

then T has a fixed point.

Proof. As a consequence of Theorem 3.2, if ϕ is defined by

$$\phi(x) = \frac{p(x, Tx)}{1 - k}, \ k \in [0, 1),$$

and since

$$p(Tx, Ty) \leq kp(x, y) \text{ for } k \in [0, 1),$$

that is

$$p(Tx, T^2x) \leq kp(x, Tx),$$

So, we have

$$\begin{aligned} p(x, Tx) &\leq \frac{p(x, Tx)}{1 - k} - \frac{p(Tx, T^2x)}{1 - k} \\ &\leq \phi(x) - \phi(Tx). \end{aligned}$$

Moreover, we can show lower semi continuity of ϕ by using assumption (v) as follows:

$$\begin{aligned}\phi(x) = \frac{p(x,Tx)}{1-k} &\leq \frac{1}{1-k}\liminf_{n\to\infty} p(x_n, Tx_n),\\ &\leq \liminf_{n\to\infty}\frac{p(x_n,Tx_n)}{1-k},\\ &= \liminf_{n\to\infty}\phi(x_n).\end{aligned}$$

Therefore, the proof for existence of fixed point is similar to the Theorem 3.2 by setting $\phi(x)$ as above. □

Remark 3.6. Theorem 3.3 extend and improve some recent works concerning extension of Banach Contraction Theorem to w-distance with graph e.g. ([15, 23, 24]).

Remark 3.7. From Remark 3.5 and the proof of lower semicontinuous of ϕ in Theorem 3.3, we observe that Proposition 2.1 in [2] can easily be strengthen upon replacing the condition $\preccurlyeq$-continuity of T by weaker condition lower semicontinuity of the map $x \mapsto d(x, Tx)$ (i.e. metric version of Definition 3.5).

4. Conclusion

In a metric space endowed with w-distance and graph, we prove some fixed point results of Caristi-type mapping for set-valued and single-valued mapping by reformulating the (OSC)-property which extend some results in [3].

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