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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

Frum-Ketkov operators which are weakly Picard

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ABSTRACT. Let (M, d) be a metric space, $X \subset M$ be a nonempty closed subset and $K \subset M$ be a nonempty compact subset. By definition, a continuous operator $f : X \to X$ is said to be a Frum-Ketkov operator if there exists $l \in]0, 1[$ such that $d(f(x), K) \leq ld(x, K)$, for every $x \in X$. In this paper, we will give sufficient conditions ensuring that a Frum-Ketkov operator is weakly Picard. Some generalized Frum-Ketkov operators are also studied.

1. INTRODUCTION

Let $(B, \|\cdot\|)$ be a Banach space, $K \subset B$ be a nonempty compact subset and $f : \tilde{B}(0; 1) \to \tilde{B}(0; 1)$ be a continuous operator, where $\tilde{B}(0; 1)$ denotes the closed unit ball in B. We will denote by $d_{\|\cdot\|}(x, K)$ the gap between a point $x \in X$ and K, generated by the norm of the space B. In [7] the following condition is assumed on f: there exists $l \in [0, 1[$ such that

$$d_{\|\cdot\|}(f(x), K) \leq ld_{\|\cdot\|}(x, K), \text{ for every } x \in \tilde{B}(0; 1). \tag{1.1}$$

The fixed point theory for this class of operators was considered by many authors, see [2], [3], [4], [7], [12], [14], [15], [25].

In this paper, we will consider a similar class of operators in a metric space. More precisely, if (M, d) is a metric space, $X \subset M$ is a nonempty closed subset and $K \subset M$ is a nonempty compact subset, then a continuous operator $f : X \to X$ is said to be a Frum-Ketkov operator if there exists $l \in]0, 1[$ such that

$$d(f(x), K) \leq ld(x, K), \text{ for every } x \in X. \tag{1.2}$$

Let us recall that if (M, d) is a metric space and $X \subset M$ is a nonempty closed subset, then $f : X \to X$ is called weakly Picard operator (WPO if the sequence of successive approximations, $\{f^n(x)\}_{n\in\mathbb{N}}$, converges for all $x \in X$ and its limit (which generally depend on x) is a fixed point of f. If f is WPO with a unique fixed point, i.e., $F_f = \{x^*\}$, then, by definition, f is called a Picard operator (PO), see [20], [24].

The purpose of this paper is to give conditions ensuring that a Frum-Ketkov operator is weakly Picard. The structure of this work is the following: 1. Introduction, 2. Preliminaries, 3. Frum-Ketkov operators in metric spaces, 4. Frum-Ketkov operators in terms of a fixed point structure, 5. Generalized Frum-Ketkov operators, 6. Non-self Frum-Ketkov operators, 7. Buley pairs.

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2. PRELIMINARIES

Let (M,d) be a metric space. Then, we denote by $P(M)$ the family of all nonempty subsets of M, by $P_{cl}(M)$ the family of all nonempty closed subsets of M and by $P_{cp}(M)$ the family of all nonempty compact subsets of M. For $x_0 \in M$ and $R > 0$, the symbol $\tilde{B}(x_0;R)$ denotes the closed ball centered in x_0 with radius R. The space of all continuous operators $f : M \to M$ is denoted by $C(M,M)$, while we define the ω-limit set of $x \in M$ under f as

$$\omega_f(x) := \bigcap_{n=0}^{\infty} \overline{\{f^k(x) : k \geq n\}},$$

where f^k is the iterate of order k of f. Notice that an useful characterization of the ω-limit set is

$$\omega_f(x) = \{x^* \in M : \text{ there exists } n_k \to \infty \text{ such that } f^{n_k}(x) \to x^*\}.$$

For some considerations on the set $\omega_f(x)$ see [5], [10], [11], [15], [24].

In the context of a Banach space $(B, \|\cdot\|)$, the symbol $P_{cv}(B)$ denotes the family of all nonempty convex subsets of B, while $P_{cp,cv}(B) := P_{cp}(B) \cap P_{cv}(B)$.

