CARPATHIAN J. MATH. 36 (2020), No. 2, 277 - 286
Online version at https://www.carpathian.cunbm.utcluj.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

Fixed points results in modular vector spaces with applications to quantum operations and Markov operators

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ABSTRACT. Recently, researchers are showing more interest on both modular vector spaces and modular function spaces. Looking at the number of results it is pertinent to say that, exploration in this direction especially in the area of fixed point theory and applications is still ongoing, many good results can still be unveiled. As a contribution from our part, we study some fixed point results in modular vector spaces associated with order relation. As an application, we were able to study the existence of fixed point(s) of both depolarizing quantum operation and Markov operators through modular functions/modular spaces. The awareness on the importance of quantum theory and Economics globally were the sole motivations of the application choices in our work. Our work complement the existing results. In fact, it adds to the number of application areas that modular vector/function spaces covered.

1. INTRODUCTION

It was recorded that, introduction of the vector space that consist of sequences was associated to the work of Orlicz [23], as we can observe below

$$\text{(OS)} \qquad \ell_{p(\cdot)} = X = \Big\{ \{u_l\} \subset \mathbb{R}^{\mathbb{N}} : \sum_{l=0}^{\infty} |\gamma\, u_l|^{p(l)} < \infty \ \text{ for some } \gamma > 0 \Big\},$$

for $\{p(l)\} \subset [1, \infty)$. After the introduction of the space X, many researchers undergo an extensive study on both the topological and geometrical structures of the space X, see [10, 13, 14, 16, 17, 21, 26, 27].
From 1950 to 1951, Nakano [19, 20] coined the definition of the modular function we use today and which specifically contained the spaces $\ell_{p(\cdot)}$ introduced by Orlicz. Looking at the variable exponent spaces (VES) [4], one can easily observe that, $\ell_{p(\cdot)}$ is a special case.

As time pass by, the applications of those spaces were unveiled especially in the area of material sciences. As such, the area received more attention from many researchers globally, which in turn boost the quantity of published results. Kováčik and Rákosník [11] tackled the topological structures/properties involving the VES; the concern on hydrodynamics of electrorheological fluids by Rajagopal and Ružička [24, 25] motivate them to initiate a mathematical investigation in the area. Due to their studies, other researchers follow their foot print and establish good results that are essential today in the area of VES. For example, mathematically, one can use the partial differential equations with non-standard growth to illustrate the behavior of the non-Newtonian fluids obtainable in the Rajagopal-Ružička model, whose solutions is typically found in Sobolev spaces of

Received: 01.11.2019. In revised form: 12.06.2020. Accepted: 12.06.2020
2010 *Mathematics Subject Classification.* 47H09, 47H10, 11F03, 81P45.
Key words and phrases. *Order relation, Electrorheological fluids, quantum state, fixed point, Markov operator, modular vector spaces, monotone mapping, ρ-asymptotically contractive mapping.*
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variable exponents. No doubt, experts in the area of medicine, defense industry and civil engineering are currently enjoying the Electrorheological fluids applications.

In this paper, with keen interest to the work of Abdou et al [1], Kamihigashi et al [6] and Batsari et al [28], we used the modular function and a binary relation to establish some fixed point results. And later, we give some applications through quantum operations and Markov operators.

As the nature of our work inclined more on the fixed point theory of metric type, the reader can consult the books written by Khamsi and Kirk [8] and the recent one written by Khamsi and Kozlowski [9].

2. Notations and definitions

As stated above that the space $X = \ell_{p(\cdot)}$ is clearly a special case of the VES, see [23]. With this motivation, Nakano [19, 20] come up with the structure of the modular vector.

Definition 2.1. Consider a real vector space X. A function $\rho : X \to [0, +\infty]$ is referred to as regular modular if the following conditions hold:

(1) $\rho(z) = 0$ iff $z = 0$.
(2) $\rho(z) = \rho(-z)$.
(3) $\rho(\beta z + (1-\beta)t) \leq \rho(z) + \rho(t)$, $\beta \in [0, 1]$ and for every $z, t \in X$.

By replacing (3) with the below inequality

(4) $\rho(\beta z + (1-\beta)t) \leq \beta\rho(z) + (1-\beta)\rho(t)$, for any $z, t \in X$ and $\beta \in [0, 1]$,

then we called the function ρ regular convex modular.

