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Dedicated to Prof. Billy E. Rhoades on the occasion of his 90^{th} *anniversary*

A Stackelberg-population competition model via variational inequalities and fixed points

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ABSTRACT. In this paper, we introduce and study a new Stackelberg-population competition model which captures the desired features of both population games and Stackelberg competition model within the same framework. We obtain some characterization results for the Stackelberg-population equilibrium response set and the Stackelberg-population equilibrium leader set by using the variational inequality technique and Brouwer's fixed point theorem. We also show an existence theorem of Nash equilibrium for Stackelberg-population competition model under some mild conditions. Finally, we give an example to illustrate our main results.

1. INTRODUCTION

Population games provide a unified framework for studying strategic interactions with the following properties: (i) the number of agents is large; (ii) individual agents are small; (iii) agents interact anonymously; (iv) the number of roles is finite; (v) payoffs are continuous. As pointed out by Sandholm [28]: "Applications of population games range from economics (externalities, macroeconomic spillovers, centralized markets) to biology (animal conflict, genetic natural selection), transportation science (highway network congestion, mode choice), and computer science (selfish routing of Internet traffic)". We note that various theoretical results, numerical algorithms and applications have been studied extensively for population games in the literature (see, for example, [2, 6, 8, 15, 17, 25, 27, 28, 29, 30] and the references therein).

Following Sandholm [28], we recall some definitions concerned with the population games. Let $\mathcal{P} = \{1, \cdots, P\}$ be a society, where $P \geq 1$ is the number of populations. The set of strategies for population $p \in \mathcal{P}$ is denoted by $S^p = \{1, 2, \cdots, n^p\}$, where n^p is the number of strategies in S^p. The total number of pure strategies in society $\mathcal{P}$ is $n = \sum_{p\in\mathcal{P}} n^p$. During the game play, for each $p \in \mathcal{P}$, the set of population states is denoted by

$$X^p = \left\{ x^p = (x_1^p, x_2^p, \cdots, x_{n^p}^p) \in \mathbb{R}_+^{n^p} : \sum_{i\in S^p} x_i^p = 1 \right\},$$

where the nonnegative scalar $x_i^p \in [0,1]$ represents the share of members in population p choosing strategy $i \in S^p$. In order to describe behavior in all populations at once, the set of social states is $X = \prod_{p\in\mathcal{P}} X^p = \{x = (x^1, x^2, \cdots, x^P) \in \mathbb{R}_+^n : x^p \in X^p\}$.

Now we take into account the payoff function of the population games. For each population p and each strategy $i \in S^p$, a continuous map $F_i^p : X \to \mathbb{R}$ is a payoff function

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for strategy $i \in S^p$, and the the payoff functions for all strategies in S^p is denoted by $F^p = (F_1^p, F_2^p, \cdots, F_{n^p}^p) : X \to \mathbb{R}^{n^p}$. Then a population game can be identified with its payoff function: $F = (F^1, F^2, \cdots, F^P) : X \to \mathbb{R}^n$.

A social state $x \in X$ is said to be a Nash equilibrium of the population game with payoff function F if in each population, every strategy in use earns the maximal payoff:

$$NE(F) = \{x \in X : x_i^p > 0 \Rightarrow F_i^p(x) \geq F_j^p(x),\ \forall i, j \in S^p,\ \forall p \in \mathcal{P}\}.$$

On the other hand, it is well known that Stackelberg equilibrium problems play an important role in the study of economics, finance, risk management, design of mechanical structures, migration problems, transportation, internet advertising, resources allocation, minimax mathematical programming and decision science (see, for example, [1, 3, 7, 12, 13, 14, 18, 19, 22, 24] and the references therein). However, in some practical world, leaders and followers may appear in the form of populations. Thus, one natural question is: can we extend the population game to the Stackelberg-population competition model? The main purpose of this paper is to make an attempt in this new direction.

Following the work of [22, 28], we are now to propose a Stackelberg-population competition model as follows. Consider a hierarchical society $\mathcal{H}$, where there is a single population in the leader level, called the leader population, and P populations in the follower level, called the follower populations $\mathcal{P}$. The set of strategy for the leader population is denoted as $S^0 = \{1, 2, \cdots, m\}$ and the set of population state for the leader population is defined as

$$X = \left\{ x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}_+^m : \sum_{i \in S^0} x_i = 1 \right\},$$

where x_i is the proportion of agents who choose strategy $i \in S^0$.

