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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} anniversary

Modified two-step extragradient method for solving the pseudomonotone equilibrium programming in a real Hilbert space

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ABSTRACT. The purpose of this paper is to come up with an inertial extragradient method for dealing with a class of pseudomonotone equilibrium problems. This method can be a view as an extension of the paper title "A new two-step proximal algorithm of solving the problem of equilibrium programming" by Lyashko and Semenov et al. (Optimization and Its Applications in Control and Data Sciences: 315—325, 2016). The theorem of weak convergence for solutions of the pseudomonotone equilibrium problems is well-established under standard assumptions placed on cost bifunction in the structure of a real Hilbert spaces. For a numerical experiment, we take up a well-known Nash Cournot equilibrium model of electricity markets to support the well-established convergence results and be adequate to see that our proposed algorithms have a competitive superiority over the time of execution and the number of iterations.

1. INTRODUCTION

The equilibrium problem (shortly, EP) [5] is also described as the Ky Fan inequality were firstly studied in [10]. The equilibrium problems had a significant impact and influence in the advancement of different branches of pure and applied sciences. It has been determined that the equilibrium problem theory set up a novel and unanimous handling of a wide class of problems which occurred in economics, finance, image reconstruction, ecology, transportation network, elasticity and optimization. It has been established that equilibrium problems take into account variational inequalities, fixed point, Nash equilibrium and game theory as special cases (see e.g.,[5, 11, 27]). Consequently, equilibrium problems cover a wide range of applications. For the study of the solution problem of an equilibrium problem, numerical iterative methods are effective and useful. Certainly, widely known problems came into existence in different branches of science can be studied by using algorithms which are iterative in their nature. In recent years, many methods have been developed to solve equilibrium problems in finite and infinite dimensional spaces (for instance, see [9, 13, 26, 24, 25, 1]).

For the numerical solution of equilibrium problem, two practical techniques are particularly important, one of them is the proximal point method (shortly, PPM) [19] and other one is auxiliary problem principle [17], both are used to solve equilibrium problems. The proximal point method (PPM) method was basically set up by Martinet for monotone variational inequality problems and afterwards, it was extended by Rockafellar [28] in the case of monotone operators. In addition, Moudafi [19] expanded the PPM

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to EPs for involving monotone bifunctions. Furthermore, Konnov [14] presented another explanation of the PPM with weaker conditions. The proximal point method is commonly imposed to monotone EPs, i.e. the bifunction of an equilibrium problem has to be monotone. Thus, each regularized subproblem turns into strongly monotone, and so its solution exists and is unique. This will not ensure the existence of the solution if the bifunction is more general, like pseudomonotone. On the other side, another well-known approach is the auxiliary problem principle, which is based on the perception to establish a new problem which is identical and generally simpler to work out that correlated to our initial problem. This concept started by Cohen [7] for optimization problems and later presented for variational inequality problems [8]. Additionally, Mastroeni [17] take the auxiliary problem principle into equilibrium problems involving strongly monotone bifunctions.

In this paper, we concentrate on the proximal point method, consisting of extragradient methods which are well-known and essentially accessible to implement owing to their easier numerical computation. As we know, the earliest established projection method for variational inequality problems is the gradient projection method. After that, many alternative projection methods were established such as the extragradient method [15], the subgradient extragradient method [6], Popov's extragradient method [23], Tseng's extragradient method [30], projection and contraction schemes [12] and others hybrid and projected gradient methods. The first consideration respecting to the extragradient method [15], is needed to figure out twice the orthogonal projection onto C per each iteration. So, in case that the set C is not "simple" to project onto it, a minimal distance problem has to be resolved (twice) in order to gain the next iteration, a fact that might affect the efficiency and applicability of the method. Evenly, an initial step to conquer this difficulty, Censor et al. in [6] introduced the subgradient extragradient scheme in which the second projection onto C is replaced by a specific subgradient projection which can be efficiently computed. On the other hand, let us point out inertial-type methods, based on the heavy ball methods of the two-order time dynamical system, Polyak [22] begin with and looked at an inertial extrapolation as an acceleration strategy to deal with the smooth convex minimization problem. The inertial algorithm is a two-step iterative process, and the next iteration is determined by making use of the earlier two iterations and it can be considered as a way of speeding up iterative sequence, look at [4, 22].

In this paper, we modify the result that presented in [16] by employing the subgradient technique [6] which have the improvements on Lyashko et al. [16] i.e. solve a minimal distance problem onto half plane which is formerly pointed out in the above passage. Additionally, at the same moment, the inertial method is also connected to the proposing method to speed up the iterative sequence. The theorem of weak convergence for the solution of the equilibrium problems involving pseudomonotone and Lipschitz-type constants of a cost bifunction is well-formed under standard assumptions placed on bifunction. Furthermore, some numerical results are also shown to look at the performance of our proposed method.

The rest of this paper is organized as follows: In Section 2, we provide some definitions and preliminary results which are making to be used entire paper. Section 3, consists of an algorithm for pseudomonotone bifunction, and we provide the weak convergence theorem for the proposed algorithm. Finally, in Section 4, we give the numerical experiments to illustrate the computational performance of the suggested algorithms on a test problem which is modelled from a Nash Cournot equilibrium model of electricity markets, in comparisons with other well-known algorithms.

2. Preliminaries

In this part, we give some preliminary results that will be useful in the demonstration of our upcoming results. From now on, let C be closed and convex subset of a Hilbert space $\mathbb{H}$ with inner product $\langle .,. \rangle$ and norm $\|.\|$ respectively. Let $\mathbb{R}$ and $\mathbb{N}$ be two sets of all real numbers and all positive integers respectively. While $\{x_n\}$ is a sequence in $\mathbb{H}$, we denote the strong convergence and weak convergence of x_n to $x \in H$ as $n \to \infty$ by $x_n \to x$ and $x_n \rightharpoonup x$ respectively. Finally, $EP(f, C)$ stand for the solution set of the equilibrium problem inside C and p is a member of $EP(f, C)$.

Definition 2.1 (Equilibrium Problem[5]). Let C be a nonempty closed convex subset of $\mathbb{H}$. Let f be a bifunction from $C \times C$ to the set of real numbers $\mathbb{R}$ such that $f(x, x) = 0$ for all $x \in C$. The equilibrium problem (EP) for the bifunction f on C is to

$$\text{find} \quad p \in C \quad \text{such that} \quad (p, y) \geq 0, \forall y \in C. \tag{2.1}$$

Definition 2.2. Let C be a closed convex subset in $\mathbb{H}$, we denote the metric projection on C by $P_C(x)$, $\forall x \in \mathbb{H}$, i.e.,

$$P_C(x) = \text{arg min}\{\|y - x\| : y \in C\}.$$

We now recall classical concepts of monotonicity of nonlinear operators.

Definition 2.3. [18]A mapping $F : \mathbb{H} \to \mathbb{H}$ is said to be

(i) *strongly monotone* on C if

$$\langle F(x) - F(y), x - y \rangle \geq \gamma \|x - y\|^2, \ \forall x, y \in C;$$

(ii) *monotone* on C if

$$\langle F(x) - F(y), x - y \rangle \geq 0, \ \forall x, y \in C;$$

(iii) *strongly pseudomonotone* on C if

$$\langle F(y), x - y \rangle \geq 0 \Longrightarrow \langle F(x), x - y \rangle \geq \gamma \|x - y\|^2, \ \forall x, y \in C;$$

(iv) *pseudomonotone* on C if

$$\langle F(y), x - y \rangle \geq 0 \Longrightarrow \langle F(x), x - y \rangle \geq 0, \ \forall x, y \in C;$$

(v) *L-Lipschitz continuous* on C if there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \ \forall x, y \in C.$$

Analogous to the above definitions, we have the following concepts for equilibrium problems (see [5] and the references therein).