Example 2.1. Let $X = [0, 1]$ define the functional $\rho : X \to [0, +\infty]$ by

$$\rho(x) = |x|, \ \forall\, x \in [0, 1],$$

it is easy to check that ρ is a regular convex modular finctional.

Example 2.2. [15] Below are definitions/formulations of two important modular functions:

(1) The Orlicz modular on the set of measurable real valued functions X is defined as

$$\rho(f) = \int_{\mathbb{R}} \psi(|f(t)|)dm(t),$$

where m denotes the lebesgue measure in $\mathbb{R}$ whereas $\psi : \mathbb{R} \to [0, \infty)$ is continuous, $\psi(v) = 0$ iff $v = 0$ and $\psi(t) \to \infty$ as $t \to \infty$.

(2) The Musielak-Orlicz modular function is given as

$$\rho(f) = \int_{\Omega} \psi(\omega, f(\omega))d\mu(\omega),$$

where μ is a $\sigma-$finite measure on Ω and $\psi : \Omega \times \mathbb{R} \to [0, \infty)$ satisfy the followings:
(i) $\psi(\omega, v)$ is a continuous even function which is non decreasing for $v > 0$, $\psi(\omega, 0) = 0$, $\psi(\omega, v) > 0$ for $v \neq 0$ and $\psi(\omega, v) \to \infty$ as $v \to \infty$.
(ii) $\psi(\omega, v)$ is a measurable function of ω for every $v \in \mathbb{R}$.

Many good examples involving modular vector spaces and modular function spaces can be found in the book [12].

Definition 2.2. Consider a vector space X and ρ its associated regular convex modular function. The set

$$X_\rho = \{z \in X : \lim_{\gamma \to 0^+} \rho(\gamma z) = 0\}$$

is a vector subspace of X which is usually known as the associated modular vector space. On it, the Luxemburg norm is define as

$$\|z\|_\rho = \inf\{\gamma > 0 : \ \rho\left(\frac{z}{\gamma}\right) \leq 1\},$$

for any $z \in X_\rho$.

In the following definition, we give the necessary tools used throughout.

Definition 2.3. Let ρ be a regular convex modular defined on a vector space X.

(1) We will say that a sequence $\{u_l\}_{l\in\mathbb{N}} \subset X_\rho$ ρ-converges to u if and only if $\lim\limits_{l\to\infty} \rho(u_l - u) = 0$. u is conventionally known as the ρ-limit of $\{u_l\}$.

(2) We will say that a sequence $\{u_l\}_{l\in\mathbb{N}} \subset X_\rho$ is ρ-Cauchy if and only if

$$\lim_{h,l\to\infty} \rho(u_h - z_l) = 0.$$

(3) We will say that $C \subset X_\rho$ is ρ-closed if the ρ-limit of any ρ-convergent sequence of C belongs to C.

(4) We will say that $C \subset X_\rho$ is ρ-complete if any ρ-Cauchy sequence $\{u_l\} \subset C$ is ρ-convergent and its ρ-limit belongs to C.

(5) We will say that $C \subset X_\rho$ is ρ-bounded if

$$\delta_\rho(C) = \sup\{\rho(z - t) : z, t \in C\} < \infty.$$

(6) ρ is said to satisfy the Fatou property if and only if for any two ρ-convergent sequences $\{u_l\}, \{s_l\} \subseteq X_\rho$ ρ-converging to u and s respectively, we have

$$\rho(u - s) \leq \liminf_{l\to\infty} \rho(u_l - s_l).$$

In general if $\lim\limits_{l\to\infty} \rho(\gamma(u_l - u)) = 0$, for some $\gamma > 0$, then we may not have $\lim\limits_{l\to\infty} \rho(\gamma(u_l - u)) = 0$, for all $\gamma > 0$. This little problem is fixed by assuming that, the modular function ρ satisfies the Δ_2-condition, i.e. $\lim\limits_{l\to\infty} \rho(\gamma(u_l - u)) = 0$ for some $\gamma > 0$ implies $\lim\limits_{l\to\infty} \rho(\gamma(u_l - u)) = 0$ for all $\gamma > 0$. In particular, we have

