CARPATHIAN J. MATH. **Online version at** `https://www.carpathian.cunbm.utcluj.ro`
36 (2020), No. 2, 303 - 312 **Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401**

Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

On $(\psi, \varphi)^2$- contractive maps

RAKESH TIWARI[1], MOHAMMAD SAEED KHAN[2], SHOBHA RANI[1] and VLADIMIR RAKOČEVIĆ[3]

ABSTRACT. In this paper, we introduce the notion of generalized weakly contractive mapping of quadratic type and we prove fixed point results for this type of mapping. We justify our result by suitable examples and show that this mapping satisfies properties P and Q. Among other things, as corollaries we recover Banach and Kannan fixed point theorems.

1. INTRODUCTION AND PRELIMINARIES

Let X be a set. We will denote the set all fixed points of a self mapping T from X into itself by $F(T)$, i.e., $F(T) = \{z \in X : Tz = z\}$.

It is obvious that if z is a fixed point of T then it is also a fixed point of T^n for each $n \in N$, i. e.,$F(T) \subset F(T^n)$ if $F(T) \neq \emptyset$. However converse is false.

Definition 1.1. ([16]) Let T be a self mapping of metric space with fixed point set $F(T) \neq \varnothing$. Then T is said to have property P if $F(T^n) = F(T)$, for each $n \in N$. Equivalently, a mapping has property P if every periodic point is a fixed point.

Also a pair of maps S and T have property Q if $F(S) \bigcap F(T) = F(S^n) \bigcap F(T^n)$ for each $n \in \mathbb{N}$.

Famous Banach fixed point theorem, usually called the Banach contraction principle, from 1922 marks the beginning of the fixed point theory in metric spaces.

Definition 1.2. Let (X, d) be a metric space. A mapping $f : X \to X$ is a *contraction* if there exists some $q \in [0, 1)$ such that

$$d(f(x), f(y)) \leq q \cdot d(x, y), \quad \text{for all } x, y \in X. \tag{1.1}$$

We shall also use the term *q–contraction* for contraction. ß

We observe that every contraction is a continuous mapping. The following theorem shows the existence and uniqueness of a fixed point of an arbitrary contraction on a complete metric space.

Theorem 1.1 (Banach contraction principle, [4]). *If (X, d) is a complete metric space and $f : X \to X$ is a contraction, then the mapping f has a unique fixed point $z \in X$.*

In 1968, Kannan's [18] proved the following famous fixed point theorem.

Received: 17.03.2020. In revised form: 18.05.2020. Accepted: 25.05.2020
2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. *Contractive mapping, weak contraction, fixed point.*
Corresponding author: Vladimir Rakočević; vrakoc@sbb.rs

Theorem 1.2. *If (X,d) is a complete metric, $0 \leq q < 1/2$ and $f : X \to X$ is a map which satisfies the condition*

$$d(f(x), f(y)) \leq q[d(x, f(x)) + d(y, f(y))], \quad \textit{for all } x, y \in X, \tag{1.2}$$

then the mapping f has a unique fixed point.

In 1972, Chatterjee [9] proved the following variant of Kannan's [18] fixed point theorem.

Theorem 1.3. *If (X,d) is a complete metric, $0 \leq q < 1/2$ and $f : X \to X$ is a map which satisfies the condition*

$$d(f(x), f(y)) \leq q[d(x, f(y)) + d(y, f(x))], \quad \textit{for all } x, y \in X, \tag{1.3}$$

then the mapping f has a unique fixed point.

For more details on the advances of fixed point theory can be found in [1, 5, 20, 24, 26, 27]. In 1977, Alber et al. [2] generalized Banachs contraction principle by introducing the concept of weak contraction mappings in Hilbert spaces. Weak contraction principle states that every weak contraction mapping on a complete Hilbert space has a unique fixed point. Rhoades [25] extended weak contraction principle in Hilbert spaces to metric spaces. Since then, many authors (for example, [3, 10, 12, 21, 22]) obtained generalizations and extensions of the weak contraction principle. Khan et al. [19] obtained fixed point theorems in metric spaces by introducing the concept of altering distance mappings. In particular, Choudhury et al.[11] obtained a generalization of the weak contraction principle in metric spaces by using altering distance mappings.

Definition 1.3. [19] A mapping $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance mapping if the following properties are satisfied
(a) ψ is strictly monotone increasing and continuous
(b) $\psi(t) = 0$ if and only if $t = 0$.
We say that a mapping $\varphi : [0,\infty) \to [0,\infty)$ is almost altering distance if φ is lower continuous and satisfies the condition (b).