Definition 2.1. Let (M,d) be a metric space. Then $f : M \to M$ is called:

(a) l-contraction if $l \in]0,1[$ and $d(f(x),f(y)) \leq ld(x,y)$, for every $x,y \in M$;

(b) contractive if $d(f(x),f(y)) < d(x,y)$, for every $x,y \in M$ with $x \neq y$;

(c) nonexpansive if $d(f(x),f(y)) \leq d(x,y)$, for every $x,y \in M$;

(d) quasinonexpansive if $F_f \neq \emptyset$ and, if $x^* \in F_f$ then $d(f(x),x^*) \leq d(x,x^*)$, for every $x \in M$.

If $f : M \to M$ is a weakly Picard operator, then we can define the operator

$$f^\infty : M \to M, \text{ given by } f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

In the case of a Picard operator with $F_f = \{x^*\}$, we have that $f^\infty(x) = x^*$, for every $x \in M$. For related notions and results concerning generalized contractions and the theory of WPO see [1], [16], [20], [23], [24], ...

Finally, we will recall the concept of fixed point structure. Let X be a nonempty set, $\emptyset \neq S(X) \subset P(X)$ and $M_1(Y) \subset \mathbb{M}(Y,Y) := \{f : Y \to Y : f \text{ is an operator }\}$, where $Y \subset X$. Then, the triple $(X, S(X), M_1)$ is a fixed point structure if

$$Y \in S(X) \text{ and } f \in M_1(Y) \Rightarrow F_f \neq \emptyset.$$

For examples of fixed point structures see [18], [19].

3. FRUM-KETKOV OPERATORS IN METRIC SPACES

Let (M,d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Then, a continuous operator $f : X \to X$ is said to be a Frum-Ketkov (l,K)-operator if $l \in]0,1[$ and

$$d(f(x),K) \leq ld(x,K), \text{ for every } x \in X. \tag{3.3}$$

We will present first some examples and remarks on this class of operators.

Remark 3.1. (1) If $X \subset K$, then each continuous operator $f : X \to X$ is a Frum-Ketkov (l,K)-operator. In this case, the fixed point theory for Frum-Ketkov operators is the fixed point theory of continuous operators on compact metric spaces. Thus, Frum-Ketkov condition is effective if $X \neq K$. If M is an infinite dimensional Banach space and $X := \tilde{B}(x_0;R)$ (where $x_0 \in M$ and $R > 0$), then $X \neq K$.

(2) Let $X \in P_{cl}(M)$ and $f : X \to X$ be an l-contraction. Let x^* be the unique fixed point of f. Let $K := \{x^*\}$. Then f is a Frum-Ketkov (l,K)-operator.

(3) Let $M := \mathbb{R}^2$, $d := d_{\|\cdot\|_2}$, $X = [0,1] \times [0,1]$ and $K := [0,1] \times \{0\}$. Let $f := (f_1, f_2) : X \to X$ be a continuous mapping. Then, we have:

(i) $d(f(x_1,x_2),K) = f_2(x_1,x_2)$;

(ii) $d((x_1,x_2),K) = x_2$.

Thus, f is a Frum-Ketkov (l,K)-operator if and only if $f_2(x_1,x_2) \leq lx_2$, for every $(x_1,x_2) \in X$.

For example, we can take $f_1(x_1,x_2) = f_1(x_1)$ a continuous mapping and $f_2(x_1,x_2) = \frac{1}{2}x_2$. Then $f(x_1,x_2) := \left(f_1(x_1), \frac{1}{2}x_2\right)$ is a Frum-Ketkov $\left(\frac{1}{2}, K\right)$-operator.

Moreover, in particular, if we consider $f(x_1,x_2) := \left(1-x_1, \frac{1}{2}x_2\right)$, then $F_f = \left\{\left(\frac{1}{2}, 0\right)\right\}$, $f^{2n}(x_1,x_2) = \left(x_1, \frac{1}{2^n}x_2\right)$ and $f^{2n+1}(x_1,x_2) = \left(1-x_1, \frac{1}{2^n}x_2\right)$, for $n \in \mathbb{N}$. It is clear that, if $x_1 \neq \frac{1}{2}$, then f is not asymptotically regular at (x_1,x_2).