$$\lim_{l\to\infty} \|u_l - u\|_\rho = 0 \ \text{ if and only if } \ \lim_{l\to\infty} \rho(\gamma(u_l - u)) = 0, \ \text{for all } \gamma > 0,$$

for any $\{u_l\} \subseteq X_\rho$ and $u \in X_\rho$. In other words, we will have ρ-convergence and $\|.\|_\rho$ convergence to be equivalent if and only if the modular function ρ satisfies the Δ_2-condition. Another similar property is also extensively used. Indeed, we say that, the modular function ρ satisfies the Δ_2-type condition if we can find a real number $K \geq 0$ satisfying

$$\rho(2z) \leq K\ \rho(z),$$

for any $z \in X_\rho$. For more information concerning the study of Δ_2-condition, the reader can check [9, 12, 18]. Moreover, it is well known [9] that, the modular function ρ satisfies the Fatou property if and only if the ρ-balls are ρ-closed. Recall that, a ρ-ball take the below definition

$$B_\rho(z, m) = \{s \in X_\rho : \ \rho(z - s) \leq m\},$$

for $z \in X_\rho$ and $m \geq 0$.

Definition 2.4. Let ρ be a regular convex modular defined on a vector space X. Assume that $\preceq$ is a partial order defined within X.

(i) We say that ρ is monotone if
$$s \preceq t \preceq v \quad \text{implies} \quad \max\{\rho(t-s), \rho(v-t)\} \leq \rho(v-s),$$
for any $s, t, v \in X_\rho$.

(ii) A sequence $\{u_l\}$ in X_ρ is said to be monotone increasing (resp. decreasing) if $u_l \preceq u_{l+1}$ (resp. $u_{l+1} \preceq u_l$), for any $l \in \mathbb{N}$.

Definition 2.5. Let ρ be a regular convex modular defined on a vector space X. Assume that $\preceq$ is a partial order defined on X. Let $\emptyset \neq C \subset X_\rho$.

(i) A self mapping $U : C \to C$ is referred to as monotone if
$$s \preceq z \quad \text{implies} \quad U(s) \preceq U(z),$$
for any $z, s \in C$.

(ii) For a mapping $U : C \to C$, a point $s \in C$ is said to be a fixed point of U if and only if $U(s) = s$. A point $s \in C$ will be called ρ-globally stable fixed point of U in C if s is a fixed point of U and $\lim\limits_{l \to +\infty} \rho(s - U^l(z)) = 0$, for any $z \in C$.

(iii) A mapping $U : C \to C$ is said to be ρ-asymptotically contractive if and only if
$$\lim_{l \to +\infty} \rho(U^l(s) - U^l(z)) = 0,$$
for all $s, z \in C$.

(iv) A mapping $U : C \to C$ is said to be monotone ρ-asymptotically contractive if and only if
$$\lim_{l \to +\infty} \rho(U^l(s) - U^l(z)) = 0,$$
for $s, z \in C$ comparable elements.

Remark 2.1. Let ρ be a regular convex modular defined on a vector space X. Let $\emptyset \neq C \subset X_\rho$. Note that if $U : C \to C$ is ρ-asymptotically contractive, then U has at most one fixed point which is ρ-globally stable. Indeed, let s and z be two fixed points. Then $\rho(U^l(s) - U^l(z)) = \rho(s - z)$, for any $l \in \mathbb{N}$. Since $\lim\limits_{l \to +\infty} \rho(U^l(s) - U^l(z)) = 0$, we conclude that $\rho(s - z) = 0$, and hence $s = z$. In addition, if s is a fixed point of U, then we have
$$\lim_{l \to +\infty} \rho(U^l(s) - U^l(z)) = \lim_{l \to +\infty} \rho(s - U^l(z)) = 0,$$
for all $z \in C$.

