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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

Iterating nonlinear contractive mappings in Banach spaces

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ABSTRACT. We introduce a new class of nonlinear contractive mappings in Banach spaces, study their iterates and establish a fixed point theorem for them.

1. INTRODUCTION AND PRELIMINARIES

Let $(X, \|\cdot\|)$ be a Banach space and let K be a nonempty and closed subset of X. Following [3, 7], we denote by $\mathcal{F}$ the set of continuous functions $f : X \to [0, \infty)$ with $f(0) = 0$ satisfying the following two conditions:

(P1) For each positive number ϵ, there is a positive number δ such that for each pair of points $x, y \in K$ satisfying $f(x - y) \le \delta$, we have $\|x - y\| \le \epsilon$;

(P2) The function $(x, y) \mapsto f(x - y), x, y \in K$, is uniformly continuous on the set $K \times K$ and for each point $z \in K$, the function $x \mapsto f(x - z),\ x \in D$, is bounded on every bounded set $D \subset K$.

Let Ψ denote the set of decreasing functions $\psi : [0, \infty) \to [0, 1]$ satisfying $\psi(t) < 1$ for each positive number t.

We begin by recalling the following result, which has recently been established by Reich and Zaslavski [8].

Theorem 1.1. *Let $f \in \mathcal{F}$ and $\psi \in \Psi$ be given, and let $A : K \to K$ be a continuous mapping such that*

$$f(Ax - Ay) \le \psi(f(x - y))f(x - y) \tag{1.1}$$

for all $x, y \in K$. Then the mapping A has a unique fixed point $x_A \in K$ and $A^i x \to x_A$ as $i \to \infty$ for each $x \in K$, uniformly on bounded subsets of K.

We also recall the following definition.

Definition 1.1. [1, 5] A self-mapping A of a metric space (X, d) is said to be p-continuous, where $p = 1, 2, 3, \ldots$, if $A^p x_n \to Az$ whenever $\{x_n\}$ is a sequence in X such that $A^{p-1}x_n \to z$.

In this paper we introduce a new class of nonlinear contractive mappings in Banach spaces, study their iterates and establish a fixed point theorem for them (see Sections 2 and 3 below). We conclude our paper with a few pertinent remarks and examples (see Section 4 below).

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2. Fixed points

Let $f \in \mathcal{F}$ and let $\psi, \phi, \varphi \in \Psi$ be such that

$$\psi(t) + \phi(t) + \varphi(t) < 1 \tag{2.2}$$

for all positive numbers t. A mapping $A : K \to K$ is said to be a *nonlinear contractive mapping* of Reich type (*cf.* [6]) if

$$f(Ax - Ay) \leq \psi(f(x-y))f(x-y) + \phi(f(x-Ax))f(x-Ax) + \varphi(f(y-Ay))f(y-Ay) \tag{2.3}$$

for all $x, y \in K$.

Theorem 2.2. *Let $A : K \to K$ be a p-continuous nonlinear contractive mapping of Reich type. Then A has a unique fixed point $x_A \in K$ and $A^n x \to x_A$ as $n \to \infty$ for each $x \in K$.*

Proof. Let $x \in K$. We claim that

$$f(A^{n+1}x - A^{n+2}x) \leq f(A^n x - A^{n+1}x), \tag{2.4}$$

for all $n \geq 0$. Indeed, if inequality (2.4) were not true, then there would exist $n_0 \in \mathbb{N}$ such that

$$f(A^{n_0+1}x - A^{n_0+2}x) > f(A^{n_0}x - A^{n_0+1}x). \tag{2.5}$$

Since $\varphi \in \Psi$, it would follow from (2.5) that

$$\varphi(f(A^{n_0+1}x - A^{n_0+2}x)) \leq \varphi(f(A^{n_0}x - A^{n_0+1}x)). \tag{2.6}$$

From (2.3) we would then obtain

$$\begin{aligned} f(A^{n_0+1}x - A^{n_0+2}x) &\leq \psi(f(A^{n_0}x - A^{n_0+1}x))f(A^{n_0}x - A^{n_0+1}x) \\ &+ \phi(f(A^{n_0}x - A^{n_0+1}x))f(A^{n_0}x - A^{n_0+1}x) \\ &+ \varphi(f(A^{n_0+1}x - A^{n_0+2}x))f(A^{n_0+1}x - A^{n_0+2}x) \end{aligned}$$

and using (2.6) we would have

$$\begin{aligned} f(A^{n_0+1}x - A^{n_0+2}x) &\leq \frac{\psi(f(A^{n_0}x - A^{n_0+1}x)) + \phi(f(A^{n_0}x - A^{n_0+1}x))}{1 - \varphi(f(A^{n_0}x - A^{n_0+1}x))} \\ &\quad \times f(A^{n_0}x - A^{n_0+1}x) \\ &\leq f(A^{n_0}x - A^{n_0+1}x). \end{aligned}$$