Definition 2.4. A bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is said to be

(i) *strongly monotone* on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma \|x - y\|^2, \ \forall x, y \in C;$$

(ii) *monotone* on C if

$$f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C;$$

(iii) *strongly pseudomonotone* on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq -\gamma \|x - y\|^2, \ \forall x, y \in C;$$

(iv) *pseudomonotone* on C if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0, \ \forall x, y \in C;$$

(v) *Lipschitz-type condition* on C if there exist two positive constants c_1, c_2 such that

$$f(x, z) \leq f(x, y) + f(y, z) + c_1\|x - y\|^2 + c_2\|y - z\|^2, \ \forall x, y, z \in C.$$

Remark 2.1. From the Definitions 2.4, the following implications holds.

$$(i) \implies (ii) \implies (iv) \quad \text{and} \quad (i) \implies (iii) \implies (iv).$$

Remark 2.2. The converses of the above implication is not true in general.

Further, we recall that the *subdifferential* of a convex function $g : C \to \mathbb{R}$ at $x \in C$ is defined by

$$\partial g(x) = \{w \in \mathbb{H} : g(y) - g(x) \geq \langle w, y - x\rangle, \forall\, y \in C\},$$

and the *normal cone* of C at $x \in C$ is defined by

$$N_C(x) = \{w \in \mathbb{H} : \langle w, y - x\rangle \leq 0, \forall\, y \in C\}.$$

Lemma 2.1 ([29], Page 97)**.** *Let C be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$ and $g : C \to \mathbb{R}$ be a convex, subdifferentiable, lower semicontinuous function on C. Then, z is a solution to the following convex optimization problem min$\{g(x) : x \in C\}$ if and only if $0 \in \partial g(z) + N_C(z)$, where $\partial g(z)$ and $N_C(z)$ denotes the subdifferential of g at z and the normal cone of C at z respectively.*

Lemma 2.2 ([3], Page 31)**.** *For all $x, y \in \mathbb{H}$ and $\mu \in \mathbb{R}$, the following equality hold:*

$$\|\mu x + (1-\mu)y\|^2 = \mu\|x\|^2 + (1-\mu)\|y\|^2 - \mu(1-\mu)\|x - y\|^2.$$

Lemma 2.3. [2] *Let ϕ_n, δ_n and β_n be sequences in $[0, +\infty)$ such that*

$$\phi_{n+1} \leq \phi_n + \beta_n(\phi_n - \phi_{n-1}) + \delta_n,\ \forall n \geq 1 \quad \text{and} \quad \sum_{n=1}^{+\infty} \delta_n < +\infty,$$

and there exists a real number β with $0 \leq \beta_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Then the following relation is true:

(i) $\sum_{n=1}^{+\infty} [\phi_n - \phi_{n-1}]_+ < \infty, where [s]_+ := \max\{s, 0\}$;
(ii) *There exists $\phi^* \in [0, +\infty)$ such that $\lim_{n\to+\infty} \phi_n = \phi^*$.*

Lemma 2.4. [21] *Let C be a nonempty set of $\mathbb{H}$ and $\{x_n\}$ be a sequence in $\mathbb{H}$ such that the following two conditions hold:*

(i) *For every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*
(ii) *Every sequentially weak cluster point of $\{x_n\}$ is in C.*

Then, $\{x_n\}$ converges weakly to a point in C.

3. An algorithm for a class of pseudomonotone equilibrium problem

In this section, we propose our main result which is a modification of algorithm 1 (see [16]) to find an approximate solution of a class of pseudomonotone problem (EP). Our proposed iterative method consists of two strong convex optimization problems with an subgradient and inertial technique which is adopted to speed up the iterative process, so we called it an "Modified two-step subgradient extragradient method" for equilibrium programming.

Assumption 1. *We assume that the bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$, satisfies the following conditions:*

A_1. *$f(x, x) = 0, \forall x \in C$ and f is pseudomontone on C.*
A_2. *f satisfies the Lipschitz-type conditions with two constants c_1 and c_2.*
A_3. *$\lim_{n\to\infty} \sup f(x_n, y) \leq f(p, y)$ for each $y \in C$ and $\{x_n\} \subset C$ with $x_n \rightharpoonup p$.*
A_4. *$f(x, .)$ is convex and subdifferentiable on C for every fixed $x \in C$.*

The following is the algorithm in detail:

Algorithm 1 (Modified two-step subgradient extragradient method)

Initialization: Choose $x_{-1}, x_0, y_0 \in \mathbb{H}$, $\lambda > 0$, and $\alpha_n \in [0, \sqrt{5}-2)$. Set

$$w_0 = x_0 + \alpha_0(x_0 - x_{-1}),$$
$$x_1 = Prox_{\lambda f(y_0,\,.)} w_0 = \arg\min_{y \in C}\{\lambda f(y_0, y) + \frac{1}{2}\|w_0 - y\|^2\},$$
$$y_1 = Prox_{\lambda f(y_0,\,.)} x_1 = \arg\min_{y \in C}\{\lambda f(y_0, y) + \frac{1}{2}\|x_1 - y\|^2\}.$$

Iterative steps: For given x_n, y_n and y_{n-1} for $n \geq 1$. Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$

and construct half space

$$H_n = \{z \in \mathbb{H} : \langle w_n - \lambda \upsilon_{n-1} - y_n, z - y_n \rangle \leq 0\},$$

where $\upsilon_{n-1} \in \partial f(y_{n-1}, y_n)$.

Step 1: Compute

$$x_{n+1} = Prox_{\lambda f(y_n,\,.)} w_n = \arg\min_{y \in H_n}\{\lambda f(y_n, y) + \frac{1}{2}\|w_n - y\|^2\}.$$

Step 2: Compute

$$y_{n+1} = Prox_{\lambda f(y_n,\,.)} x_{n+1} = \arg\min_{y \in C}\{\lambda f(y_n, y) + \frac{1}{2}\|x_{n+1} - y\|^2\}.$$

Step 3: If $x_{n+1} = w_n$ and $y_n = y_{n-1}$, then stop and y_n is the solution of an equilibrium problem, otherwise set $n := n+1$ and go back Step 1.

Lemma 3.5. *From Algorithm 1 we have the following useful inequality.*

$$\lambda f(y_n, y) - \lambda f(y_n, x_{n+1}) \geq \langle w_n - x_{n+1}, y - x_{n+1} \rangle, \ \forall y \in H_n.$$

Proof. From Lemma 2.1 and definition of x_{n+1} in Algorithm 1, we have

$$0 \in \partial_2\Big\{\lambda f(y_n, y) + \frac{1}{2}\|w_n - y\|^2\Big\}(x_{n+1}) + N_{H_n}(x_{n+1}).$$

Thus, there exist $\omega \in \partial f(y_n, x_{n+1})$ **and** $\overline{\omega} \in N_C(x_{n+1})$ **such that**

$$\lambda\omega + x_{n+1} - w_n + \overline{\omega} = 0.$$

Thus, we have

$$\langle w_n - x_{n+1}, y - x_{n+1} \rangle = \lambda\langle \omega, y - x_{n+1} \rangle + \langle \overline{\omega}, y - x_{n+1} \rangle, \ \forall y \in H_n.$$

Since $\overline{\omega} \in N_{H_n}(x_{n+1})$ then $\langle \overline{\omega}, y - x_{n+1} \rangle \leq 0$ for all $y \in H_n$. This implies that

(3.2) $$\lambda\langle \omega, y - x_{n+1} \rangle \geq \langle w_n - x_{n+1}, y - x_{n+1} \rangle, \ \forall y \in H_n.$$

From $\omega \in \partial f(y_n, x_{n+1})$ and the definition of subdifferential, we have

(3.3) $$f(y_n, y) - f(y_n, x_{n+1}) \geq \langle \omega, y - x_{n+1} \rangle.$$

Combining (3.2) and (3.3) we obtain

(3.4) $$\lambda f(y_n, y) - \lambda f(y_n, x_{n+1}) \geq \langle w_n - x_{n+1}, y - x_{n+1} \rangle, \ \forall y \in H_n.$$