Example 2.3. Let $\mathcal{K}$ denote the vector space consisting of 2×2 real valued matrices. Let $\rho : \mathcal{K} \to [0, +\infty]$ be any regular convex modular. Suppose the nonempty set $\mathcal{C} \subseteq \mathcal{K}$ is ρ-bounded and convex. Fix $B \in \mathcal{C}$ and $p \in (0, 1]$. If the mapping $U : \mathcal{C} \to \mathcal{C}$ is defined by
$$U(A) = p\, B + (1-p)\, A,$$
then U is ρ-asymptotically contractive and B is the ρ-global stable fixed point of U. Indeed, it is obvious that $U(B) = B$. Moreover, we have
$$\rho(U(A) - U(A')) = \rho\Big((1-p)\, (A - A')\Big),$$
for any $A, A' \in \mathcal{C}$. Using the convexity of ρ, we get
$$\rho(U(A) - U(A')) \leq (1-p)\, \rho\Big(A - A'\Big),$$
which implies
$$\rho(U^l(A) - U^l(A')) \leq (1-p)^l\, \rho\Big(A - A'\Big),$$

for any $A, A' \in \mathcal{C}$ and $l \in \mathbb{N}$. Since $\rho(A - A') < +\infty$ and $(1-p) < 1$, we conclude that

$$\lim_{l \to +\infty} \rho(U^l(A) - U^l(A')) = 0,$$

this is to say U is ρ-asymptotically contractive. Using Remark 2.1, we conclude that, B is the ρ-globally stable fixed point of U.

3. Main results

Throughout we assume that, ρ is a regular convex and monotone modular defined on the vector space X, and $\preceq$ is a partial order defined on X.

Theorem 3.1. *Let $C \subset X_\rho$ be a nonempty subset. Let $U : C \to C$ be a monotone mapping. Assume there exist $s, t \in C$ such that, the conditions*

$$\lim_{i \to +\infty} \rho(U^i(s) - U^i(t)) = 0, \tag{3.1}$$

$$U^i(s) \preceq t, \ \forall\, i \in \mathbb{N}, \tag{3.2}$$

$$t \preceq U(t), \tag{3.3}$$

are satisfied. Then U has a fixed point.

Proof. Let the conditions (3.1)-(3.3) hold. Since U is monotone and that condition (3.3) holds, we have $\{U^i(t)\}$ is monotone increasing. Using (3.2), we have

$$U^i(s) \preceq t \preceq U(t) \preceq U^i(t),$$

for any $i \geq 1$. The monotonicity of ρ implies

$$0 \leq \rho(t - U(t)) \leq \rho(U^i(s) - U^i(t))$$

for any $i \geq 1$. The condition (3.1) implies $\rho(t - U(t)) = 0$, i.e., $U(t) = t$. □

Example 3.4. Consider the collection $\mathcal{M}$ of 2×2 matrices with real entries. Define a mapping $U : \mathcal{M} \to \mathcal{M}$ by

$$U\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2a & b \\ c & 2d \end{bmatrix}. \tag{3.4}$$

Define a relation $\preceq$ on $\mathcal{M}$ by $A \preceq B$ iff $a_{ij} \geq b_{ij}$ for $A, B \in \mathcal{M}$ and $i, j \in \{1, 2\}$. Let the function $\rho : \mathcal{M} \to \mathbb{R}_+$ be defined as

$$\rho(A) = \sum_{j=1}^{2}\sum_{i=1}^{2} |a_{ij}|.$$

So with the above definitions, all conditions and assumptions of the above Theorem 3.1 are satisfied.

Proof. The relation $\preceq$ can easily be seen as a partial order on $\mathcal{M}$. Moreover, it is obvious that the mapping U is monotone. From the properties of the absolute value function, it is trivial for the functional ρ to be a regular convex and monotone modular. Using similar procedure as in Example 2.3, condition (3.1) follows. Conditions (3.2) and (3.3) easily follow from the definitions of U and $\preceq$.
Furthermore, using

$$t = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \textit{ and } s = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix},$$

one can check to see that U(t)=t and the fixed point set of U denoted by $F(U)$ is

$$F(U) = \left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},\ a, b \in \mathbb{R}\right\}.$$

□

Next we assume a property satisfied by most of the modular function spaces when partially ordered by the almost-pointwise order:

Property (UBY): For any increasing sequence $\{u_l\} \subset X_\rho$ which is ρ-convergent to $u \in X_\rho$, we have $u_l \preceq u$, for any $l \in \mathbb{N}$. Moreover, if there exists $z \in X_\rho$ such that $u_l \preceq z$, for any $l \in \mathbb{N}$, then $u \preceq z$ holds.

Theorem 3.2. *Assume X_ρ is ρ-complete. Let $C \subset X_\rho$ be nonempty and ρ-closed. Let $U : C \to C$ be a monotone ρ-asymptotically contractive mapping. Assume there exist $t, z \in C$ such that*

$$t \preceq U(t), \tag{3.5}$$

$$U^l(t) \preceq z, \ \forall l \in \mathbb{N}. \tag{3.6}$$

Then U has a fixed point.