Then concerned the follower populations $\mathcal{P} = \{1, 2, \cdots, P\}$, the set of strategies for population $p \in \mathcal{P}$ is denoted by $S^p = \{1, 2, \cdots, n^p\}$, where n^p is the number of strategies in S^p, and $n = \sum_{p \in \mathcal{P}} n^p$ is the total number of strategies in the follower populations $\mathcal{P}$. The set of population states for population p is defined by

$$Y^p = \left\{ y^p = (y_1^p, y_2^p, \cdots, y_{n^p}^p) \in \mathbb{R}_+^{n^p} : \sum_{i \in S^p} y_i^p = 1 \right\},$$

where the scalar y_i^p represents the share of members in population p choosing strategy $i \in S^p$. Moreover, the set of social states in the follower level is denoted by $Y = \prod_{p \in \mathcal{P}} Y^p = \{y = (y^1, y^2, \cdots, y^P) \in \mathbb{R}_+^n : y^p \in Y^p\}$. It is easy to see that both X and Y are bounded, closed and convex.

If we take the set of strategies as fixed, we can identify a game with payoff functions of leader and follower populations. Let $F_i : X \times Y \to \mathbb{R}$ be the payoff function to strategy $i \in S^0$ and $F = (F_1, F_2, \cdots, F_m) : X \times Y \to \mathbb{R}^m$ be the leader population's payoff function for all strategies in S^0.

Let $G_i^p : X \times Y \to \mathbb{R}$ be the payoff function for any strategy $i \in S^p$ and any population p in the follower populations $\mathcal{P}$. Then the payoff functions for all strategies in S^p can be defined as $G^p = (G_1^p, G_2^p, \cdots, G_{n^p}^p) : X \times Y \to \mathbb{R}^{n^p}$ and the payoff function for the follower populations $\mathcal{P}$ can be denoted as $G = (G^1, G^2, \cdots, G^P) : X \times Y \to \mathbb{R}^n$.

Concerning the Stackelberg-population competition model, the leader population moves first and followed by the follower populations. Thus, the first step is to determine the Stackelberg-population equilibrium response set as

$$R_{SPE}(x) = \left\{ y \in Y : y_i^p > 0 \Rightarrow G_i^p(x, y) \geq G_j^p(x, y),\ \forall\, p \in \mathcal{P},\ \forall\, i, j \in S^p \right\}$$

for each fixed $x \in X$. Now assume that $R_{SPE}(x) \neq \emptyset$ for each $x \in X$, the concluding step (for the leader population) is to determine the Stackelberg-population equilibrium leader set as

$$S_{SPE} = \left\{ x \in X : x_i > 0 \Rightarrow F_i(x, r(x)) \geq F_j(x, r(x)),\ \forall\, i, j\ \in S^0 \right\},$$

where $r(x)$ is a selection of the set-valued mapping $R_{SPE}(x)$.

Clearly, if the leader population and the follower populations are in the same level, then the Stackelberg-population competition model reduces to the classical population game. We would like to mention that the Stackelberg-population competition model captures the desired features of both population games and Stackelberg competition model within the same framework. Thus, it is important and interesting to study the Stackelberg-population competition model under some mild conditions.

The rest of this paper is organized as follows. The next section presents some basic concepts and necessary lemmas. In Section 3, by employing the variational inequality technique and Brouwer's fixed point theorem, some characterization results are given for the Stackelberg-population equilibrium response set and the Stackelberg-population equilibrium leader set. Moreover, an existence theorem of Nash equilibrium for Stackelberg-population competition model is obtained under some mild conditions. Finally, we summarize this paper in Section 4.

2. Preliminaries

In this section, we recall some basic known notions and lemmas that are essential for our further results.

Definition 2.1. Let $K \subset \mathbb{R}^n$ be a nonempty set. A function $f : K \to \mathbb{R}^n$ is said to be pseudomonotone if for any $x, y \in K$,

$$\langle f(x), y - x \rangle \geq 0 \Rightarrow \langle f(y), y - x \rangle \geq 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

Definition 2.2. Let X and Y be two metric spaces. A set-valued mapping $F : X \to 2^Y$ is said to be lower semicontinuous (l.s.c.) at $x_0 \in X$ if, for any $y \in F(x_0)$ and any neighborhood $V(y)$ of y, there exists a neighborhood $U(x_0)$ of x_0 such that $F(x) \bigcap V(y) \neq \emptyset$ for every $x \in U(x_0)$.