□

Lemma 3.6. *From Algorithm 1 we also have the following useful inequality.*

$$\lambda f(y_n, y) - \lambda f(y_n, y_{n+1}) \geq \langle x_{n+1} - y_{n+1}, y - y_{n+1} \rangle, \ \forall y \in C.$$

Proof. From Lemma 2.1 and definition of y_{n+1} in Algorithm 1, we have

$$0 \in \partial_2\Big\{\lambda f(y_n, y) + \frac{1}{2}\|x_{n+1} - y\|^2\Big\}(y_{n+1}) + N_C(y_{n+1}).$$

Thus, there exist $\omega \in \partial f(y_n, y_{n+1})$ **and** $\overline{\omega} \in N_C(y_{n+1})$ **such that**

$$\lambda\omega + y_{n+1} - x_{n+1} + \overline{\omega} = 0.$$

Thus, we have

$$\langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle = \lambda\langle \omega, y - y_{n+1}\rangle + \langle \overline{\omega}, y - y_{n+1}\rangle, \ \forall y \in C.$$

Since $\overline{\omega} \in N_C(y_{n+1})$ implies $\langle \overline{\omega}, y - y_{n+1}\rangle \leq 0$ for all $y \in C$. This implies that

$$\lambda\langle \omega, y - y_{n+1}\rangle \geq \langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle, \ \forall y \in C. \tag{3.5}$$

From $\omega \in \partial f(y_n, y_{n+1})$ and the definition of subdifferential, we have

$$f(y_n, y) - f(y_n, y_{n+1}) \geq \langle \omega, y - y_{n+1}\rangle. \tag{3.6}$$

Combining (3.5) and (3.6) we obtain

$$\lambda f(y_n, y) - \lambda f(y_n, y_{n+1}) \geq \langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle, \ \forall y \in C. \tag{3.7}$$

□

Lemma 3.7. *Let $\{x_n\}$ and $\{y_n\}$ generated from the Algorithm 1 the following relation holds.*

$$\lambda\big\{f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)\big\} \geq \langle w_n - y_n, x_{n+1} - y_n\rangle.$$

Proof. It follows from Algorithm 1 and $x_{n+1} \in H_n$, by the definition of H_n implies that $\langle w_n - \lambda\upsilon_{n-1} - y_n, x_{n+1} - y_n\rangle \leq 0$. Thus, we get

$$\lambda\langle \upsilon_{n-1}, x_{n+1} - y_n\rangle \geq \langle w_n - y_n, x_{n+1} - y_n\rangle. \tag{3.8}$$

By $\upsilon_{n-1} \in \partial f(y_{n-1}, y_n)$ and the definition of subdifferential, we have

$$f(y_{n-1}, y) - f(y_{n-1}, y_n) \geq \langle \upsilon_{n-1}, y - y_n\rangle, \ \forall y \in \mathbb{H}.$$

Put $y = x_{n+1}$ in the above expression

$$f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) \geq \langle \upsilon_{n-1}, x_{n+1} - y_n\rangle, \ \forall y \in \mathbb{H}. \tag{3.9}$$

By combining (3.8) and (3.9) we obtain

$$\lambda\big\{f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)\big\} \geq \langle w_n - y_n, x_{n+1} - y_n\rangle.$$

□

Next, we discuss different possible stopping criterion for Algorithm 1 and also provide the proof for readable purpose.

Lemma 3.8. *If $x_{n+1} = y_n = w_n$, in Algorithm 1 then $y_n \in EP(f, C)$.*

Proof. From Lemma 3.5, we have

$$\lambda f(y_n, y) - \lambda f(y_n, x_{n+1}) \geq \langle w_n - x_{n+1}, y - x_{n+1}\rangle, \ \forall y \in H_n.$$

From above hypothesis and condition A_1 in (Assumption 1) implies $\lambda f(y_n, y) \geq 0$, and further due to step size $\lambda > 0$ with $C \subset H_n$ implies that $y_n \in EP(f, C)$. □

Lemma 3.9. *If $x_{n+1} = y_{n+1} = y_n$ in Algorithm 1 then $y_n \in EP(f, C)$.*

Proof. From Lemma 3.6, we have

$$\lambda f(y_n, y) - \lambda f(y_n, y_{n+1}) \geq \langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle, \ \forall y \in C.$$

From above hypothesis and condition A_1 in (Assumption 1) implies $\lambda f(y_n, y) \geq 0$, and further due to step size $\lambda > 0$, implies that $y_n \in EP(f, C)$. □

Lemma 3.10. *If $x_{n+1} = w_n$ and $y_n = y_{n-1}$ as in Step 3 of Algorithm 1, then $y_n \in EP(f, C)$.*

Proof. By $x_{n+1} = w_n$ in Lemma 3.5, we obtain

$$\lambda f(y_n, y) - \lambda f(y_n, x_{n+1}) \geq 0, \ \forall y \in H_n. \tag{3.10}$$

Since $x_{n+1} \in H_n$, we have

$$\lambda\{f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)\} \geq \langle w_n - y_n, x_{n+1} - y_n\rangle.$$

From given $y_n = y_{n-1}$ and $x_{n+1} = w_n$, with hypothesis A_1 in (Assumption 1) above implies that

$$\lambda f(y_n, x_{n+1}) \geq \|w_n - y_n\|^2 \geq 0. \tag{3.11}$$

From expression (3.10) and (3.11) implies that $y_n \in EP(f, C)$. □

Lemma 3.11. *Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is a bifunction satisfying the conditions (A_1)-(A_4) in (Assumption 1). Assume that the solution set $EP(f, C)$ is nonempty. Then, for all $p \in EP(f, C)$, we have*

$$\begin{aligned} \|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 &- (1 - 4c_1\lambda)\|w_n - y_n\|^2 - (1 - 2c_2\lambda)\|x_{n+1} - y_n\|^2 \\ &+ 4c_1\lambda\|w_n - y_{n-1}\|^2. \end{aligned} \tag{3.12}$$

Proof. Substituting $y = p$ into Lemma 3.5, we obtain

$$\lambda f(y_n, p) - \lambda f(y_n, x_{n+1}) \geq \langle w_n - x_{n+1}, p - x_{n+1}\rangle, \ \forall y \in H_n. \tag{3.13}$$

Since $p \in EP(f, C)$, we have $f(p, y_n) \geq 0$. Thus $f(y_n, p) \leq 0$, due to pseudomonotonicity of bifunction f. Thus, from (3.13) we get

$$\langle w_n - x_{n+1}, x_{n+1} - p\rangle \geq \lambda f(y_n, x_{n+1}). \tag{3.14}$$

The Lipschitz-type continuity of f leads to

$$f(y_{n-1}, x_{n+1}) \leq f(y_{n-1}, y_n) + f(y_n, x_{n+1}) + c_1\|y_{n-1} - y_n\|^2 + c_2\|y_n - x_{n+1}\|^2. \tag{3.15}$$

From relation (3.14) and (3.15) implies that

$$\begin{aligned} \langle w_n - x_{n+1}, x_{n+1} - p\rangle &\geq \lambda\{f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)\} \\ &\quad - c_1\lambda\|y_{n-1} - y_n\|^2 - c_2\lambda\|y_n - x_{n+1}\|^2. \end{aligned} \tag{3.16}$$

Since $x_{n+1} \in H_n$ and by Lemma 3.7, we have

$$\lambda\{f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)\} \geq \langle w_n - y_n, x_{n+1} - y_n\rangle. \tag{3.17}$$

From (3.16) and (3.17), we obtain

$$\begin{aligned} \langle w_n - x_{n+1}, x_{n+1} - p\rangle &\geq \langle w_n - y_n, x_{n+1} - y_n\rangle \\ &\quad - c_1\lambda\|y_{n-1} - y_n\|^2 - c_2\lambda\|y_n - x_{n+1}\|^2. \end{aligned} \tag{3.18}$$