Proof. Since $t \preceq U(t)$, then $\{U^l(t)\}$ is monotone increasing. Since U is monotone, the condition (3.5) implies $U^{l+h}(t) \preceq U^l(z)$, for any $h, l \in \mathbb{N}$. Therefore, we have

$$U^l(t) \preceq U^{l+h}(t) \preceq U^l(z), \tag{I}$$

for any $h, l \in \mathbb{N}$. Since ρ is monotone, we get

$$\rho(U^l(t) - U^{l+h}(t)) \leq \rho(U^l(t) - U^l(z)),$$

for any $h, l \in \mathbb{N}$. Since t and z are comparable and U is monotone ρ-asymptotically contractive, we get

$$\lim_{n\to+\infty} \rho(U^l(t) - U^l(z)) = 0.$$

Let $\varepsilon > 0$. There exists $l_0 \in \mathbb{N}$ such that

$$\rho(U^l(t) - U^l(z)) < \varepsilon,$$

for any $l \geq l_0$. Hence

$$\rho(U^l(t) - U^{l+h}(t)) < \varepsilon,$$

for any $l \geq l_0$ and $h \in \mathbb{N}$, i.e., $\{U^l(t)\}$ is ρ-Cauchy. Since X_ρ is ρ-complete, there exists $\hat{t} \in X_\rho$ such that $\{U^l(t)\}$ is ρ-convergent to $\hat{t}$. Since C is ρ-closed, $\hat{t} \in C$. From the inequalities (I) and the property (UBY), we get

$$U^l(t) \preceq \hat{t} \preceq U^l(z),$$

for any $l \in \mathbb{N}$. Since U is monotone, we get

$$U^{l+1}(t) \preceq U(\hat{t}) \preceq U(U^{l-1}(z)) = U^l(z), \ l \geq 1$$

Using the property (UBY), we conclude that $\hat{t} \preceq U(\hat{t})$. Putting everything together, we obtain

$$U^l(t) \preceq \hat{t} \preceq U(\hat{t}) \preceq U^l(z),$$

for any $l \in \mathbb{N}$. The monotonicity of ρ implies

$$\rho(\hat{t} - U(\hat{t})) \leq \rho(U^l(t) - U^l(z)),$$

for any $l \in \mathbb{N}$. Since $\lim\limits_{l\to\infty} \rho(U^l(t) - U^l(z)) = 0$, we conclude that $\rho(\hat{t} - U(\hat{t})) = 0$, i.e. $U(\hat{t}) = \hat{t}$. In other words, $\hat{t}$ is a fixed point of U which completes the proof of Theorem 3.2. □

4. Some applications

4.1. **Quantum operations.**

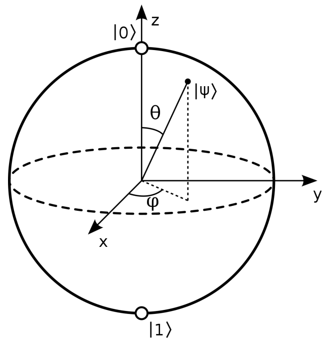

FIGURE 1. **Bloch sphere**

Let $\mathcal{Q}$ denote the vector space formed from the collection of quantum states of a two-state quantum system [22]. Consider the two-state quantum system representation known as the Bloch sphere and denote it by $\mathcal{B}$ (see Fig. 1). Each quantum state $|\psi\rangle$ of the system has a density matrix representation $\xi_{|\psi\rangle}$ given as

$$\xi_{|\psi\rangle} = \frac{1}{2}\begin{pmatrix} 1+\gamma\cos\theta & \gamma e^{-i\varphi}\sin\theta \\ \gamma e^{i\varphi}\sin\theta & 1-\gamma\cos\theta \end{pmatrix},$$

where $\gamma \in [0,1]$, $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$. Inspired by an example introduced by Orlicz in [23] (see (OS)), we define the modular function on $\mathcal{Q}$ by

$$\rho(\xi) = \rho\left(\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\right) = \sum_{j=1}^{2}\left(\sum_{i=1}^{2}|m_{ij}|^{p_{ij}}\right),$$