Inspired on the work on weak contraction mappings and generalized weak contraction mappings, in this paper we introduce and study generalized weakly contractive mapping of quadratic type. We prove fixed point results for this type of mapping, justify our result by suitable examples and show that this mapping satisfies properties P and Q. Among other things, as corollaries we recover Banach and Kannan fixed point theorems.

Definition 1.4. Let the mapping $T : X \to X$ in metric space (X,d). Then T is called generalized weakly contractive mapping of quadratic type, shortly $(\psi,\varphi)^2$- contractive map, if

$$\psi(d^2(Tx,Ty)) \leq \psi(P_T(x,y)) - \varphi(Q_T(x,y)), \tag{1.4}$$

for all $x, y \in X$, where

$$P_T(x,y) = \max\left\{ d^2(x,y), d(x,Ty) \cdot d(y,Tx), \frac{d^2(x,Tx) + d^2(y,Ty)}{2} \right\},$$

$$Q_T(x,y) = \max\{d^2(x,y), d(x,Ty) \cdot d(y,Tx), d(x,Tx) \cdot d(y,Ty)\},$$

ψ is an altering distance mapping and φ is an almost altering distance mapping.

Conditions (1.4) is similar to Ciric's (see [7, 8]) contractive conditions that do not ensure unique fixed point.

Example 1.1. If f is a q–contraction, then f is $(\psi,\varphi)^2$- contractive map with $\psi(t)=t$ and $\varphi(t)=(1-q^2)t$. This follows by (1.1), since we have

$$\begin{aligned} d^2(f(x),f(y)) &\leq q^2\cdot d^2(x,y)\leq q^2Q_f(x,y)\\ &= Q_f(x,y)-(1-q^2)Q_f(x,y)\\ &\leq P_f(x,y)-(1-q^2)Q_f(x,y), \end{aligned}$$

for all $x,y\in X$.

Example 1.2. Kannan contraction (1.3) f is $(\psi,\varphi)^2$- contractive map with $\psi(t)=t$ and $\varphi(t)=(1-4q^2)t$. This follows from (1.3), since

$$\begin{aligned} d^2(f(x),f(y)) &\leq q^2\cdot[d(x,f(x))+d(y,f(y))]^2\\ &\leq 2q^2\cdot[d^2(x,f(x))+d^2(y,f(y))]\\ &\leq 4q^2P_f(x,y)=P_f(x,y)-(1-4q^2)P_f(x,y),\\ &\leq P_f(x,y)-(1-4q^2)Q_f(x,y), \end{aligned} \tag{1.5}$$

for all $x,y\in X$.

2. Main results

In this section, we give the main results of our paper.

Theorem 2.4. *Let (X,d) be a complete metric space and T a $(\psi,\varphi)^2$- contractive map. Then T has a unique fixed point $p\in X$, and for any $x\in X$ the sequence of iterates T^nx converges to p.*

Proof. Let $x_0\in X$. We define a sequence $\{x_n\}$ in X such that $x_{n+1}=Tx_n$, for all $n\geq 0$. If there exist a positive integer N such that $x_N=x_{N+1}$, then x_N is a fixed point of T. Hence we shall assume that $x_n\neq x_{n+1}$, for all $n\geq 0$.
From (1.4), we have for all $n\geq 0$,

$$\begin{aligned} \psi(d^2(x_{n+1},x_{n+2})) &= \psi(d^2(Tx_n,Tx_{n+1}))\\ &\leq \psi(P_T(x_n,x_{n+1})-\varphi(Q_T(x_n,x_{n+1})) \end{aligned} \tag{2.6}$$

$$\leq \psi\left(\max\left\{(d^2(x_n,x_{n+1}),\frac{d^2(x_n,x_{n+1})+d^2(x_{n+1},x_{n+2})}{2}\right\}\right)$$
$$-\varphi(\max\{d^2(x_n,x_{n+1}),d(x_n,x_{n+1})\cdot d(x_{n+1},x_{n+2})\}).$$

Suppose that $d(x_n,x_{n+1})\leq d(x_{n+1},x_{n+2})$ for some positive integer n. Then by (2.6) we have

$$\psi(d^2(x_{n+1},x_{n+2}))\leq\psi(d^2(x_{n+1},x_{n+2}))-\varphi(d(x_n,x_{n+1})\cdot d(x_{n+1},x_{n+2})). \tag{2.7}$$