We have the following facts:

$$-2\langle w_n - x_{n+1}, x_{n+1} - p\rangle = -\|w_n - p\|^2 + \|x_{n+1} - w_n\|^2 + \|x_{n+1} - p\|^2.$$

$$2\langle w_n - y_n, x_{n+1} - y_n\rangle = \|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 - \|w_n - x_{n+1}\|^2.$$

From the above last two inequalities and (3.18) we obtain

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\leq \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2 - \|w_n - y_n\|^2 - \|x_{n+1} - y_n\|^2 \\ &\quad + \|x_{n+1} - w_n\|^2 + 2c_1\lambda\|y_{n-1} - y_n\|^2 + 2c_2\lambda\|y_n - x_{n+1}\|^2 \\ &= \|w_n - p\|^2 - \|w_n - y_n\|^2 - (1 - 2c_2\lambda)\|x_{n+1} - y_n\|^2 + 2c_1\lambda\|y_{n-1} - y_n\|^2. \end{aligned} \tag{3.19}$$

We have the following inequality

$$\|y_{n-1}-y_n\|^2 \le \big(\|y_{n-1}-w_n\|+\|w_n-y_n\|\big)^2 \le 2\|y_{n-1}-w_n\|^2+2\|w_n-y_n\|^2.$$

From the above inequality and equation (3.19) implies the required result.

$$\|x_{n+1}-p\|^2 \le \|w_n-p\|^2-(1-4c_1\lambda)\|w_n-y_n\|^2-(1-2c_2\lambda)\|x_{n+1}-y_n\|^2 \quad +4c_1\lambda\|w_n-y_{n-1}\|^2.$$

□

Let us formulate the main convergence result of this work.

Theorem 3.1. *Let $f:\mathbb{H}\times\mathbb{H}\to\mathbb{R}$ is a bifunction satisfying the conditions (A_1)-(A_4) in (Assumption 1). Let $\{x_n\}$ be a sequences in $\mathbb{H}$ generated by Algorithm 1, where the sequence α_n is non-decreasing and with λ be a positive real number such that*

$$0<\lambda\le\frac{\frac{1}{2}-2\alpha-\frac{1}{2}\alpha^2}{c_2(1-\alpha)^2+2c_1(1+\alpha+\alpha^2+\alpha^3)} \quad \textit{and} \quad 0\le\alpha_n\le\alpha<\sqrt{5}-2.$$

Then, $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ converges weakly to an element of $EP(f,C)$.

Proof. From Lemma 3.11 and adding $4c_1\lambda\|w_{n+1}-y_n\|^2$ in both sides, we obtain

$$\|x_{n+1}-p\|^2+4c_1\lambda\|w_{n+1}-y_n\|^2 \le \|w_n-p\|^2-(1-4c_1\lambda)\|w_n-y_n\|^2$$

$$-(1-2c_2\lambda)\|x_{n+1}-y_n\|^2 \quad +4c_1\lambda\|w_n-y_{n-1}\|^2+4c_1\lambda\|w_{n+1}-y_n\|^2. \tag{3.20}$$

By the definition of w_n in Algorithm 1, we have

$$\|w_n-p\|^2=\|x_n+\alpha_n(x_n-x_{n-1})-p\|^2=\|(1+\alpha_n)(x_n-p)-\alpha_n(x_{n-1}-p)\|^2$$

$$=(1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2. \tag{3.21}$$

By the definition of w_{n+1} in Algorithm 1, we obtain

$$\|w_{n+1}-y_n\|^2=\|x_{n+1}+\alpha_{n+1}(x_{n+1}-x_n)-y_n\|^2=\|(1+\alpha_{n+1})(x_{n+1}-y_n)-\alpha_{n+1}(x_n-y_n)\|^2$$

$$=(1+\alpha_{n+1})\|x_{n+1}-y_n\|^2-\alpha_{n+1}\|x_n-y_n\|^2+\alpha_{n+1}(1+\alpha_{n+1})\|x_{n+1}-x_n\|^2$$

$$\le(1+\alpha_n)\|x_{n+1}-y_n\|^2+\alpha_n(1+\alpha_n)\|x_{n+1}-x_n\|^2. \tag{3.22}$$

From expression (3.20), (3.21) and (3.22) implies that

$$\|x_{n+1}-p\|^2+4c_1\lambda\|w_{n+1}-y_n\|^2$$

$$\le(1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2$$

$$+4c_1\lambda\|w_n-y_{n-1}\|^2-(1-4c_1\lambda)\|w_n-y_n\|^2-(1-2c_2\lambda)\|x_{n+1}-y_n\|^2$$

$$+4c_1\lambda(1+\alpha_n)\|x_{n+1}-y_n\|^2+4c_1\lambda\alpha_n(1+\alpha_n)\|x_{n+1}-x_n\|^2 \tag{3.23}$$

$$\le(1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2$$

$$+4c_1\lambda\|w_n-y_{n-1}\|^2+4c_1\lambda\alpha_n(1+\alpha_n)\|x_{n+1}-x_n\|^2$$

$$-(1-4c_1\lambda)\|w_n-y_n\|^2-(1-2c_2\lambda-4c_1\lambda(1+\alpha_n))\|x_{n+1}-y_n\|^2 \tag{3.24}$$

$$\le(1+\alpha_{n+1})\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2$$

$$+4c_1\lambda\|w_n-y_{n-1}\|^2+4c_1\lambda\alpha_n(1+\alpha_n)\|x_{n+1}-x_n\|^2$$

$$-\frac{\big(1-2c_2\lambda-4c_1\lambda(1+\alpha_n)\big)}{2}\Big[2(\|x_{n+1}-y_n\|^2+\|w_n-y_n\|^2)\Big]. \tag{3.25}$$

Next, put

$$\sigma_n=\frac{1-2c_2\lambda-4c_1\lambda(1+\alpha_n)}{2},$$

and due to the inequality

$$2\|x_{n+1}-y_n\|^2+2\|w_n-y_n\|^2\geq \|x_{n+1}-w_n\|^2,$$

the expression (3.25) becomes

$$\begin{aligned}\Delta_{n+1}\leq{}&\Delta_n+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2+4c_1\lambda\alpha_n(1+\alpha_n)\|x_{n+1}-x_n\|^2\\&-\sigma_n\|x_{n+1}-w_n\|^2,\end{aligned}\tag{3.26}$$

where

$$\Delta_n=\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+4c_1\lambda\|w_n-y_{n-1}\|^2.$$

Furthermore, by the definition w_{n+1} and using Cauchy inequality, we have

$$\begin{aligned}\|x_{n+1}-w_n\|^2&=\|x_{n+1}-x_n-\alpha_n(x_n-x_{n-1})\|^2\\&=\|x_{n+1}-x_n\|^2+\alpha_n^2\|x_n-x_{n-1}\|^2-2\alpha_n\langle x_{n+1}-x_n,x_n-x_{n-1}\rangle\\&\geq\|x_{n+1}-x_n\|^2+\alpha_n^2\|x_n-x_{n-1}\|^2-2\alpha_n\|x_{n+1}-x_n\|\|x_n-x_{n-1}\|\\&\geq\|x_{n+1}-x_n\|^2+\alpha_n^2\|x_n-x_{n-1}\|^2-\alpha_n\|x_{n+1}-x_n\|^2-\alpha_n\|x_n-x_{n-1}\|^2\\&\geq(1-\alpha_n)\|x_{n+1}-x_n\|^2+(\alpha_n^2-\alpha_n)\|x_n-x_{n-1}\|^2.\end{aligned}\tag{3.27}$$