where $\xi \in \mathcal{Q}$, $m_{ij} \in \mathbb{C}$ and $p_{ij} \ge 1$, for all $i,j \in \{1,2\}$. It is easy to check that ρ is a regular convex modular. For different partial orders on $\mathcal{B}$, we recommend the paper by Coecke and Martin [2]. Here we consider the spectral partial order $\preceq$ defined by $\xi \preceq \xi'$ if and only if the line segment from the center of the Bloch sphere to the point ξ' passes through ξ, where the center of the Bloch sphere is known as the completely mixed quantum state represented by the matrix $I/2$. Note that if ξ is defined by (γ,θ,φ) and ξ' is defined by $(\gamma',\theta',\varphi')$, then $\xi \preceq \xi'$ if and only if $\gamma \le \gamma'$, $\theta = \theta'$ and $\varphi = \varphi'$. The modular ρ is easily checked to be monotone with respect to the partial order $\preceq$. Consider the depolarizing quantum operation U on the Bloch sphere defined by $U(\xi) = \frac{p}{2}I + (1-p)\xi$, with the depolarizing parameter $p \in [0,1]$. If ξ is given by (γ,θ,φ), then

$$U(\xi) = \frac{1}{2}\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} + \frac{1-p}{2}\begin{pmatrix} 1+\gamma\cos\theta & \gamma e^{-i\varphi}\sin\theta \\ \gamma e^{i\varphi}\sin\theta & 1-\gamma\cos\theta \end{pmatrix}$$

which implies

$$U(\xi) = \frac{1}{2}\begin{pmatrix} 1+\gamma(1-p)\cos\theta & \gamma(1-p)e^{-i\varphi}\sin\theta \\ \gamma(1-p)e^{i\varphi}\sin\theta & 1-\gamma(1-p)\cos\theta \end{pmatrix}.$$

Clearly, the angles θ and ϕ are not affected by the depolarizing quantum operation U. It is then easy to check that U is monotone. According to Example 2.3, U is ρ-asymptotically contractive and $I/2$ is the ρ-global stable fixed point of U.

4.2. **Nonlinear Markov operators.** Let $\mathcal{F}$ denote the set whose elements are the probability distribution functions defined on the set of real numbers $\mathbb{R}$; one can verify that, each $f \in \mathcal{F}$ is increasing and upper semi-continuous function from the set $\mathbb{R}$ to the closed interval [0,1], and also satisfy the below conditions

$$\lim_{x \downarrow -\infty} f(x) = 0, \text{ for each } x \in \mathbb{R} \tag{4.7}$$

$$\lim_{x \uparrow +\infty} f(x) = 1, \text{ for each } x \in \mathbb{R}. \tag{4.8}$$

Note that, $\mathcal{F}$ is a convex set [3]. The self mapping $U : \mathcal{F} \to \mathcal{F}$ is mostly referred to as nonlinear Markov operator [6].
Consider the order relation $\preceq$ defined on $\mathcal{F}$ by

$$f \preceq g \; iff \; g(x) \leq f(x), \; \forall x \in \mathbb{R}. \tag{4.9}$$

The order $\preceq$ defined above is called a stochastic dominance.

Suppose $U : \mathcal{F} \to \mathcal{F}$ is monotone. For $f, g \in \mathcal{F}$ and $x \in \mathbb{R}$, define

$$\rho(f-g) = \int_{\mathbb{R}} |f(x) - g(x)|^{p(x)} dx, \; p(x) > 1, \; \forall \, x \in \mathbb{R}. \tag{4.10}$$

Definition 4.6. A function $\Phi_b : X \to \mathbb{R}_+ \cup \{\infty\}$ for $b \in B \subseteq \mathbb{R}$ is **one dimensional**, if $b \in B$ does not affect the images/values of
Φ_b; $\Phi_b = \Phi_{b'} \; \forall b, b' \in B$.

Theorem 4.3. *Let U be monotone and ρ-asymptotically contractive mapping on the modular space $\mathcal{F}_\rho$. If there exist $f, \overline{f} \in \mathcal{F}$ such that,*

$$f \preceq Uf, \tag{4.11}$$

$$U^i f \preceq \overline{f}. \tag{4.12}$$

Then, U has a global stable fixed point $f^* \in \mathcal{F}$.