However, this inequality would contradict inequality (2.5). Therefore we conclude that (2.4) is true, as claimed. Next, we intend to show that

$$\lim_{n\to\infty} f(A^n x - A^{n+1}x) = 0. \tag{2.7}$$

Indeed, if (2.7) were not true, then by (2.4) it would follow that there exists $\varepsilon > 0$ such that the decreasing sequence $\{f(A^n x - A^{n+1}x)\}$ converges to ε and

$$f(A^n x - A^{n+1}x) \geq \epsilon \tag{2.8}$$

for all $n \geq 0$. From relation (2.3) we now obtain that

$$\begin{aligned} f(A^{n+1}x - A^{n+2}x) &\leq \psi(f(A^n x - A^{n+1}x))f(A^n x - A^{n+1}x) \\ &+ \phi(f(A^n x - A^{n+1}x))f(A^n x - A^{n+1}x) \\ &+ \varphi(f(A^{n+1}x - A^{n+2}x))f(A^{n+1}x - A^{n+2}x) \end{aligned}$$

and since $\psi, \phi, \varphi \in \Psi$, from (2.8) we also see that

$$\begin{aligned} f(A^{n+1}x - A^{n+2}x) &\leq \psi(\epsilon)f(A^n x - A^{n+1}x) \\ &+ \phi(\epsilon)f(A^n x - A^{n+1}x) \\ &+ \varphi(\epsilon)f(A^{n+1}x - A^{n+2}x). \end{aligned}$$

Hence

$$f(A^{n+1}x - A^{n+2}x) \leq \frac{\psi(\epsilon)+\phi(\epsilon)}{1-\varphi(\epsilon)} f(A^n x - A^{n+1}x). \tag{2.9}$$

Taking $n \to \infty$ on both sides of this inequality, we conclude that (2.7) is true, as claimed. Now it follows from property (P1) and (2.7) that

$$\|A^n x - A^{n+1}x\| \to 0 \text{ as } n \to \infty. \tag{2.10}$$

We claim that $\{A^n x\}$ is a Cauchy sequence. Indeed, if this were not true, there would exist $\epsilon > 0$ and two sequences of natural numbers $\{n_k\}$ and $\{m_k\}$ such that

$$\|A^{n_k}x - A^{m_k}x\| > \epsilon, \tag{2.11}$$

for all $k \in \mathbb{N}$. From (2.11) and property (P1) it would then follow that there exists $\delta > 0$ such that

$$f(A^{n_k}x - A^{m_k}x) > \delta \tag{2.12}$$

for all $k \in \mathbb{N}$. Since $\psi \in \Psi$, it follows that

$$\psi(f(A^{n_k}x - A^{m_k}x)) \leq \psi(\delta) \tag{2.13}$$

for all $k \in \mathbb{N}$. Next, we note that from property (P2) and (2.10) it follows that there exists $k_0 \in \mathbb{N}$ such that

$$|f(A^{n_k+1}x - A^{m_k+1}x) - f(A^{n_k}x - A^{m_k}x)| \leq \frac{\delta(1-\psi(\delta))}{2}, \tag{2.14}$$

for all $k \geq k_0$. However, using (2.3), (2.7) and (2.13) we see that

$$\begin{aligned} f(A^{n_k}x - A^{m_k}x) &- f(A^{n_k+1}x - A^{m_k+1}x) \\ &\geq f(A^{n_k}x - A^{m_k}x) - \psi(\delta) f(A^{n_k}x - A^{m_k}x) \\ &- \phi(f(A^{n_k}x - A^{n_k+1}x)) f(A^{n_k}x - A^{n_k+1}x) \\ &- \varphi(f(A^{m_k}x - A^{m_k+1}x)) f(A^{m_k}x - A^{m_k+1}x) \\ &> \delta(1-\psi(\delta))/2 \end{aligned}$$