(4) Let $M := \mathbb{R}^2$, $d := d_{\|\cdot\|_2}$, $X = \mathbb{R}^2 \backslash B(0;1)$ and $K := \tilde{B}(0;1)$. Let $f := (f_1,f_2) : X \to X$ defined by $f(x) = \frac{1}{2}A\left(x + \frac{x}{\|x\|_2}\right)$, where A is the rotation matrix of an angle $\alpha \in \left]0, \frac{\pi}{2}\right[$, i.e.,

$$A := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

It is clear that $F_f = \emptyset$ and we have

$$\|f(x)\|_2 = \left\|\frac{1}{2}\left(x + \frac{x}{\|x\|_2}\right)\right\|_2 = \frac{1}{2}\|x\|_2 + \frac{1}{2}$$

and

$$d(f(x),K) = \|f(x)\|_2 - 1 = \frac{1}{2}(\|x\|_2 - 1) = \frac{1}{2}d(x,K).$$

Hence f is a Frum-Ketkov $\left(\frac{1}{2}, K\right)$-operator. It easy to see that

$$\begin{aligned} \|f^n(x)\|_2 &= \left(\frac{1}{2}\right)^n \|x\|_2 + \left(\frac{1}{2}\right)^n + \ldots + \frac{1}{2} = \\ &= \left(\frac{1}{2}\right)^n \|x\|_2 + 1 - \left(\frac{1}{2}\right)^n \to 1 \text{ as } n \to +\infty \end{aligned}$$

and

$$\|f^{n+1}(x) - f^n(x)\|_2 = \sqrt{\|f^{n+1}(x)\|_2^2 + \|f^n(x)\|_2^2 - 2\|f^{n+1}(x)\|_2\|f^n(x)\|_2\cos\alpha}.$$

Therefore

$$\|f^{n+1}(x) - f^n(x)\|_2 \to \sqrt{2 - 2\cos\alpha} > 0,$$

proving that f is not asymptotically regular.

(5) Let $f : X \to X$ be a Frum-Ketkov (l,K)-operator. Let $g : X \to K$ be an operator such that $d(f(x),K) = d(f(x),g(x))$, for every $x \in X$. Then, $\overline{g(X)} \in P_{cp}(X)$ and $d(f(x),g(x)) \leq l d(x,g(x))$, for every $x \in X$. Moreover, if f is g-asymptotically regular, then f is asymptotically regular. Indeed, for every $x \in X$, we have:

$$d(f^{n+1}(x), f^n(x)) \leq d(f^{n+1}(x), g(f^n(x))) + d(f^n(x), g(f^n(x))) \leq$$

$$(l+1)d(f^n(x), g(f^n(x))) \to 0, \text{ as } n \to \infty.$$

(6) Let $f : X \to X$ be a Frum-Ketkov (l, K)-operator. Suppose there exists a continuous operator $\rho : M \to K$ such that $d(x, K) = d(x, \rho(x))$, for every $x \in X$. Then, we have

$$d(f(x), \rho(f(x))) \leq ld(x, \rho(x)), \text{ for each } x \in X.$$

This implies that

$$d(f^n(x), \rho(f^n(x))) \leq l^n d(x, \rho(x)) \to 0, \text{ as } n \to \infty, \text{ for each } x \in X.$$

As a consequence, f is ρ-asymptotically regular.

The main result of this section is the following theorem.

Theorem 3.1. *Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Let $f : X \to X$ be a Frum-Ketkov (l, K)-operator. Then, the following conclusions hold:*

(i) $\omega_f(x) \neq \emptyset$ and $\omega_f(x) \in X \cap K$, for every $x \in X$;

(ii) $F_f \subset X \cap K$;

(iii) $f(X \cap K) \subset X \cap K$;

(iv) if f is asymptotically regular, then $\omega_f(x) \subset F_f$, for every $x \in X$ and, as a consequence, $F_f \neq \emptyset$;

(v) if, in addition, f is quasinonexpansive, then f is WPO.

Proof. (i) Let $x \in X$ be arbitrary chosen. Since K is compact, there exists $y_n \in K$ such that $d(f^n(x), K) = d(f^n(x), y_n)$. By Frum-Ketkov condition, we obtain that $d(f^n(x), K) \to 0$ as $n \to \infty$. Using again the compactness of K, we can find a subsequence y_{n_i} which converges to an element $y^*(x) \in K$ as $n_i \to \infty$. As a consequence, $f^{n_i}(x)$ converges to $y^*(x) \in X \cap K$ as $n_i \to \infty$.

(ii) Let $x \in F_f$. Then $d(x, K) = d(f(x), K) \leq ld(x, K)$. Since $l < 1$ we get that $d(x, K) = 0$, showing that $x \in K$.