We say that F is l.s.c. on X if it is l.s.c. at each point $x \in X$.

Lemma 2.1. [4] *A set valued mapping $F : X \to 2^Y$ is l.s.c. at $x_0 \in X$ if and only if for any sequence $\{x_n\} \subset X$ with $x_n \to x_0$ and for any $y \in F(x_0)$, there exists $\{y_n\} \subset F(x_n)$ such that $y_n \to y$.*

Definition 2.3. Let X and Y be two metric spaces. A single-valued mapping $f : X \to Y$ is said to be a selection of a set-valued mapping $F : X \to 2^Y$ if $f(x) \in F(x)$ for every $x \in X$.

Lemma 2.2. [20] *Let F be an l.s.c. set-valued mapping with closed convex values from a compact metric space X to a Banach space Y. Then F has a continuous selection.*

Definition 2.4. Let $K \subset \mathbb{R}^n$ be a nonempty set. The metric projection $P_K : \mathbb{R}^n \to K$ of $x \in \mathbb{R}^n$ to K is defined by

$$P_K(x) = \{y \in K : \|y - x\| = \inf_{z \in K} \|x - z\|\}, \quad \forall x \in K.$$

It is well known that $P_K(x) \neq \emptyset$ when K is closed. Furthermore, if K is closed and convex, then we have the following characterization.

Lemma 2.3. [21] *Assume that $K \subset \mathbb{R}^n$ is a closed, convex and nonempty subset. Then*

$$x = P_K(y) \Leftrightarrow \langle y - x, x - z\rangle \geq 0, \quad \forall z \in K.$$

Moreover, the mapping P_K is non-expansive, i.e.

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

3. Main results

In this section, we first show characterization results for the Stackelberg-population equilibrium response set and Stackelberg-population equilibrium leader set.

For any given $x \in X$, to determine the elements of $R_{SPE}(x)$, we define the Stackelberg-population variational response set as follows:

$$R_{SPV}(x) = \{\bar{y} \in Y : \langle G(x, \bar{y}), y - \bar{y}\rangle \leq 0, \forall y \in Y\}.$$

Theorem 3.1. *For every fixed $x \in X$, one has $R_{SPE}(x) = R_{SPV}(x)$.*

Proof. Suppose that $\bar{y} \in R_{SPE}(x)$. Then, for each $p \in \mathcal{P}$ and $i \in S^p$,

$$\bar{y}^p_i > 0 \Rightarrow G^p_i(x, \bar{y}) \geq G^p_j(x, \bar{y}), \quad \forall j \in S^p.$$

This shows that

$$\bar{y}^p_i G^p_i(x, \bar{y}) \geq \bar{y}^p_i G^p_j(x, \bar{y}), \quad \forall j \in S^p.$$

Summing with respect to $i \in S^p$ on both side of the above inequality and noting that $\sum_{i\in S^p} \bar{y}^p_i = 1$, we have

$$\langle G^p(x, \bar{y}), \bar{y}^p\rangle \geq G^p_j(x, \bar{y}), \quad \forall j \in S^p.$$

For any $y^p = (y^p_1, y^p_2, \cdots, y^p_{n^p}) \in Y^p$, multiplying both sides of above inequality by y^p_j, summing with respect to $j \in S^p$ and noting that $\sum_{j\in S^p} y^p_j = 1$, one has

$$\langle G^p(x, \bar{y}), \bar{y}^p\rangle \geq \langle G^p(x, \bar{y}), y^p\rangle, \quad \forall p \in \mathcal{P}.$$

Thus, for each $p \in \mathcal{P}$ and $y^p \in Y^p$, we know that $\langle G^p(x, \bar{y}), y^p - \bar{y}^p\rangle \leq 0$ is true and so $\bar{y} = (\bar{y}^1, \bar{y}^2, \cdots, \bar{y}^P) \in R_{SPV}(x)$.