for all large enough $k \in \mathbb{N}$. Since this contradicts (2.14), we conclude that $\{A^n x\}$ is indeed a Cauchy sequence, as claimed. Therefore there exists a point $x_A \in K$ such that

$$x_A = \lim_{n\to\infty} A^n x. \tag{2.15}$$

Since $A^{p-1}A^n x \to x_A$, the p-continuity of A implies that $\lim_{n\to\infty} A^p A^n x = Ax_A$. Thus x_A is a fixed point of A. Its uniqueness follows from (2.3) because if y_A is any fixed point of A, then

$$\begin{aligned} f(x_A - y_A) &= f(Ax_A - Ay_A) \\ &\leq \psi(f(x_A - y_A)) f(x_A - y_A) \\ &+ \phi(f(x_A - Ax_A)) f(x_A - Ax_A) \\ &+ \varphi(f(y_A - Ay_A)) f(y_A - Ay_A) \\ &= \psi(f(x_A - y_A)) f(x_A - y_A), \end{aligned}$$

which implies that $x_A = y_A$. □

3. UNIFORM CONVERGENCE

In this section we continue to use the notations, definitions and assumptions introduced in the previous two sections. In addition, we assume that the following condition holds:

(P3) the function f is bounded on bounded subsets of $K - K$.

Theorem 3.3. *Assume that $A : K \to K$ is a p-continuous nonlinear contractive mapping of Reich type which is bounded on bounded subsets of K and that $x_A \in K$ satisfies*

$$Ax_A = x_A. \tag{3.16}$$

Then $A^n x \to x_A$ as $n \to \infty$ for all $x \in K$, uniformly on bounded subsets of K.

Proof. Set $A^0 x := x$ for all $x \in K$ and let $\epsilon, M > 0$. It has already been shown in the proof of Theorem 2.1 that for all $x \in K$, we have

$$f(A^{n+1}x - A^{n+2}x) \leq f(A^n x - A^{n+1}x) \text{ for all integers } n \geq 0 \tag{3.17}$$

and

$$\lim_{n\to\infty} \|A^n x - x_A\| = 0. \tag{3.18}$$

Since A is bounded on bounded sets, there exists a number $M_0 > M$ such that

$$\|Ax - x_A\| \leq M_0 \text{ for each } x \in K \text{ satisfying } \|x - x_A\| \leq M. \tag{3.19}$$

By (P3), there exists a number $M_1 > M_0$ such that

$$f(z_1 - z_2) \leq M_1 \text{ for all } z_1, z_2 \in K \text{ satisfying } \|z_1\|, \|z_2\| \leq \|x_A\| + M_0. \tag{3.20}$$

By (P1), there is $\delta_0 \in (0, \epsilon/2)$ such that

$$\text{if } z_1, z_2 \in K \text{ and } f(z_1 - z_2) \leq \delta_0, \text{ then } \|z_1 - z_2\| \leq \epsilon/2. \tag{3.21}$$

By (P2), there exists $\delta_1 \in (0, \delta_0)$ such that

$$|f(z_1 - z_2) - f(\xi_1 - \xi_2)| \leq 4^{-1}\delta_0(1 - \psi(\delta_0)) \tag{3.22}$$

for all $z_1, z_2, \xi_1, \xi_2 \in K$ satisfying

$$\|z_i - \xi_i\| \leq 2\delta_1, \; i = 1, 2.$$

Property (P1) implies that there exists a positive number

$$\delta < (1 - \psi(\delta_0))\delta_0/4$$

such that

$$\text{if } z_1, z_2 \in K \text{ and } f(z_1 - z_2) \leq \delta, \text{ then } \|z_1 - z_2\| \leq \delta_1. \tag{3.23}$$

Choose a natural number $n_0 > 4$ such that

$$(\psi(\delta) + \phi(\delta) + \varphi(\delta))^{n_0 - 1} < \delta M_1^{-1}. \tag{3.24}$$

and assume that

$$x \in K \text{ and } \|x - x_A\| \leq M. \tag{3.25}$$

We claim that there exists an integer $i \in [0, n_0]$ such that

$$f(A^i x - A^{i+1}x) \leq \delta.$$

Suppose to the contrary that no such integer exists. Then for each integer $i \in [0, n_0]$,