From (3.26) and (3.27) implies that

$$\begin{aligned}\Delta_{n+1}\leq{}&\Delta_n+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2+4c_1\lambda\alpha_n(1+\alpha_n)\|x_{n+1}-x_n\|^2\\&-\sigma_n(1-\alpha_n)\|x_{n+1}-x_n\|^2-\sigma_n(\alpha_n^2-\alpha_n)\|x_n-x_{n-1}\|^2\\\leq{}&\Delta_n+\eta_n\|x_n-x_{n-1}\|^2-\zeta_n\|x_{n+1}-x_n\|^2,\end{aligned}\tag{3.28}$$

where

$$\eta_n:=\alpha_n(1+\alpha_n)+\sigma_n\alpha_n(1-\alpha_n),\quad \zeta_n:=\sigma_n(1-\alpha_n)-4c_1\lambda\alpha_n(1+\alpha_n).$$

Further, we take

$$\Gamma_n=\Delta_n+\eta_n\|x_n-x_{n-1}\|^2.$$

It follows from (3.28) that

$$\begin{aligned}&\Gamma_{n+1}-\Gamma_n\\&=\|x_{n+1}-p\|^2-\alpha_{n+1}\|x_n-p\|^2+\eta_{n+1}\|x_{n+1}-x_n\|^2+4c_1\lambda\|w_{n+1}-y_n\|^2\\&\quad-\|x_n-p\|^2+\alpha_n\|x_{n-1}-p\|^2-\eta_n\|x_n-x_{n-1}\|^2-4c_1\lambda\|w_n-y_{n-1}\|^2\\&=\|x_{n+1}-p\|^2-(1+\alpha_{n+1})\|x_n-p\|^2+\alpha_n\|x_{n-1}-p\|^2+4c_1\lambda\|w_{n+1}-y_n\|^2\\&\quad-4c_1\lambda\|w_n-y_{n-1}\|^2-\eta_n\|x_n-x_{n-1}\|^2+\eta_{n+1}\|x_{n+1}-x_n\|^2\\&\leq-\zeta_n\|x_{n+1}-x_n\|^2+\eta_{n+1}\|x_{n+1}-x_n\|^2\\&=-(\zeta_n-\eta_{n+1})\|x_{n+1}-x_n\|^2.\end{aligned}\tag{3.29}$$

Further, we compute

$$\begin{aligned}
&\zeta_n - \eta_{n+1}\\
&= \sigma_n(1-\alpha_n) - 4c_1\lambda\alpha_n(1+\alpha_n) - \alpha_{n+1}(1+\alpha_{n+1}) - \sigma_{n+1}\alpha_{n+1}(1-\alpha_{n+1})\\
&\geq \sigma_n(1-\alpha_n) - 4c_1\lambda\alpha_n(1+\alpha_n) - \alpha_n(1+\alpha_n) - \sigma_n\alpha_n(1-\alpha_n)\\
&\geq \sigma_n(1-\alpha)^2 - 4c_1\lambda\alpha(1+\alpha) - \alpha(1+\alpha)\\
&\geq \frac{1-2c_2\lambda-4c_1\lambda(1+\alpha)}{2}(1-\alpha)^2 - 4c_1\lambda\alpha(1+\alpha) - \alpha(1+\alpha)\\
&= \Big(\frac{1}{2} - 2\alpha - \frac{1}{2}\alpha^2\Big) - \lambda\Big(c_2\big(1-\alpha\big)^2 + 2c_1\big(1+\alpha+\alpha^2+\alpha^3\big)\Big)\\
&\geq 0.
\end{aligned} \tag{3.30}$$

By expression (3.29) and (3.30) for some $\delta \geq 0$, implies that

$$\Gamma_{n+1} - \Gamma_n \leq -(\zeta_n - \eta_{n+1})\|x_{n+1} - x_n\|^2 \leq -\delta\|x_{n+1} - x_n\|^2 \leq 0. \tag{3.31}$$

So expression (3.31) implies that the sequence $\{\Gamma_n\}$ is nonincreasing. Further, from definition of Γ_{n+1} we have

$$\begin{aligned}
\Gamma_{n+1} &= \|x_{n+1} - p\|^2 - \alpha_{n+1}\|x_n - p\|^2 + \eta_{n+1}\|x_{n+1} - x_n\|^2 + 4c_1\lambda\|w_{n+1} - y_n\|^2\\
&\geq -\alpha_{n+1}\|x_n - p\|^2.
\end{aligned} \tag{3.32}$$

Also, from Γ_n we have

$$\begin{aligned}
\Gamma_n &= \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \eta_n\|x_n - x_{n-1}\|^2 + 4c_1\lambda\|w_n - y_{n-1}\|^2\\
&\geq \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2.
\end{aligned} \tag{3.33}$$

By equation (3.33) implies that

$$\begin{aligned}
\|x_n - p\|^2 &\leq \Gamma_n + \alpha_n\|x_{n-1} - p\|^2\\
&\leq \Gamma_1 + \alpha\|x_{n-1} - p\|^2\\
&\leq \cdots \leq \Gamma_1(\alpha^{n-1} + \cdots + 1) + \alpha^n\|x_0 - p\|^2\\
&\leq \frac{\Gamma_1}{1-\alpha} + \alpha^n\|x_0 - p\|^2.
\end{aligned} \tag{3.34}$$

From expression (3.32) and (3.34), we obtain

$$\begin{aligned}
-\Gamma_{n+1} &\leq \alpha_{n+1}\|x_n - p\|^2\\
&\leq \alpha\|x_n - p\|^2\\
&\leq \alpha\frac{\Gamma_1}{1-\alpha} + \alpha^{n+1}\|x_0 - p\|^2.
\end{aligned} \tag{3.35}$$

It follows from (3.31) and (3.35) that

$$\begin{aligned}
\delta\sum_{n=1}^{k}\|x_{n+1} - x_n\| &\leq \Gamma_1 - \Gamma_{k+1}\\
&\leq \Gamma_1 + \alpha\frac{\Gamma_1}{1-\alpha} + \alpha^{n+1}\|x_0 - p\|^2\\
&\leq \frac{\Gamma_1}{1-\alpha} + \|x_0 - p\|^2,
\end{aligned} \tag{3.36}$$

letting $k \to \infty$ in above expression (3.36), we have

$$\sum_{n=1}^{\infty}\|x_{n+1} - x_n\| < +\infty \quad \text{implies} \quad \|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.37}$$

By expression (3.27) and (3.37) we obtain

$$\|x_{n+1} - w_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.38}$$

Next, equation (3.35) also implies that

$$-\Delta_{n+1} \le \alpha \frac{\Gamma_1}{1-\alpha} + \alpha^{n+1}\|x_0 - p\|^2 + \eta_{n+1}\|x_{n+1} - x_n\|^2. \tag{3.39}$$

Further, from expression (3.25) we have

$$\begin{aligned} &\big(1 - 2c_2\lambda - 4c_1\lambda(1+\alpha)\big)\Big[\|x_{n+1} - y_n\|^2 + \|w_n - y_n\|^2\Big] \\ &\le \Delta_n - \Delta_{n+1} + \alpha(1+\alpha)\|x_n - x_{n-1}\|^2 + 4c_1\lambda\alpha(1+\alpha)\|x_{n+1} - x_n\|^2. \end{aligned} \tag{3.40}$$

Now, we fix a number $k \in \mathbb{N}$ and consider above inequality for all number $1, 2, \cdots, k$. Summing up them, we obtain

$$\begin{aligned} &\big(1 - 2c_2\lambda - 4c_1\lambda(1+\alpha)\big)\sum_{n=1}^{k}\Big[\|x_{n+1} - y_n\|^2 + \|w_n - y_n\|^2\Big] \\ &\le \Delta_0 - \Delta_{k+1} + \alpha(1+\alpha)\sum_{n=1}^{k}\|x_n - x_{n-1}\|^2 + 4c_1\lambda\alpha(1+\alpha)\sum_{n=1}^{k}\|x_{n+1} - x_n\|^2 \\ &\le \Delta_0 + \alpha\frac{\Gamma_1}{1-\alpha} + \alpha^{k+1}\|x_0 - p\|^2 + \eta_{k+1}\|x_{k+1} - x_k\|^2 \\ &\quad + \alpha(1+\alpha)\sum_{n=1}^{k}\|x_n - x_{n-1}\|^2 + 4c_1\lambda\alpha(1+\alpha)\sum_{n=1}^{k}\|x_{n+1} - x_n\|^2, \end{aligned} \tag{3.41}$$

letting $k \to \infty$ in above expression we have

$$\sum \|x_{n+1} - y_n\|^2 = \sum \|w_n - y_n\|^2 < +\infty, \tag{3.42}$$

and implies that

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = \lim_{n\to\infty}\|w_n - y_n\| = 0. \tag{3.43}$$