Proof. From the above definition of the modular ρ (4.10), it is clear that, ρ is not one dimensional. So, another technique of proof is required other than those discussed earlier that involved a one dimensional modular function. Henceforth, if we write $f \leq g$ we mean $f(x) \leq g(x), \; \forall \, x \in \mathbb{R}$. Now, from (4.11) and (4.9) we have $Uf \leq f$.
Now, let

$$f^* = inf_{i \in \mathbb{N}} \{U^i f\}, \tag{4.13}$$

where the infimum is taken point-wise. Clearly, f^* satisfies (3.2) and for U being ρ-asymptotically contractive condition (3.1) holds, with reference to **Theorem** 3.1, for us to conclude f^* is a fixed point of U, it suffices to show condition (3.3) holds.

We proceed by showing $f^* \in \mathcal{F}$. Since each $f \in \mathcal{F}$ is increasing, then f^* is increasing too. Considering (4.11) - (4.13), we have $\overline{f} \leq f^* \leq U^i f \leq f \; \forall i \in \mathbb{N}$. Hence, $f^*(x) \in [0,1]$ for all $x \in \mathbb{R}$. Moreover, $0 \leq \lim_{x \downarrow -\infty} f^*(x) \leq \lim_{x \downarrow -\infty} f(x) = 0$ and $1 = \lim_{x \uparrow +\infty} \overline{f}(x) \leq \lim_{x \uparrow +\infty} f^*(x) \leq 1$. Therefore, $\lim_{x \uparrow +\infty} f^*(x) = 1$ and $\lim_{x \downarrow -\infty} f^*(x) = 0$. Note that, for f^* being an infimum of upper semi-continuous functions, it is upper semi-continuous too. Thus, $f^* \in \mathcal{F}$.
Next, for U being monotone, $f^* \leq U^i f \; \forall i \in \mathbb{N}$ and $U^{i+1} f \leq U^i f$, we have $Uf^* \leq U^{i+1} f \; \forall i \in \mathbb{N}$. Since $\{U^i f\}$ is decreasing with respect to $\leq$, then $\inf_{i \in \mathbb{N}} U^{i+1} f = \inf_{i \in \mathbb{N}} U^i f = f^*$, which implies $Uf^* \leq f^*$. Thus, $f^* \preceq Uf^*$; condition (3.3) holds. Therefore, from the proof of **Theorem** 3.1, f^* is a fixed point of U (invariant probability distribution function). As U is ρ-asymptotically contractive, then f^* is a global stable fixed point of U (Remark 2.1). $\square$

Definition 4.7. Suppose S denote a topological space endowed with borel sets $\mathfrak{B}$. Let $\mathfrak{P}_S$ denote the space of probability measures/distributions on $(S, \mathfrak{B})$. A sequence $\{\mu_n\} \subset \mathfrak{P}_S$ is considered to be tight if, for all $\epsilon > 0$, there exists a compact $K \subset S$ such that $\mu_n(S \backslash K) < \epsilon$ for all n, see [7].

One way to ensure the existence of f satisfying (4.12) is by assuming that $\{T^i f\}$ is "tight" (with $\{Tif\}$ viewed as a sequence of probability measures).

5. Conclusion

As modular functions were utilized in establishing the fixed point results, the method/technique proved to be promising. The consequent applications to both Markov and Quantum operations will no doubt serve as an opener to more application results using modular vector/function spaces.

It will be interesting if more general quantum operations were studied in modular vector/function spaces, as fixed points of either the quantum states or density operators representations can be of valuable importance in quantum information theory.

Acknowledgements. The authors acknowledge the financial support provided by King Mongkut's University of Technology Thonburi through the "KMUTT 55^{th} Anniversary Commemorative Fund". Umar Batsari Yusuf was supported by the Petchra Pra Jom Klao Doctoral Academic Scholarship for Ph.D. Program at KMUTT. Moreover, the second author was supported by Theoretical and Computational Science (TaCS) Center, under Computational and Applied Science for Smart Innovation Cluster (CLASSIC), Faculty of Science, KMUTT.