$$f(A^i x - A^{i+1}x) > \delta. \tag{3.26}$$

Let $i \in \{0, \ldots, n_0 - 1\}$. By (2.3), (3.17) and (3.26), we have

$$f(A^{i+1}x - A^{i+2}x)$$

$$\leq \psi(f(A^i x - A^{i+1}x))f(A^i x - A^{i+1}x) + \phi(f(A^i x - A^{i+1}x))f(A^i x - A^{i+1}x) + \varphi(f(A^{i+1}x - A^{i+2}x))f(A^{i+1}x - A^{i+2}x)$$
$$\leq \psi(\delta)f(A^i x - A^{i+1}x) + \phi(\delta)f(A^i x - A^{i+1}x) + \varphi(\delta)f(A^{i+1}x - A^{i+2}x)$$
$$\leq f(A^i x - A^{i+1}x)(\psi(\delta) + \phi(\delta) + \varphi(\delta)). \tag{3.27}$$

In view of (3.27),

$$f(A^{n_0-1}x - A^{n_0}x) \leq f(x - Ax)(\psi(\delta) + \phi(\delta) + \varphi(\delta))^{n_0-1}. \tag{3.28}$$

By (3.19), (3.20) and (3.25),

$$f(x - Ax) \leq M_1. \tag{3.29}$$

It follows from (3.26), (3.28) and (3.29) that

$$\delta < f(A^{n_0-1}x - A^{n_0}x) \leq (\psi(\delta) + \phi(\delta) + \varphi(\delta))^{n_0-1}M_1.$$

This, however, contradicts (3.24). The contradiction we have reached shows that there indeed exists an integer $j \in \{0, \ldots, n_0\}$ such that

$$f(A^j x - A^{j+1}x) \leq \delta.$$

When combined with (3.17) this implies that

$$f(A^i x - A^{i+1}x) \leq \delta \text{ for all integers } i \geq n_0. \tag{3.30}$$

By (3.23) and (3.30),

$$\|A^i x - A^{i+1}x\| \leq \delta_1 \text{ for all integers } i \geq n_0. \tag{3.31}$$

Let the integers m_1 and m_2 satisfy

$$n_0 \leq m_1 < m_2.$$

We claim that

$$f(A^{m_1}x - A^{m_2}x) \leq \delta_0.$$

Suppose to the contrary that

$$f(A^{m_1}x - A^{m_2}x) > \delta_0. \tag{3.32}$$

By (2.3), (3.30) and (3.32), we have

$$f(A^{m_1+1}x - A^{m_2+1}x)$$
$$\leq \psi(f(A^{m_1}x - A^{m_2}x))f(A^{m_1}x - A^{m_2}x)$$
$$+ \phi(f(A^{m_1}x - A^{m_1+1}x))f(A^{m_1}x - A^{m_1+1}x)$$
$$+ \varphi(f(A^{m_2}x - A^{m_2+1}x))f(A^{m_2}x - A^{m_2+1}x)$$
$$\leq \psi(f(A^{m_1}x - A^{m_2}x))f(A^{m_1}x - A^{m_2}x) + 2\delta$$
$$\leq \psi(\delta_0)f(A^{m_1}x - A^{m_2}x) + 2\delta.$$

Using the above relation, (3.32) and the choice of δ, we obtain

$$f(A^{m_1}x - A^{m_2}x) - f(A^{m_1+1}x - A^{m_2+1}x)$$
$$\geq (1 - \psi(\delta_0))f(A^{m_1}x - A^{m_2}x) - 2\delta \geq (1 - \psi(\delta_0))\delta_0 - 2\delta \geq 2^{-1}\delta_0(1 - \psi(\delta_0)). \tag{3.33}$$

In view of (3.23) and (3.31),

$$\|A^{m_1}x - A^{m_1+1}x\| \leq \delta_1, \ \|A^{m_2}x - A^{m_2+1}x\| \leq \delta_1. \tag{3.34}$$

It now follows from (3.22) and (3.34) that

$$|f(A^{m_1}x - A^{m_2}x) - f(A^{m_1+1}x - A^{m_2+1}x)| \leq 4^{-1}\delta_0(1 - \psi(\delta_0)).$$

This, however, contradicts (3.33). The contradiction we have reached proves that indeed we have

$$f(A^{m_1}x - A^{m_2}x) \leq \delta_0. \tag{3.35}$$

By (3.21) and (3.35),

$$\|A^{m_1}x - A^{m_2}x\| \leq \epsilon/2$$

for all pairs of integers $m_1, m_2 \geq n_0$. When combined with (3.18) this implies that

$$\|A^m x - x_A\| \leq \epsilon$$

for all integers $m \geq n_0$. This completes the proof of Theorem 3.3. □

We now present the following corollary of Theorems 2.2 and 3.3. It concerns nonlinear contractive mappings of Kannan [4] type.