Hence, $\varphi(d(x_n,x_{n+1})\cdot d(x_{n+1},x_{n+2}))\leq 0$, which implies that $d(x_n,x_{n+1})\cdot d(x_{n+1},x_{n+2})=0$ or $x_{n+1}=x_{n+2}$, contradicting our assumption that $x_n\neq x_{n+1}$, for each n. Therefore,

$$d(x_{n+1},x_{n+2})<d(x_n,x_{n+1}),\qquad \text{for all}\quad n\geq 0 \tag{2.8}$$

and $\{(d(x_n,x_{n+1})\}$ is a monotone decreasing sequence of positive real number. Hence there exists an $r\geq 0$ such that

$$\lim_{n\to\infty}d(x_n,x_{n+1})=r. \tag{2.9}$$

From (2.6) and (2.8) we have for all $n\geq 0$,

$$\psi(d^2(x_{n+1},x_{n+2}))\leq\psi(d^2(x_n,x_{n+1}))-\varphi(d^2(x_n,x_{n+1})).$$

Taking the limit as $n \to \infty$ in the above inequality and using the continuities of φ and ψ, we have

$$\psi(r^2) \le \psi(r^2) - \varphi(r^2),$$

that is $\varphi(r^2) \le 0$, which is contradiction unless $r = 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.10}$$

Now we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exists an $\epsilon \in (0,1)$ for which we find two sequences of positive integer $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integer k

$$n(k) > m(k) > k,\ d^2(x_{m(k)}, x_{n(k)}) \ge \epsilon \quad \text{and} \quad d^2(x_{m(k)}, x_{n(k)-1}) < \epsilon.$$

Now

$$\epsilon \le d^2(x_{m(k)}, x_{n(k)}) \le (d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}))^2,$$

that is

$$\epsilon \le d^2(x_{m(k)}, x_{n(k)}) < (\epsilon + d(x_{n(k)-1}, x_{n(k)}))^2.$$

Taking the limit as $k \to \infty$ in the above inequality and using (2.10), we have

$$\epsilon \le \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \le \epsilon^2 < \epsilon, \tag{2.11}$$

which is a contradiction with our assumption on ϵ. Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exist a $p \in X$ such that $x_n \to p$ as $n \to \infty$.
Putting $x = x_n$, $y = p$ in (1.4) we have

$$\begin{aligned} \psi(d^2(x_{n+1}, Tp)) &= \psi(d^2(Tx_n, Tp)) \\ &\le \psi(P(x_n, p)) - \varphi(Q(x_n, p)) \end{aligned}$$

Taking $n \to \infty$ in the above inequality and using (2.10) and the continuities of φ and ψ, we obtain

$$\psi(d^2(p, Tp)) \le \psi(\frac{1}{2} d^2(p, Tp)),$$

which implies that $d^2(p, Tp) = 0$, or $p = Tp$. Hence p is a fixed point of T.

Next we establish that the fixed point is unique. Let p and q be two fixed points of T and suppose that $p \ne q$ Then putting $x = p$ and $y = q$ in (1.4), we have

$$\psi(d^2(p, q)) = \psi(d^2(Tp, Tq)) \quad \le \quad \psi(P(p, q)) - \varphi(Q(p, q)),$$

that is

$$\psi(d^2(p, q)) \le \psi(d^2(p, q)) - \varphi(d^2(p, q)),$$

which is contradiction by virtue of a property of φ. Therefore $p = q$ and the fixed point is unique. $\square$

The following corollary shows a relation between T^n and T in the case when T^n is a $(\psi, \varphi)^2$- contractive.

Corollary 2.1. *Let (X, d) be a complete metric space and let $T : X \to X$ is a mapping such that T^n be a $(\psi, \varphi)^2$- contractive map for some $n \in \mathbb{N}$, then T has a unique fixed point in X.*

Proof. From previous theorem T^n has a unique fixed point, say $p \in X$. Then $T^n p = p$ implies $T^{n+1} p = Tp$ and Tp is also fixed point of T^n. Since T^n has a unique fixed point, we have $p = Tp$. To show the uniqueness fixed point of T, we suppose that there exists $q \in X$ such that $Tq = q$. Now, $T^n q = q$, and since T^n has unique fixed point it must be $q = p$. $\square$

Now we recover the following well known result.