Conversely, assume that $\bar{y} \in R_{SPV}(x)$. Then

$$\langle G(x, \bar{y}), y - \bar{y}\rangle \leq 0, \quad \forall y \in Y.$$

For any $p \in \mathcal{P}$ and $y^p \in Y^p$, take $y = (y^p; \bar{y}^{-p}) \in Y$, where $-p = \mathcal{P}\backslash\{p\}$. Then

$$\langle G^p(x, \bar{y}), \bar{y}^p\rangle \geq \langle G^p(x, \bar{y}), y^p\rangle, \quad \forall y^p \in Y^p,\ \forall p \in \mathcal{P}.$$

Since $\bar{y}^p \in Y^p$, without loss of generality, assume that $\bar{y}^p_i > 0$ for a certain $i \in S^p$. For any $j \in S^p$, set

$$\hat{y}^p = (\bar{y}^p_1, \bar{y}^p_2, \cdots, \bar{y}^p_{i-1}, 0, \bar{y}^p_{i+1}, \cdots, \bar{y}^p_{j-1}, \bar{y}^p_i + \bar{y}^p_j, \bar{y}^p_{j+1}, \cdots, \bar{y}^p_{n^p}).$$

Then $\hat{y}^p \in Y^p$. Replacing y^p by $\hat{y}^p$ in the above inequality, we have

$$\langle G^p(x, \bar{y}), \bar{y}^p\rangle \geq \langle G^p(x, \bar{y}), \hat{y}^p\rangle.$$

This shows that

$$G^p_i(x, \bar{y})\bar{y}^p_i + G^p_j(x, \bar{y})\bar{y}^p_j \geq G^p_j(x, \bar{y})(\bar{y}^p_i + \bar{y}^p_j), \quad \forall j \in S^p$$

and so

$$(G^p_i(x, \bar{y}) - G^p_j(x, \bar{y}))\bar{y}^p_i \geq 0, \quad \forall j \in S^p.$$

Since $\bar{y}^p_i > 0$, it clearly implies that

$$G^p_i(x, \bar{y}) \geq G^p_j(x, \bar{y}), \quad \forall j \in S^p.$$

That is,

$$\bar{y}_i^p > 0 \Rightarrow G_i^p(x, \bar{y}) \geq G_j^p(x, \bar{y}), \quad \forall j \in S^p.$$

Due to the arbitrariness of $p \in \mathcal{P}$ and $i \in S^p$, we have $\bar{y} \in R_{SPE}(x)$. This completes the proof. □

Assume that $R_{SPV}(x) \neq \emptyset$ for every $x \in X$ and there is a continuous selection $r : X \to Y$ of the set-valued mapping $R_{SPV}(x)$. Then the Stackelberg-population variational leader set can be defined as

$$S_{SPV} = \{\bar{x} \in X : \langle F(\bar{x}, r(\bar{x})), x - \bar{x} \rangle \leq 0, \forall x \in X\}.$$

Similar to the proof of Theorem 3.1, we have the following result for the Stackelberg-population variational leader set.

Theorem 3.2. *Assume that $R_{SPV}(x)$ is nonempty and there exists a continuous selection $r(x)$ of $R_{SPV}(x)$ for every $x \in X$. Then $S_{SPE} = S_{SPV}$.*

Since the equivalence of variational inequalities and projection mappings under some assumption, it is more easier to compute the Stackelberg-population variational response set $R_{SPV}(x)$ than the Stackelberg-population equilibrium response set $R_{SPE}(x)$. We can locate the elements of $R_{SPE}(x)$ among the points in $R_{SPV}(x)$ which can be characterized by fixed points of the metric projection mapping into the social states Y.