(iii) If $x \in X \cap K$, then $d(f(x), K) \leq ld(x, K) = 0$. Thus, $f(x) \in X \cap K$.

(iv) If f is asymptotically regular and continuous, then $\omega_f(x) \subset F_f$, for each $x \in X$. Indeed, for $x \in X$, let $x^* \in \omega_f(x)$. Then, there exists $n_i \to \infty$ such that $f^{n_i}(x) \to x^*$. By the asymptotically regularity of f, using the continuity assumption, we get

$$d(x^*, f(x^*)) \leq d(x^*, f^{n_i}(x)) + d(f^{n_i+1}(x), f^{n_i}(x)) + d(f(f^{n_i}(x)), f(x^*)) \to 0,$$

as $n_i \to \infty$. Thus $x^* \in F_f$.

(v) Let $x \in X$ and $f^{n_i}(x) \to y^*(x)$ (see (i)). By the quasinonexpansivity assumption on f, we get that the sequence $(d(f^n(x), y^*(x)))_{n \in \mathbb{N}}$ is decreasing. Hence, it is convergent to an element $d \in [0, \infty[$. Since the subsequence $d(f^{n_i}(x), y^*(x))$ converges to 0, it follows that $d = 0$. □

Remark 3.2. Instead of (iv) in the above theorem, we can consider each of the following assumptions:

(iv′) (see [22]) there exist $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta : X \to \mathbb{R}_+$ such that:

(a) $t_n \in \mathbb{R}_+$, $\alpha(t_n) \to 0$ as $n \to \infty \Rightarrow t_n \to 0$ as $n \to \infty$;

(b) $\alpha(d(x, f(x)) \leq \beta(x) - \beta(f(x))$, for each $x \in X$.

(iv″) (see Remark 3.1 (5)) there exists $k > 0$ and $\rho \in C(M, K)$ such that

$$d(x, K) = d(x, \rho(x)) \text{ and } d(x, f(x)) \leq kd(x, \rho(x)), \text{ for each } x \in X.$$

4. Frum-Ketkov operators in terms of a fixed point structure

Let $f : X \to X$ be a Frum-Ketkov operator. By Theorem 3.1 we have that $X \cap K \neq \emptyset$, $f(X \cap K) \subset X \cap K$ and $F_f \subset X \cap K$.

Consider now $(M, S(M), M_1)$ be a fixed point structure. If $X \cap K \in S(M)$ and the restriction of f to $X \cap K$ belongs to $M_1(X \cap K)$, then $F_f \neq \emptyset$. Thus, we can prove the following result.

Theorem 4.2. *Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Suppose that $f : X \to X$ is a contractive Frum-Ketkov operator. Then $f|_{X\cap K}$ is a Picard operator.*

Proof. We consider on M the fixed point structure $(M, P_{cp}(M), M_1)$, where for $Y \subset M$ we define

$$M_1(Y) := \{g : Y \to Y : \text{ g is a contractive operator}\}.$$

Since $X \cap M \in P_{cp}(M)$ and $f|_{X\cap K} \in M_1(X \cap M)$, the conclusion follows by the definition of a fixed point structure. □

Another result of this type is the following.

Theorem 4.3. *Let $(B, \|\cdot\|)$ be a Banach space, $X \in P_{cl,cv}(B)$ and $K \in P_{cp,cv}(B)$. Suppose that $f : X \to X$ is a Frum-Ketkov operator. Then $F_f \neq \emptyset$.*

Proof. The conclusion follows using Schauder's fixed point structure, see [18]. □

Let $(B, \|\cdot\|)$ be a Banach space, $K \in P_{cp,cv}(B)$ and $f \in C^1(B, B)$ be a Frum-Ketkov operator. The problem is in which conditions f is a PO ? (see [21]).

5. Generalized Frum-Ketkov operators

We present first the notion of generalized Frum-Ketkov operator.

Definition 5.2. Let (M, d) be a metric space, $X \in P_{cl}(M)$ and let $K \in P_{cp}(M)$. Then, $f : X \to X$ is generalized Frum-Ketkov operator if f is continuous and $d(f^n(x), K) \to 0$ as $n \to \infty$, for every $x \in X$.