Corollary 2.2. (Bryant [6]) *If (X, d) is a complete metric space and $f : X \to X$ is a mapping such that f^n is a contraction for some $n \in \mathbb{N}$, then f has a unique fixed point in X.*

Proof. By Example 1.5 and Corollary 3.18. □

Corollary 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ is a $(\psi, \varphi)^2$- contractive map. Then T has a property P.*

Proof. By Corollaryy 3.18. □ □

Now we can prove the following common fixed point result.

Theorem 2.5. *Let (X, d) is a complete metric space. Let $S, T : X \to X$ be a self mappings such that, for all $x, y \in X$*

$$\psi(d^2(Sx, Ty)) \leq \psi\left(\max\left\{d^2(x, y), d(x, Ty) \cdot d(y, Sx), \frac{d^2(x, Sx) + d^2(y, Ty)}{2}\right\}\right)$$
$$-\varphi(\max\{d^2(x, y), d(x, Ty) \cdot d(y, Sx), d(x, Sx) \cdot d(y, Ty)\}), \tag{2.12}$$

ψ is an altering distance mapping and φ is an almost altering distance mapping. Then S and T have a unique common fixed point. Moreover, any fixed point of S is a fixed point of T and conversely.

Proof. Let $x_0 \in X$.
We define a sequence$\{x_n\}$ by $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$, $n \geq 0$. If there exist a positive integer n such that $x_{2n} = x_{2n+1}$, then x_{2n} is a fixed point of S and hence a fixed point of T. A similar conclusion holds if $x_{2n+1} = x_{2n+2}$, for some n. Therefore we may assume that $x_n \neq x_{n+1}$, for all $n \geq 0$.
By (2.12)

$$\psi(d^2(x_{2n+1}, x_{2n+2})) = \psi(d^2(Sx_{2n}, Tx_{2n+1}))$$
$$\leq \psi\left(\max\left\{d^2(x_{2n}, x_{2n+1}), \frac{d^2(x_{2n}, x_{2n+1}) + d^2(x_{2n+1}, x_{2n+2})}{2}\right\}\right) \tag{2.13}$$
$$-\phi(\max\{d^2(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}) \cdot d(x_{2n+1}, x_{2n+2})\})).$$

Suppose that $d(x_{2n}, x_{2n+1}) \leq d(x_{2n+1}, x_{2n+2})$ for some positive integer n. Then from (2.13) we have

$$\psi(d^2(x_{2n+1}, x_{2n+2})) \leq \psi(d^2(x_{2n+1}, x_{2n+2})) - \phi(d(x_{2n}, x_{2n+1}) \cdot d(x_{2n+1}, x_{2n+2})),$$

that is $\phi(d(x_{2n}, x_{2n+1}) \cdot d(x_{2n+1}, x_{2n+2})) \leq 0$, which implies that $d^2(x_{2n+1}, x_{2n+2}) = 0$ or $x_{2n+1} = x_{2n+2}$, contradicting our assumption that $x_n \neq x_{n+1}$, for each n.
Therefore $d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1})$, for all $n \geq 0$.
Similarly, $d(x_{2n+2}, x_{2n+3}) < d(x_{2n+1}, x_{2n+2})$, for all $n \geq 0$.
Thus $\{(d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists an $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{2.14}$$

From (2.13) we have for all $n \geq 0$,

$$\psi(d^2(x_{2n+1}, x_{2n+2})) \leq \psi(d^2(x_{2n}, x_{2n+1})) - \phi(d^2(x_{2n}, x_{2n+1})).$$

Taking the limit as $n \to \infty$ in the above inequality and using the continuities of ϕ and ψ, we have

$$\psi(r^2) \leq \psi(r^2) - \phi(r^2)$$

that is $\phi(r) \leq 0$, which is contradiction unless $r = 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.15}$$

Now, as in the proof of Theorem 2.4, we can prove that that $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exist a $p \in X$ such that $x_n \to p$ as $n \to \infty$.
Putting $x = x_{2n}$, $y = p$ in (2.12), we have

$$\begin{aligned}
\psi(d^2(x_{2n+1}, Tp)) &= \psi(d^2(Sx_{2n}, Tp)) \\
&\leq \psi\Bigg(\max\Bigg\{d^2(x_{2n}, p), d(x_{2n}, Tp)\cdot d(p, x_{2n+1})), \\
&\quad \frac{d^2(x_{2n}, x_{2n+1}) + d^2(p, Tp)}{2}\Bigg\}\Bigg) - \varphi(\max\{d^2(x_{2n}, p), \\
&\quad d(x_{2n}, Tp)\cdot d(p, x_{2n+1})), d(x_{2n}, x_{2n+1})\cdot d(p, Tp)\}).
\end{aligned}$$

Taking $n \to \infty$ in the above inequality and using the continuities of φ and ψ, we get

$$\psi(d^2(p, Tp)) \leq \psi\left(\frac{d^2(p, Tp)}{2}\right) - \varphi(0),$$

which implies that $d^2(p, Tp) = 0$, or $p = Tp$. Similarly we get that p is a fixed point of S. Hence p is a fixed point of T and S.