Also, from above equation expression we can easily derive the following

$$\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|x_n - w_n\| = \lim_{n\to\infty}\|y_{n-1} - y_n\| = 0. \tag{3.44}$$

Furthermore, by the definition w_n and using Cauchy inequality we have

(3.45)

$$\begin{aligned} \|w_n - y_{n-1}\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - y_{n-1}\|^2 \\ &= \|(1+\alpha_n)(x_n - y_{n-1}) - \alpha_n(x_{n-1} - y_{n-1})\|^2 \\ &= (1+\alpha_n)\|x_n - y_{n-1}\|^2 - \alpha_n\|x_{n-1} - y_{n-1}\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \\ &\le (1+\alpha_n)\|x_n - y_{n-1}\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \\ &\le (1+\alpha)\|x_n - y_{n-1}\|^2 + \alpha(1+\alpha)\|x_n - x_{n-1}\|^2. \end{aligned}$$

Now, summing up equation (3.45) for $n = 1, 2\cdots, k$, we obtain

$$\sum_{n=1}^{k}\|w_n - y_{n-1}\|^2 \le (1+\alpha)\sum_{n=1}^{k}\|x_n - y_{n-1}\|^2 + \alpha(1+\alpha)\sum_{n=1}^{k}\|x_n - x_{n-1}\|^2. \tag{3.46}$$

The above equation with (3.37) and (3.42) implies that

$$\sum \|w_n - y_{n-1}\|^2 < +\infty. \tag{3.47}$$

Further, equation (3.23) gives that

$$\|x_{n+1} - p\|^2 \leq (1+\alpha)\|x_n - p\|^2 - \alpha\|x_{n-1} - p\|^2 + \alpha(1+\alpha)\|x_n - x_{n-1}\|^2 + 4c_1\lambda\|w_n - y_{n-1}\|^2, \tag{3.48}$$

above equation with (3.37), (3.47) and Lemma 2.3, implies that limit of $\|x_n - p\|$ exists. Further, equation (3.44) implies that the limit of $\|w_n - p\|$ and $\|y_n - p\|$ exists. This implies that, the sequences $\{x_n\}$, $\{w_n\}$ and $\{y_n\}$ are bounded, and for every $p \in EP(f, C)$, the $\lim_{n\to\infty} \|x_n - p\|^2$ exists. Now, further we show that for very sequential weak cluster point of the sequence $\{x_n\}$ is in $EP(f, C)$. Assume that p is a weak cluster point of $\{x_n\}$, i.e., there exists a subsequence, denoted by $\{x_{n_k}\}$, of $\{x_n\}$ weakly converging to p. Then $\{y_{n_k}\}$ also weakly converges to p and $p \in C$. Let us show that $p \in EP(f, C)$. By Lemma 3.5, the Lipschitz-type condition of bifunction and Lemma 3.7, we have

$$\begin{aligned} \lambda f(y_{n_k}, y) &\geq \lambda f(y_{n_k}, x_{n_k+1}) + \langle w_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle \\ &\geq \lambda f(y_{n_k-1}, x_{n_k+1}) - \lambda f(y_{n_k-1}, y_{n_k}) - c_1\lambda\|y_{n_k-1} - y_{n_k}\|^2 \\ &\quad - c_2\lambda\|y_{n_k} - x_{n_k+1}\|^2 + \langle w_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle \\ &\geq \langle w_{n_k} - y_{n_k}, x_{n_k+1} - y_{n_k}\rangle - c_1\lambda\|y_{n_k-1} - y_{n_k}\|^2 \\ &\quad - c_2\lambda\|y_{n_k} - x_{n_k+1}\|^2 + \langle w_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle, \end{aligned} \tag{3.49}$$

where y is any element in H_n. It follows from (3.43), (3.44), (3.38) and the boundness of $\{x_n\}$ that the right-hand side of the last inequality tends to zero. Using $\lambda > 0$, the condition (A_3) in (Assumption 1) and $y_{n_k} \rightharpoonup p$, we have

$$0 \leq \limsup_{k\to\infty} f(y_{n_k}, y) \leq f(z, y), \ \forall y \in H_n.$$

Since $p \in C \subset H_n$, we have $f(p, y) \geq 0, \forall y \in C$. This shows that $p \in EP(f, C)$. Thus Lemma 2.4, ensures that $\{w_n\}$, $\{x_n\}$ and $\{y_n\}$ converges weakly to p as $n \to \infty$. □

Note: If we assume that the bifunction $f(x, y) := \langle F(x), y - x\rangle$ for all $x, y \in C$, then the equilibrium problem convert into the above variational inequality problem with $L = \frac{1}{2}c_1 = \frac{1}{2}c_2$. It follows from the definition of y_{n+1} in the Algorithm 1 and the above definition of bifunction f such that

(3.50)

$$\begin{aligned} &y_{n+1} \\ &= \arg\min_{y\in C}\Big\{\lambda f(y_n, y) + \frac{1}{2}\|x_{n+1} - y\|^2\Big\} \\ &= \arg\min_{y\in C}\Big\{\lambda\langle F(y_n), y - y_n\rangle + \frac{1}{2}\|x_{n+1} - y\|^2 + \frac{\lambda^2}{2}\|F(y_n)\|^2\Big\} \\ &= \arg\min_{y\in C}\Big\{\lambda\langle F(y_n), y - x_{n+1}\rangle + \frac{1}{2}\|x_{n+1} - y\|^2 + \frac{\lambda^2}{2}\|F(y_n)\|^2 + \lambda\langle F(y_n), x_{n+1} - y_n\rangle\Big\} \\ &= \arg\min_{y\in C}\Big\{\frac{1}{2}\|x_{n+1} - \lambda F(y_n) - y\|^2\Big\} \end{aligned}$$

(3.51)

$$= P_C(x_{n+1} - \lambda F(y_n)).$$

Similarly to the expression (3.51), x_{n+1} in Algorithm 1 reduces to

$$x_{n+1} = P_{H_n}(w_n - \lambda F(y_n)).$$

Assumption 2. *We assume that F satisfying the following assumptions:*

F_1. *F is pseudomonotone on C and $VI(F,C)$ is nonempty;*
F_2. $\limsup_{n\to\infty}\langle F(x_n), y - x_n\rangle \le \langle F(x^*), y - x^*\rangle$ *for every* $y \in C$ *and* $\{x_n\} \subset C$ *satisfying* $x_n \rightharpoonup x^*$.
F_3. *F is L-Lipschitz continuous on C for some positive constant* $L > 0$.

Corollary 3.1. *Suppose that $F : C \to \mathbb{E}$ satisfying the conditions (F_1, F_2, F_3). Let $\{w_n\}$, $\{x_n\}$, $\{y_n\}$ be the sequences generated as follows.*

i. Choose $x_{-1}, x_0, y_0 \in \mathbb{H}$, $\lambda > 0$, *and* $\alpha_n \in [0, \sqrt{5} - 2)$. *Set*

$$w_0 = x_0 + \alpha_0(x_0 - x_{-1}),$$

$$x_1 = P_C(w_0 - \lambda F(y_0)) \quad \text{and} \quad y_1 = P_C(x_1 - \lambda F(y_0)).$$

ii. For given x_n, y_n *and* y_{n-1} *for* $n \ge 1$. *compute*

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$

and construct half space

$$H_n = \{z \in \mathbb{H} : \langle w_n - \lambda F(y_{n-1}) - y_n, z - y_n\rangle \le 0\}.$$

iii. Compute

$$\begin{cases} x_{n+1} = P_C\big(w_n - \lambda F(y_n)\big), \\ y_{n+1} = P_{H_n}\big(x_{n+1} - \lambda F(y_n)\big). \end{cases}$$

Then, the sequence $\{w_n\}$, $\{x_n\}$ and $\{y_n\}$ converges weakly to the solution p of $VI(F,C)$.