References

[1] Abdou, A. A. N. and Khamsi, M. A., *Fixed point theorems in modular vector spaces*, J. Nonlinear Sci. Appl., **10** (2017), 4046–4057

[2] Coecke, B. and Martin, K., *A partial order on classical and quantum states*, *Research Report PRG-RR-02-07*, Oxford University Computing Laboratory, Oxford, United Kingdom, 2002

[3] Cozman, F. G., *Sets of probability distributions and independence*, SIPTA Summer School Tutorials, July 2 - 8 (2008), Montpellier, France, 2008

[4] Diening, L., Harjulehto, P., Hästö, P. and Ružička, M., *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Note in Mathematics Vol. **2017**, Springer, Berlin, 2011

[5] Eisner, T., Farkas., B., Haase, M. and Nagel, R., *Operator theoretic aspects of ergodic theory*, Vol. **272** of Graduate Texts in Mathematics, Springer International Publishing, Cham, 2015

[6] Kamihigashi, T. and Stachurski, J., *Simple fixed point results for order-preserving self-maps and applications to nonlinear Markov operators*, Fixed Point Theory and Appl., **2013** (2013), 10 pp.

[7] Kamihigashi, T. and Stachurski, J., *Stochastic stability in monotone economies*, Theor. Econ., **9** (2014), 383–407

[8] Khamsi, M. A., and Kirk, W. A., *An introduction to metric spaces and fixed point theory*, John Wiley, New York, 2001

[9] Khamsi, M. A. and Kozlowski, W. M., *Fixed point theory in modular function spaces*, Birkhauser, New York, 2015

[10] Klee, V., *Summability in $\ell(p_{11}, p_{21}, \cdots)$ spaces*, Stud. Math., **25** (1965), 277–280

[11] Kováčik, O. and Rákosník, J., *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J., **41** (1991), 592–618

[12] Kozlowski, W. M., *Modular function spaces*, Series of monographs and textbooks in pure and applied mathematics, Vol. **122**, Dekker, New York/Basel, 1988

[13] Kirk, W. A., Hong-Kun, Xu., *Asymptotic pointwise contractions*, Nonlinear Anal., **69** (2008), 4706–4712

[14] Kuaket, K. and Kumam, P., *Fixed points of asymptotic point wise contractions in modular spaces*, Appl. Math. Lett., **24** (2011), 1795–1798

[15] Kumam, P., *Fixed point theorems for nonexpansive mappings in modular spaces*, Arch. Math. (BRONO), **40** (2004), 345–353

[16] Mongkolkeha, C., Sintunavarat, W. and Kumam, P., *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl., **2011** (2011), 9 pp.

[17] Mongkolkeha, C., Sintunavarat, W. and Kumam, P., *Correction: Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl., **2012** (2012), 3 pp.
[18] Musielak, J., *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1983
[19] Nakano, H., *Modulared Semi-ordered Linear Spaces*, Maruzen Co., Tokyo, 1950
[20] Nakano, H., *Topology of linear topological spaces*, Maruzen Co. Ltd., Tokyo, 1951
[21] Nakano, H., *Modulared sequence spaces*, Proc. Japan Acad., **27** (1951), 508–512
[22] Nielsen, M. A. and Chuang, I. L., *Quantum Computation and Quantum Information*, 10^{th} anniversary edition, Cambridge University Press, Cambridge, United Kingdom, 2010
[23] Orlicz, W., *Über konjugierte Exponentenfolgen*, Studia Math., **3** (1931), 200–211
[24] Rajagopal, K. and Ružička, M., *On the modeling of electrorheological materials*, Mech. Research Comm., **23** (1996), 401–407
[25] Ružička, M., *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, **1748**, Springer-Verlag, Berlin, 2000
[26] Sundaresan, K., *Uniform convexity of Banach spaces* $\ell(\{p_i\})$, Studia Math., **39** (1971), 227–231
[27] Waterman, D., Ito, T., Barber, F. and Ratti, J., *Reflexivity and Summability: The Nakano* $\ell(p_i)$ *spaces*, Stud. Math., **33** (1969), 141–146
[28] Batsari, U. Y., Kumam, P. and Dhompongsa, S., *Fixed points of terminating mappings in partial metric spaces*, J. Fixed Point Theory Appl., **21** (2019), 20 pp.
[29] Yusuf, U. B., Kumam, P. and Yoo-Kong, S., *Some generalised fixed point theorems applied to quantum operations*, Symmetry (MDPI), **12** (2020), 16 pp.

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