Suppose that p and q are common fixed point of S and T, and $p \neq q$. From (2.12), we have

$$\begin{aligned}
\psi(d^2(p, q)) &= \psi(d^2(Sp, Tq)) \\
&\leq \psi\left(\max\left\{d^2(p, q), d(p, Tq)\cdot d(q, Sp), \frac{d^2(p, Sp) + d^2(q, Tq)}{2}\right\}\right) \\
&- \varphi(\max\{d^2(p, q), d(p, Tq)\cdot d(q, Sp), d(p, Sp)\cdot d(q, Tq)\}) \\
&\leq \psi(d^2(p, q)) - \varphi(d^2(p, q)).
\end{aligned}$$

This is a contradiction of the property of φ. Hence $p = q$. Similarly we have that any fixed point of T is also a fixed point of S. □ □

Corollary 2.4. *Let (X, d) be a complete metric space. Let $S, T : X \to X$ be self mappings such that, for all $x, y \in X$,*

$$\begin{aligned}
\psi(d^2(S^m x, T^n y)) \leq \psi\left(\max\left\{d^2(x, y), d(x, T^n y)\cdot d(y, S^m x), \frac{d^2(x, S^m x) + d^2(y, T^n y)}{2}\right\}\right) \\
-\varphi(\max\{d^2(x, y), d(x, T^n y)\cdot d(y, S^m x), d(x, S^m x)\cdot d(y, T^n y)\}),
\end{aligned} \tag{2.16}$$

m and n are fixed positive integers, ψ is an altering distance mapping and φ is an almost altering distance mapping. Then S and T have a unique common fixed point. Further, any fixed point of S is a fixed point of T and conversely.

Proof. Let $f = S^m$ and $g = T^n$. Then from Theorem 4.2, f and g have a common unique fixed point $p \in X$. Now, $S^{m+1}p = S^m(Sp) = Sp$ and $T^{n+1}p = T^n(Tp) = Tp$, and so $Sp = p$ and $Tp = p$. Hence, it follows that p is a common fixed point of S and T.
Suppose that p is a fixed point of S. Then p is a fixed point of f. By Theorem 4.2, p is a fixed point of g. From the uniqueness of common fixed point of f and g, it follows that p is a fixed point of T. Similarly it can be shown that any fixed point of T is also fixed point of S. Condition (2.16) implies the uniqueness of p. □ □

Corollary 2.5. *Let (X,d) be a complete metric space and S,T be self mapping of X. If S and T satisfy* (2.12)*, then S and T have a property Q.*

Proof. By Corollary 2.4. □ □

Corollary 2.6. (Corollary 15 of [23]) *Let (X,d) be a complete metric space and S, T be self mapping of X. If there exists $\alpha \in (0,1|2)$ so that*

$$d^2(Sx,Ty) \le \alpha\cdot[d^2(x,Sx)+d^2(y,Ty)] \tag{2.17}$$

holds true, for all $x,y\in X$, then S and T have a unique common fixed point.

Proof. By Corollary 1.2 and Theorem 2.5. □ □

Next example is given to support the usability of our results.

Example 2.3. Let $X=[0,+\infty)$ and $d: X\times X\to\mathbb{R}$ be given as

$$d(x,y)=\begin{cases}\sqrt{x+y}, & if\ x\neq y,\\ 0, & if\ x=y.\end{cases}$$

Then (X,d) is a complete metric space.
Let $\psi,\ \varphi:[0,\infty)\to[0,\infty)$ be defined by

$$\psi(t)=t^2,\ t\in[0,\infty)$$

and for $s\in[0,\infty)$,

$$\varphi(s)=\begin{cases}\dfrac{s^2}{2}, & if\ s\le 1,\\ \dfrac{1}{2}, & if\ s>1.\end{cases}$$

Then ψ *and* φ have the properties in Theorem 4.1.
Let $T: X\to X$ be defined as follows

$$Tx=\begin{cases}x-1, & if\ x>1,\\ 0, & if\ x\in[0,1].\end{cases}$$

Then we discuss the following cases for $x,y\in X$.