4. Computational experiment

In this section few numerical results will be presented in order to test Algorithm 1 (shortly, MTSPA) and also illustrates the comparison of our proposed algorithm with Extragradient method [25] (shortly, EgA) and Two-Step Proximal Algorithm (shortly, TSPA)[16]. The MATLAB codes run on a PC (with Intel(R) Core(TM)i3-4010U CPU @ 1.70GHz 1.70GHz, RAM 4.00 GB) under MATLAB version 9.5 (R2018b).

4.1. **Nash-Cournot equilibrium model of electricity markets.** In this experiment, we apply our proposed algorithm to a Nash-Cournot equilibrium model of electricity markets as in [24]. In this model, it is considered that there are three electricity companies i ($i = 1, 2, 3$). Each company i has its own, several generating units with index set I_i. In this experiment, suppose that $I_1 = \{1\}$, $I_2 = \{2, 3\}$ and $I_3 = \{4, 5, 6\}$. Let x_j be the power generation of units j ($j = 1, \cdots, 6$) and suppose that the electricity price p can be expressed as by

$$p = 378.4 - 2\sum_{j=1}^{6} x_j.$$

The cost of a generating unit j is illustrated as:

$$c_j(x_j) := \max\{\overset{\circ}{c}_j(x_j), \overset{\bullet}{c}_j(x_j)\},$$

with

$$\overset{\circ}{c}_j(x_j) := \frac{\overset{\circ}{\alpha}_j}{2}x_j^2 + \overset{\circ}{\beta}_j x_j + \overset{\circ}{\gamma}_j,$$

and

$$\overset{\bullet}{c}_j(x_j) := \overset{\bullet}{\alpha}_j x_j + \frac{\overset{\bullet}{\beta}_j}{\overset{\bullet}{\beta}_j + 1}\overset{\bullet}{\gamma}_j^{\frac{-1}{\overset{\bullet}{\beta}_j}}(x_j)^{\frac{(\overset{\bullet}{\beta}_j+1)}{\overset{\bullet}{\beta}_j}},$$

where the parameters values are given in $\overset{\circ}{\alpha}_j$, $\overset{\circ}{\beta}_j$, $\overset{\circ}{\gamma}_j$, $\overset{\bullet}{\alpha}_j$, $\overset{\bullet}{\beta}_j$ and $\overset{\bullet}{\gamma}_j$ are given in Table 1. Suppose for the profit of the company i is given by

$$f_i(x) := p\sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j) = \Big(378.4 - 2\sum_{l=1}^{6} x_l\Big)\sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j),$$

where $x = (x_1, \cdots, x_6)^T$ subject to the constraint $x \in C := \{x \in \mathbb{R}^6 : x_j^{\min} \le x_j \le x_j^{\max}\}$, with $x_j^{\min}$ and $x_j^{\max}$ give in Table 2.

TABLE 1. The parameters values using in above equations

j	$\overset{\circ}{\alpha}_j$	$\overset{\circ}{\beta}_j$	$\overset{\circ}{\gamma}_j$	$\overset{\bullet}{\alpha}_j$	$\overset{\bullet}{\beta}_j$	$\overset{\bullet}{\gamma}_j$
1	0.0400	2.00	0.00	2.0000	1.0000	25.0000
2	0.0350	1.75	0.00	1.7500	1.0000	28.5714
3	0.1250	1.00	0.00	1.0000	1.0000	8.0000
4	0.0116	3.25	0.00	3.2500	1.0000	86.2069
5	0.0500	3.00	0.00	3.0000	1.0000	20.0000
6	0.0500	3.00	0.00	3.0000	1.0000	20.0000

TABLE 2. The parameters values used in this example

j	1	2	3	4	5	6
x_j^{min}	0	0	0	0	0	0
x_j^{max}	80	80	50	55	30	40

Next, we define the equilibrium function f by

$$f(x, y) := \sum_{i=1}^{3} \big(\phi_i(x, x) - \phi_i(x, y)\big),$$

where

$$\phi_i(x, y) := \Big[378.4 - 2\Big(\sum_{j\notin I_i} x_j + \sum_{j\in I_i} y_j\Big)\Big]\sum_{j\in I_i} y_j - \sum_{j\in I_i} c_j(y_j).$$

The Nash-Cournot equilibrium models of electricity markets can be reformulated as an equilibrium problem:

$$\text{find} \quad x^* \in C \quad \text{such that} \quad f(x^*, y) \ge 0, \quad \forall y \in C.$$

By Lemma 7 in [24], the equilibrium model of electricity markets can be reformulated as the equilibrium problem of the form $EP(f, C)$, where $C := C_1 \times C_2 \times C_3$. Then, the bifunction f can be expressed as

$$f(x, y) := \big\langle (A + B)x + By + a,\ y - x\big\rangle + c(y) - c(x),$$

where $q^i = (q_1^i, \cdots, q_6^i)^T$ and

$$q_j^i := \begin{cases} 1 & \text{if} \quad j \in I_i \\ 0 & \text{otherwise,} \end{cases} \qquad A := 2\sum_{i=1}^{3}(1 - q_j^j)(q^i)^T, \quad B := 2\sum_{i=1}^{3}(q^i)(q^i)^T,$$

$$a := -378.4\sum_{i=1}^{3} q^i, \quad c(x) := \sum_{i=1}^{6} c_j(x_j).$$

But A is not positive semidefinite, so f is not monotone on C. However, in this case, equilibrium problem $EP(f, C)$ is equivalent to $EP(g, C)$ where the bifunction g is defined by

$$g(x, y) := \big\langle A_1 x + B_1 y + a,\ y - x\big\rangle + c(y) - c(x),$$

where $A_1 := A + \frac{3}{2}B$ and $B_1 := \frac{1}{2}B$.

4.1.1. *Algorithm nature by using different stopping criterion.* During the first experiment, we take $x_{-1} = (10, 0, 10, 1, 10, 1)^T$, $x_0 = (48, 48, 30, 27, 18, 24)^T$, $y_0 = (48, 48, 30, 27, 18, 24)^T$, $\lambda = 0.1$ and y-axes represent for the value of D_n while the x-axes represent for the number of iterations or elapsed time (in seconds). Figures 1 and 2 describes the numerical results for the comparison regarding different stopping criterion which are follows:

$$D_1 = \|x_{n+1} - y_n\|^2 + \|w_n - y_n\|^2$$

$$D_2 = \|x_{n+1} - y_{n+1}\|^2 + \|y_{n+1} - y_n\|^2$$

$$D_3 = \|x_{n+1} - w_n\|^2 + \|y_n - y_{n-1}\|^2.$$

TABLE 3. The numerical results for Figure 1-2

error	α_n	x	iter.	time	TOL
D_1	0.164	$(46.6578, 32.1726, 14.9533, 24.0724, 11.0785, 11.6547)^T$	457	19.4914	$\epsilon = 10^{-5}$
D_2	0.164	$(46.6638, 32.1766, 14.9563, 24.0841, 11.0735, 11.6686)^T$	478	22.1301	$\epsilon = 10^{-5}$
D_3	0.164	$(46.6639, 32.1796, 14.9776, 24.0496, 11.1676, 11.5678)^T$	471	23.0261	$\epsilon = 10^{-5}$