Corollary 3.1. *Let $f \in \mathcal{F}$ have property (P3), let $\phi, \varphi \in \Psi$, and let $A : K \to K$ be a continuous mapping such that*

(1) $\phi(t) + \varphi(t) < 1$ for all $t > 0$,
(2) $f(Ax - Ay) \leq \psi(f(x - Ax))f(x - Ax) + \varphi(f(y - Az))f(y - Ay)$ for all $x, y \in K$.

Then the mapping A has a unique fixed point $x_A \in K$ and $A^n x \to x_A$ as $n \to \infty$ for all $x \in K$, uniformly on bounded subsets of K.

4. Some remarks and examples

Remark 4.1. Note that Theorem 1.1 follows from Theorems 2.2 and 3.3 by setting $\psi(t) = 0$ and $\phi(t) = 0$ for all $t \in [0, \infty)$.

Remark 4.2. It is not difficult to see that, as a matter of fact, in Theorem 1.1 the continuity of the mapping A follows from the other assumptions.

Remark 4.3. A comparison of various definitions of contractive mappings can be found in [2, 9].

Remark 4.4. In the proof of Theorem 2.2 we use the p-continuity of the mapping A. Note that in the case where the mapping A satisfies condition (1.1) instead of condition (2.3), we have

$$f(A^n x - Ax_A) \leq \psi(f(A^{n-1}x - x_A))f(A^{n-1}x - x_A). \tag{4.36}$$

Therefore, in view of (2.15), we may conclude that

$$f(A^n x - Ax_A) \to 0 \text{ as } \to \infty. \tag{4.37}$$

Now property (P1) implies that

$$\|A^n x - Ax_A\| \to 0 \text{ as } n \to \infty \tag{4.38}$$

and so we have

$$\|Ax_A - x_A\| \leq \|Ax_A - A^n x_A\| + \|A^n x_A - x_A\|, \tag{4.39}$$

which implies that $Ax_A = x_A$. Therefore the condition that the mapping A is continuous is superfluous.

Example 4.1. Let $X = \mathbb{R}, K = [0,1], Ax = \frac{1}{16}x, \psi(t) = \frac{1}{2}e^{-t}, f(x) = |x|$. In this case the assumptions of Theorem 1.1 are fulfilled.

Example 4.2. Let $X = \mathbb{R}, K = [0, 4]$ and let $f(x) := x^2$ for all $x \in X$. Define a mapping $A : K \to K$ by

$$Ax := \begin{cases} 1, & x \in [0, 3], \\ 0, & x \in (3, 4]. \end{cases}$$

Then the following two statements hold true.
(a) The mapping A does not satisfy condition (1.1) and we cannot apply Theorem 1.1.
(b) The mapping A does satisfy condition (2.3) and therefore we can apply Theorems 2.2 and 3.3.

Indeed, since $f(A3 - A4) = f(3 - 4)$, the mapping A is not a nonlinear contractive mapping for which (1.1) holds. However, it is not difficult to see that

$$f(Ax - Ay) \leq \phi(f(x - Ax))f(x - Ax) + \varphi(f(y - Ay))f(y - Ay)$$

for all $x, y \in K$, where

$$\phi(t) = \varphi(t) := \begin{cases} \frac{1}{3}, & t \in [0, 3], \\ \frac{1}{t}, & t \in (3, \infty). \end{cases}$$

Example 4.3. Let $X = \mathbb{R}, K = [0, 4]$ and let $f(x) := x^2$ for all $x \in X$. Define a mapping $A : K \to K$ by

$$Ax := \begin{cases} 3, & x \in [0, 3], \\ 0, & x \in (3, 4]. \end{cases}$$

Now let

$$\phi(t) = \varphi(t) := \begin{cases} \frac{5}{6}, & t \in [0, 3], \\ \frac{1}{t}, & t \in (3, \infty). \end{cases}$$

Then the mapping A does not satisfy Kannan's contractive condition, but it does satisfy the assumptions of Theorems 2.2 and 3.3.

Remark 4.5. Regarding the relationship between continuity and p-continuity, see Examples 1.2–1.5 on page 3502 of [5].

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