1. If $y\in(1,+\infty)$ and $x>y$, then

$$\psi(d^2(Tx,Ty))=\psi(d^2(x-1,y-1))=(x+y-2)^2,$$

$$\begin{aligned}\psi(P_T(x,y)) &= \psi(\max\{x+y,\sqrt{(x+y-1)\cdot(x+y-1)},\frac{1}{2}[(2x-1)+(2y-1)]\})\\ &= \psi(\max\{x+y,\sqrt{(x+y-1)\cdot(x+y-1)},x+y-1\})\\ &= \psi(x+y)=(x+y)^2,\end{aligned}$$

and

$$\begin{aligned}\psi(Q_T(x,y)) &= \varphi(\max\{x+y,\sqrt{(x+y-1)\cdot(x+y-1)},\sqrt{(2x-1)\cdot(2y-1)}\})\\ &= \varphi(x+y)=\frac{1}{2}.\end{aligned}$$

Hence,

$$\psi(d^2(Tx,Ty))\le\psi(P_T(x,y))-\varphi(Q_T(x,y)).$$

2. If $y\in(1,+\infty)$ and $x=y$, then obviously condition (1.4) is satisfied.

3. If $y \in [0,1]$ and $x \in [1,+\infty)$, then
$\psi(d^2(Tx,Ty)) = \psi(d^2(x-1,0)) = (x-1)^2$,

$$\begin{aligned}\psi(P_T(x,y)) &= \psi(\max\{d^2(x,0), d(x,T0)\cdot d(0,Tx), \frac{1}{2}[d^2(x,Tx)+d^2(0,T0)]\})\\ &= \psi(\max\{x, \sqrt{x(x-1)}, \frac{1}{2}[x+(x-1)]\})\\ &= \psi(\max\{x, \frac{2x-1}{2}\}) = \psi(x) = x^2,\end{aligned}$$

and

$$\begin{aligned}\psi(Q_T(x,y)) &= \varphi(\max\{d^2(x,0), d(x,T0)\cdot d(0,Tx), d(x,Tx)\cdot d(0,T0))\})\\ &= \varphi(\max\{x, \sqrt{x(x-1)}, 0\}) = \varphi(x) = \frac{1}{2}.\end{aligned}$$

Thus all conditions of Theorem 4.1 are satisfied. Clearly, 0 is the unique fixed point of T.

4. If $y, x \in [0,1]$, then obviously condition (1.4) is satisfied.

Remark 2.1. Let us remark that map T in Example 2.3 is not a q–contraction. It follows from

$$\frac{d(Tn,T(n+1))}{d(n,n+1)} = \frac{d(n-1,n)}{d(n,n+1)} = \frac{\sqrt{2n-1}}{\sqrt{2n+1}} \to 1, \quad n\to\infty.$$

Furthermore, map T does not satisfy Kannan condition (1.2). It follows from

$$\begin{aligned}\frac{d(Tn,T(n+1))}{d(n,Tn)+d(n+1,T(n+1))} &= \frac{d(n-1,n)}{d(n,n-1)+d(n+1,n)}\\ &= \frac{\sqrt{2n-1}}{\sqrt{2n-1}+\sqrt{2n+1}} \to \frac{1}{2}, \quad n\to\infty.\end{aligned}$$

Finally, map T does not satisfy Chatterjee condition (1.3). It follows from

$$\frac{d(Tn,T(n+1))}{d(n,T(n+1))+d(n+1,Tn)} = \frac{d(n-1,n)}{d(n,n)+d(n+1,n-1)} = \frac{\sqrt{2n-1}}{2\sqrt{2n}} \to \frac{1}{2}, \quad n\to\infty.$$

3. Results in compact metric spaces

In this section we prove some results on compact metric spaces. These results are related to our results from the previous section.

Theorem 3.6. *Let (X,d) be a compact metric space and $T: X\to X$ be a continuous map. If*

$$\psi(d^2(Tx,Ty)) < \psi(P_T(x,y)), \tag{3.18}$$

for all $x, y \in X$, $x \neq y$, where

$$P_T(x,y) = \max\left\{d^2(x,y), d(x,Ty)\cdot d(y,Tx), \frac{d^2(x,Tx)+d^2(y,Ty)}{2}\right\},$$

and ψ is an altering distance mapping, then T has a unique fixed point.