Example 5.1. Let (M, d) be a metric space, $X \in P_{cl}(M)$, $K \in P_{cp}(M)$. Let $f : X \to X$ be a continuous operator. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function (i.e., φ is increasing and the sequence $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, for every $t > 0$) and

$$d(f(x), K) \leq \varphi(d(x, K)), \text{ for every } x \in X,$$

then f is a generalized Frum-Ketkov operator. In this case, f is called a Frum-Ketkov φ-operator.

Example 5.2. Let (M, d) be a metric space, $X \in P_{cl}(M)$, $K \in P_{cp}(M)$. Let $f : X \to X$ be a continuous operator such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, K) < \varepsilon + \delta \;\Rightarrow\; d(f(x), K) < \varepsilon.$$

Then f is a generalized Frum-Ketkov operator. In this case f is called a Frum-Ketkov MK-operator.
Indeed, by the above Meir-Keeler type condition, we get that $d(f^n(x), K) \to 0$ as $n \to \infty$, for every $x \in X$. (Proof. Since $d_n := d(f^n(x), K)$ is decreasing, we can suppose, by contradiction, that $d_n := d(f^n(x), K) \searrow \varepsilon > 0$ as $n \to \infty$. Assuming, for some $m \in \mathbb{N}^*$, that $d_m < \varepsilon + \delta$, we get, by the definition of f, that $d_{m+1} < \varepsilon$, which gives the desired contradiction.)

Example 5.3. Let (M, d) be a metric space, $X \in P_{cl}(M)$, $K \in P_{cp}(M)$. Let $f : X \to X$ be a continuous operator for which there exists $\varphi : X \to \mathbb{R}_+$ such that

$$d(f(x), K) \leq \varphi(x) - \varphi(f(x)), \text{ for all } x \in X.$$

Then f is a generalized Frum-Ketkov operator.

The main result for this section is the following theorem.

Theorem 5.4. *Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Suppose that $f : X \to X$ is a generalized Frum-Ketkov operator. Then $f|_{X \cap K}$ is a Picard operator..*

Proof. The proof follows the ideas and the approach given in Theorem 4.2. □

6. NON-SELF FRUM-KETKOV OPERATORS

Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$ such that $X \setminus K \neq \emptyset$. Suppose that $f : X \to M$ is a Frum-Ketkov (l, K)-operator. We also suppose that there exists a continuous retraction $r : M \to X$ such that:

(1) $F_f = F_{r \circ f}$, i.e., f is retractible with respect to r (see [19] and the references therein);

(2) there exists $c > 0$ with $cl < 1$ such that $d(r(x), K) \leq cd(x, K)$, for every $x \in M \setminus X$

Then, the $r \circ f$ is a self operator on X and it is a Frum-Ketkov (cl, K)-operator. As a consequence, by the results presented in Section 3, we obtain the following results.

Theorem 6.5. *Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$ such that $X \setminus K \neq \emptyset$. Suppose that $f : X \to M$ is a Frum-Ketkov (l, K)-operator. We also suppose that there exists a continuous retraction $r : M \to X$ such that:*

(1) $F_f = F_{r \circ f}$;

(2) there exists $c > 0$ with $cl < 1$ such that $d(r(x), K) \leq cd(x, K)$, for every $x \in M \setminus X$.

Then, the following conclusions hold:

(i) $\omega_{r \circ f}(x) \neq \emptyset$ and $\omega_{r \circ f}(x) \in X \cap K$, for every $x \in X$;

(ii) $F_f \subset X \cap K$;

(iii) $(r \circ f)(X \cap K) \subset X \cap K$;

(iv) if $r \circ f$ is asymptotically regular, then $\omega_{r \circ f}(x) \subset F_f$, for every $x \in X$ and, as a consequence, $F_f \neq \emptyset$;

(v) if, in addition, $r \circ f$ is quasinonexpansive, then $r \circ f$ is WPO.

Example 6.4. Let $M := \mathbb{R}^m$, $d := d_{\|\cdot\|_2}$, $K := \tilde{B}(0; 1)$ and $X := \tilde{B}(0; R) \setminus B(0; 1)$, with $R > 1$. Let $f : X \to M$ be a Frum-Ketkov (l, K)-operator and $r : M \to X$ be the radial retraction. In this case, we have that

$$d(r(x), K) \leq d(x, K), \text{ for every } x \in M \setminus K.$$

As a consequence, by the above considerations, we get that $r \circ f : X \to X$ is a Frum-Ketkov (l, K)-operator.