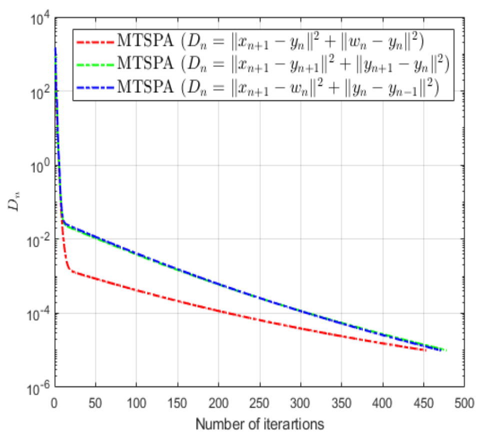

FIGURE 1. Algorithm 1 in term of number of iterations by taking different stopping criterion

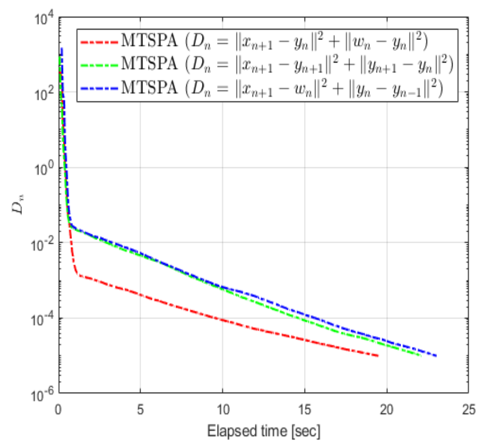

FIGURE 2. Algorithm 1 in term of elapsed time by taking different stopping criterion

4.1.2. *Algorithms nature by using different value of α_n.* During this experiment, we take $x_{-1} = (10, 0, 10, 1, 10, 1)^T$, $x_0 = (48, 48, 30, 27, 18, 24)^T$, $y_0 = (48, 48, 30, 27, 18, 24)^T$ and y-axes represent for the value of D_n while the x-axes represent for the number of iterations or elapsed time (in seconds). Figures 3 and 4 describes the numerical results for the stopping criterion (error term $D_n = \|x_{n+1} - y_n\|^2 + \|w_n - y_n\|^2$) of Algorithm 1 (shortly, MTSPA) with respect to using different values of α_n in term of no.of iterations and elapsed time respectively. The results are shown in Figures 3 and 4 and Table 4.

TABLE 4. The numerical results for Figure 3-4

λ	α_n	x	iter.	time	TOL
0.089	0.215	$(46.6535, 32.2173, 14.9333, 24.1343, 10.9688, 11.7062)^T$	412	27.706785	$\epsilon = 10^{-5}$
0.089	0.174	$(46.6538, 32.1726, 14.9563, 24.0741, 11.0705, 11.6645)^T$	453	29.946840	$\epsilon = 10^{-5}$
0.089	0.123	$(46.6539, 32.1726, 14.9774, 24.0496, 11.1620, 11.5975)^T$	514	34.186107	$\epsilon = 10^{-5}$
0.089	0.082	$(46.6540, 32.1615, 14.9882, 24.0572, 11.2093, 11.5427)^T$	569	38.273818	$\epsilon = 10^{-5}$
0.089	0.021	$(46.6540, 32.1523, 14.9973, 24.0948, 11.2467, 11.4680)^T$	658	45.240680	$\epsilon = 10^{-5}$

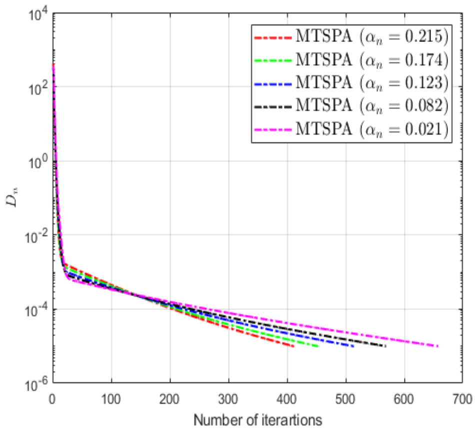

FIGURE 3. Algorithm 1 in term of number of iterations respect to different values of α_n

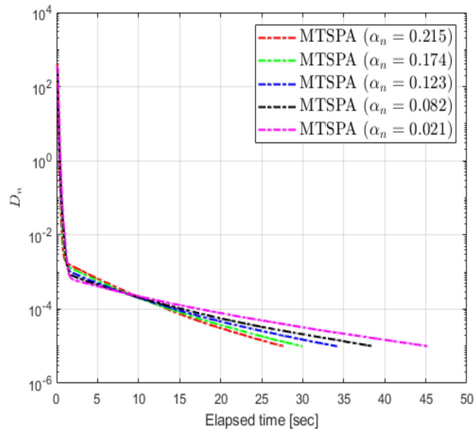

FIGURE 4. Algorithm 1 in term of elapsed time respect to different values of α_n

4.1.3. *Algorithm comparison with existing algorithms.* Figures 5 and 6 illustrate the comparison of our proposed algorithm with the already existing algorithm that appears in papers [16, 25]. For numerical results, we compare Algorithm 1 (shortly, MTSPA) with Extragradient method [25] (shortly, EgA) and Two-Step Proximal Algorithm (shortly, TSPA)[16]. The results are shown in Figures 5 and 6 and Table 5.

TABLE 5. The numerical results for Figure 5-6

Algo. name	λ	α_n	x	iter.	time	TOL
EgA	0.089	0.123	$(46.6523, 32.1467, 15.0011, 25.1426, 10.8359, 10.8359)^T$	3349	233.335939	$\epsilon = 10^{-5}$
TSPA	0.089	0.123	$(46.6541, 32.1516, 14.9981, 23.9903, 11.2888, 11.5298)^T$	724	48.766912	$\epsilon = 10^{-5}$
MTSPA	0.089	0.123	$(46.6539, 32.1726, 14.9774, 24.0496, 11.1620, 11.5975)^T$	514	34.186107	$\epsilon = 10^{-5}$

5. CONCLUSION

The paper suggests the modification of the one existing algorithm with the help of subgradient and inertial technique and theorem of weak convergence is established. Due to these actions, we improve the efficiency of the algorithm in the term of both number of iteration and elapsed time. In the end, we also discuss our results for one equilibrium model and also observe that which extrapolation factor in inertial step is work better than all others. These numerical results have also confirmed that the algorithm with inertial effects seems to work better than without inertial effects.

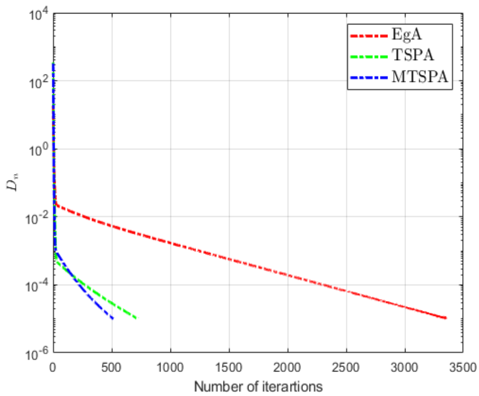

FIGURE 5. Algorithm 1 comparison with other algorithms in term of number of iterations

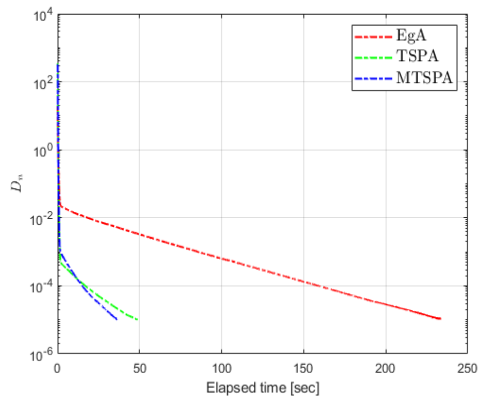

FIGURE 6. Algorithm 1 comparison with other algorithms in term of elapsed time

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