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$, for all $n \geq 0$. If there exist a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T. Hence we shall assume that $x_n \neq x_{n+1}$, for all $n \geq 0$. From (3.18), we have for all $n \geq 0$,

$$\begin{aligned}\psi(d^2(x_{n+1},x_{n+2})) &= \psi(d^2(Tx_n,Tx_{n+1})) < \psi(P_T(x_n,x_{n+1})\\ &= \psi\left(\max\left\{(d^2(x_n,x_{n+1}), \frac{d^2(x_n,x_{n+1})+d^2(x_{n+1},x_{n+2})}{2}\right\}\right).\end{aligned} \tag{3.19}$$

Suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$ for some positive integer n. Then by (3.19) we have

$$\psi(d^2(x_{n+1}, x_{n+2})) < \psi(d^2(x_{n+1}, x_{n+2})), \tag{3.20}$$

Hence, a contradiction. Therefore,

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \quad \text{for all} \quad n \geq 0 \tag{3.21}$$

and $\{(d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of positive real number. Hence there exists an $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{3.22}$$

Let $\{k(n)\}$ be a sequence of positive integers such that $\{T^{k(n)}x_0\}$ converges to $p \in X$. Hence, $x_{k(n)+1} \to Tp$ and $x_{k(n)+2} \to T^2p$, $n \to \infty$. We note that by (3.22) we have

$$\lim d(x_{k(n)}, x_{k(n)+1}) = d(p, Tp) = \lim d(x_{k(n)+1}, x_{k(n)+2}) = d(Tp, T^2p) = r$$

Now, we claim that $Tp = p$. If $Tp \neq p$, by (3.18) we have

$$\psi(d^2(Tp, T^2p)) < \psi(P_T(p, Tp)) \tag{3.23}$$

$$= \psi\left(\max\left\{d^2(p, Tp), d(p, T^2p) \cdot d(Tp, Tp), \frac{d^2(p, Tp) + d^2(Tp, T^2p)}{2}\right\}\right) = \psi(d^2(Tp, T^2p),$$

hence, a contradiction. So $Tp = p$. If there is $q \in X$ such that $Tq = q$, and $p \neq q$, then by (3.18) we have

$$\psi(d^2(p, q)) = \psi(d^2(Tp, Tq)) < \psi(P_T(p, q)) \tag{3.24}$$

$$= \psi\left(\max\left\{d^2(p, q), d(p, q) \cdot d(q, p), \frac{d^2(p, p) + d^2(q, q)}{2}\right\}\right) = \psi(d^2(p, q)),$$

hence, a contradiction. □

Now, as corollaries, for $\psi(t) = t$, by Theorem 3.6.we obtain the next results.

Corollary 3.7. (Edelstein [13]) *Let (X, d) be a compact metric space and let $f : X \to X$ be a map such that*

$$d(f(x), f(y)) < d(x, y), \quad \textit{for all } x, y \in X \textit{ with } x \neq y.$$

Then the mapping f has a unique fixed point.

Corollary 3.8. (Fisher [14], Górnicki [15]) *Let (X, d) be a compact metric space and let $f : X \to X$ be a map such that*

$$d(f(x), f(y)) < \frac{d(x, f(x)) + d(y, f(y))}{2}, \quad \textit{for all } x, y \in X, \textit{with } x \neq y. \tag{3.25}$$

Then the mapping f has a unique fixed point.

Corollary 3.9. *Let (X, d) be a compact metric space and let $f : X \to X$ be a map such that*

$$d(f(x), f(y)) < \frac{d^2(x, Tx) + d^2(y, Ty)}{2}, \quad \textit{for all } x, y \in X, \textit{ with } x \neq y. \tag{3.26}$$

Then the mapping f has a unique fixed point.

Acknowledgements. The authors are thankful to the anonymous referees for the useful suggestions and remarks that contributed to the improvement of the manuscript.

REFERENCES

[1] Agarwal, R. P., Karapinar, E., O'Regan, D. and Roldán-López-de-Hierro, A. F., *Fixed Point Theory in Metric Type Spaces*, Springer International Publishing, Switzerland, 2015

[2] Alber, Y. I. and Guerre-Delabriere, S., *Principles of weakly contractive maps in Hilbert spaces*, in Goldberg, I, Lyubich, Yu. (eds.), New Results in Operator Theory. Advances and Appl., vol. **98**, pp. 7–12 Birkhauser, Basel, 1977

[3] Aydi, H., Karapinar, E. and Shatanawi, W., *Coupled fixed point results for (ψ, f)-weakly contractive condition in ordered partial metric spaces*, Comput. Math. Appl., **62** (2011), 4449–4460