7. BULEY PAIRS

Let (M, d) be a metric space and $X \in P_{cl}(M)$. By definition (see [7]), the operators $f, g : X \to M$ form a Buley pair (f, g) (or an l-Buley pair) if the following conditions are satisfied:

(1) f is continuous;

(2) the set $\overline{g(X)}$ is compact;

(3) there exists $l \in]0, 1[$ such that $d(f(x), g(x)) \leq ld(x, g(x))$, for each $x \in X$.

Notice that, if (f, g) is a Buley pair, then $F_f = F_g$.

For a better understanding of the relations between the following conditions:
(FK) f is a Frum-Ketkov operator;
(B) there exists an operator g such that (f, g) is a Buley pair;
we give the following example.

Example 7.5. Let $M := \mathbb{R}^2$, $d := d_{\|\cdot\|_2}$, $X := \tilde{B}(0;1)$, $K := [-1,1] \times \{0\}$, $f \in C(X, X)$ and $g : X \to \mathbb{R}^2$ be given by $g(x_1, x_2) := (f_1(x_1, x_2), 0)$.
We notice that:
(a) if there exists $l \in]0,1[$ such that $|f_2(x_1, x_2)| \leq l|x_2|$ for all $(x_1, x_2) \in X$, then f is a Frum-Ketkov (l, K)-operator.
(b) if there exists $l \in]0,1[$ such that $|f_2(x_1, x_2)|^2 \leq l^2 \left[|x_1 - f_1(x_1, x_2)|^2 + x_2^2\right]$ for all (x_1, x_2), then (f, g) is an l-Buley pair.

It is obvious from the above example that Buley condition on f is less restrictive than Frum-Ketkov condition.
In [4] the following theorem was given.

Theorem 7.6. *Let E be a normed linear space and X be a contractible set, where*

$$X \in \{Y \in P(E) : \text{ there are } n \in \mathbb{N}^* \text{ and } C_1, \cdots, C_n \in P_{cl,cv}(E) \text{ such that } Y = \bigcup_{i=1}^{n} C_i\}.$$

Let $f : X \to E$ be continuous, such that $f(\partial X) \subset X$ and $g : X \to E$ be such that (f, g) is an l-Buley pair. Then $F_f \neq \emptyset$.

The problem is, if $f(X) \subset X$, in which conditions f is WPO.

Concerning this problem we have the following result.

Theorem 7.7. *Let (M, d) be a metric space and $X \in P_{cl}(M)$. Let $f, g : X \to M$ such that (f, g) is an l-Buley pair and $f(X) \subset X$. We suppose:*
(1) f is asymptotically g-regular;
(2) f is nonexpansive.
Then f is a WPO.

Proof. For each $x \in X$, we have

$$d(f^{n+1}(x), g(f^n(x))) \leq d(f^n(x), g(f^n(x))),$$

and

$$d(f^{n+1}(x), f^n(x)) \leq (l+1)d(f^n(x), g(f^n(x))).$$

Thus, by (1), we get that f is asymptotically regular. Since f is g-asymptotically regular and $g(f^n(x)) \in \overline{g(X)} \in P_{cp}(M)$, there exists $n_i \to \infty$ as $i \to \infty$, such that

$$g(f^{n_i}(x)) \to y^*(x) \in \overline{g(X)} \text{ and } f^{n_i}(x) \to y^*(x) \in X, \text{ as } i \to \infty.$$

Since f is asymptotically regular, we have that $y^*(x) \in F_f$. Since f is nonexpansive all the sequences $\{f^n(x)\}_{n \in \mathbb{N}}$ converge to $y^*(x)$. Thus, f is WPO. □

Remark 7.3. Instead of (2) we can consider the following condition
(2′) f is conditionally quasinonexpansive.

Remark 7.4. If. in Theorem 7.7. instead of (2) we consider the following condition
(2″) f is contractive,
then f is a PO.

Remark 7.5. In the case of a nonself operator $f : X \to M$, if we suppose that there exists a continuous retraction $r : M \to X$ such that f is retractible with respect to r, then $r \circ f : X \to X$ and $F_f = F_{r \circ f}$. In this case the problem is in which conditions $r \circ f$ is a WPO.

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