[4] Banach, S., *Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales*, Fund. Math., **3** (1922), 133–181

[5] Berinde, V., *Iterative Approximation of Fixed Points*, Springer, Berlin, 2007

[6] Bryant, V. W., *A remark on a fixed point theorem for iterated mappings*, Amer. Math. Monthly, **75** (1968), 399–400

[7] Ćirić, Lj., *On some maps with a nonunique fixed point*,Publ. Inst. Math. (Beograd) (N.S.), **17** (31), (1974), 52–58

[8] Ćirić, Lj., *Some recent results in metrical fixed point theory*, University of Belgrade, Beograd, 2003

[9] Chatterjea, S. K., *Fixed-point theorems*, C. R. Acad. Bulgare Sci., **25** (1972), 727–730

[10] Chi, K. P, Karapinar, E. and Thanh, T. D., *On the fixed point theorems for generalized weakly contractive mappings on partial metric spaces*, Bull. Iran. Math. Soc., **30** (2013), 369–381

[11] Choudhury, B. S., Konar, P., Rhoades, B. E. and Metiya, N., *Fixed point theorems for generalized weakly contractive mappings*, Nonlinear Anal., **74** (2011), 2116–2126

[12] Djorić, D., *Common fixed point for generalized (ψ, f)-weak contractions*, Appl. Math. Lett., **22** (2009), 1896–1900

[13] Edelstein, M., *On fixed and periodic points under contractive mappings*, J. London Math. Soc., **37** (1962), 74–79

[14] Fisher, B., *A fixed point mapping*, Bull. Calcutta Math. Soc., **68** (1976), 265–266

[15] Górnicki, J., *Fixed point theorems for Kannan type mappings*, J. Fixed Point Theory Appl., **19** (2017), 2145-2152

[16] Jeong, G. S. and Rhoades, B. E., *Maps for which $F(T) = F(T^n)$*, Fixed point Theory Appl., **6** (2006), 72–105

[17] Jeong, G. S. and Rhoades, B. E., *More maps for which $F(T) = F(T^n)$*, Demonstratio Math., **40** (2007), 671–680

[18] Kannan, R., *Some results on fixed points*, Bull. Calcutta Math. Soc., **60** (1968), 71–76

[19] Khan, M. S., Swaleh, M., and Sessa, S., *Fixed point theorems by altering distance between the points*, Bull. Aust. Math. Soc., **30** (1984), 1–9

[20] Kirk, W. and Shahzad, N., *Fixed Point Theory in Distance Spaces*, Springer International Publishing, Switzerland, 2014

[21] Lakzian, H. and Samet, B., *Fixed points for (ψ, f)-weakly contractive mappings in generalized metric spaces*, Appl. Math. Lett., **25** (2012), 902–906

[22] Lakzian, H., Rakočević, V. and Aydi, H., *Extensions of Kannan contraction via w-distances*, Aequat. Math., **93** (2019), No. 6, 1231–1244

[23] Malčeski, R., Malčeski, A., Anevska, K. and Malčeski, S., *Common Fixed Points of Kannan and Chatterjea Types of Mappings in a Complete Metric Space*, British Journal of Mathematics & Computer Science, **18** (2016), No. 2, 1–11

[24] Malkowsky, E. and Rakočević, V., *Advanced Functional Analysis*, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2019

[25] Rhoades, B. E., *Some theorems on weakly contractive maps*, Nonlinear Analysis TMA, **47** (2001), 2683–2693

[26] Rus, I. A., *Generalized Contractions and Applications* Cluj University Press, Cluj-Napoca, 2001

[27] Rus, I. A., Petrusel, A. and Petrusel, G., *Fixed Point Theory*, Cluj University Press Cluj-Napoca, 2008

[1]DEPARTMENT OF MATHEMATICS
GOVERNMENT V. Y. T. POST-GRADUATE AUTONOMOUS COLLEGE
DURG 491001, CHHATTISGARH, INDIA
Email address: rakeshtiwari66@gmail.com
Email address: shobharaniy89@gmail.com

[2]DEPARTMENT OF MATHEMATICS
COLLEGE OF SCIENCE, SULTAN QABOOS UNIVERSITY
POBOX 36, AL-KHOD,MUSCAT, OMAN
Email address: mohammad@squ.edu.om

[3]DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIŠ, FACULTY OF SCIENCES AND MATHEMATICS
VIŠEGRADSKA 33, 18000 NIŠ, SERBIA
Email address: vrakoc@sbb.rs