Theorem 3.3. *Let $G(x, y)$ be continuous. Then $R_{SPE}(x) \neq \emptyset$ for every $x \in X$.*

Proof. For any fixed $x \in X$ and $\alpha > 0$, define a mapping $A_\alpha^x : Y \to Y$ by

$$A_\alpha^x(y) = P_Y\left[y + \alpha G(x, y)\right], \quad \forall y \in Y.$$

First we show that $\bar{y} \in R_{SPV}(x)$ if and only if $A_\alpha^x(\bar{y}) = \bar{y}$. In fact, $\bar{y} \in R_{SPV}(x)$ if and only if

$$\langle \bar{y} + \alpha G(x, \bar{y}) - \bar{y}, \bar{y} - y \rangle \geq 0, \quad \forall y \in Y$$

for all/some $\alpha > 0$. Thus, according to Lemma 2.3, it is equivalent to

$$\bar{y} = P_Y\left[\bar{y} + \alpha G(x, \bar{y})\right].$$

Thus, we have

$$R_{SPV}(x) = \{\bar{y} \in Y : P_Y\left[\bar{y} + \alpha G(x, \bar{y})\right] = \bar{y}\} = \{\bar{y} \in Y : A_\alpha^x(\bar{y}) = \bar{y}\}. \tag{3.1}$$

Next we prove that $R_{SPE}(x) \neq \emptyset$. In fact, since P_Y is non-expensive and $G(\cdot, y)$ is continuous, we know that A_α^x is a continuous mapping. Clearly, Y is convex and compact. By Brouwer's fixed point theorem, the mapping A_α^x has a fixed point $y^* \in Y$ and so $y^* \in R_{SPV}(x)$. According to (3.1) and Theorem 3.1, we can see that $y^* \in R_{SPE}(x)$. □

Now we turn to characterize the Stackelberg-population equilibrium leader set S_{SPE}. To this end, we need to introduce the Stackelberg-population Minty variational response set as follows:

$$R_{SPMV}(x) = \{\bar{y} \in Y : \langle G(x, y), y - \bar{y} \rangle \leq 0, \forall y \in Y\}.$$

Theorem 3.4. *Let $-G(x, y)$ be continuous and pseudomonotone with respect to the second variable. Then $R_{SPV}(x) = R_{SPMV}(x)$.*

Proof. Let $\bar{y} \in R_{SPV}(x)$. Then the pseudomonotonicity of $-G(\cdot, y)$ shows that

$$\langle G(x, \bar{y}), y - \bar{y} \rangle \leq 0 \Rightarrow \langle G(x, y), y - \bar{y} \rangle \leq 0, \quad \forall y \in Y$$

and so $\bar{y} \in R_{SPMV}(x)$.

Conversely, assume that $\bar{y} \in R_{SPMV}(x)$. Then the convexity of Y implies that $y = \bar{y} + t(z - \bar{y}) \in Y$ for all $z \in Y$ and $t \in (0,1)$. From the fact that $\bar{y} \in R_{SPMV}(x)$, we have

$$\langle G(x, \bar{y} + t(z - \bar{y})), t(z - \bar{y})\rangle \leq 0$$

and so

$$\langle G(x, \bar{y} + t(z - \bar{y})), z - \bar{y}\rangle \leq 0.$$

Since $G(\cdot, y)$ is continuous, taking $t \to 0$, one has

$$\langle G(x, \bar{y}), z - \bar{y}\rangle \leq 0, \quad \forall z \in Y.$$

This implies that $\bar{y} \in R_{SPV}(x)$. Therefore, $R_{SPV}(x) = R_{SPMV}(x)$. □

Theorem 3.5. *Let $-G(x, y)$ be continuous and pseudomonotone with respect to y. Then the set-valued mapping $R_{SPV}(x)$ has a continuous selection $r(x)$.*

Proof. For any given $x \in X$, Theorem 3.3 shows that $R_{SPV}(x) \neq \emptyset$. By the compactness of X and Lemma 2.2, we only need to show that R_{SPV} is an l.s.c. set-valued mapping with closed and convex values. Since G is continuous, we can see that $R_{SPV}(x)$ is closed.

Now we check that $R_{SPV}(x)$ is convex. Consider the Stackelberg-population Minty variational response set $R_{SPMV}(x)$. Letting $u, v \in R_{SPMV}(x)$, one has

$$\langle G(x, y), y - u\rangle \leq 0, \quad \langle G(x, y), y - v\rangle \leq 0.$$

For any $t \in (0, 1)$, it follows that

$$\langle G(x, y), y - (tu + (1 - t)v)\rangle \leq 0$$

and so $tu + (1 - t)v \in R_{SPMV}(x)$. This means that $R_{SPMV}(x)$ is convex. By Theorem 3.4, we know that $R_{SPV}(x)$ is also convex.

Next we show that R_{SPV} is l.s.c.. In fact, let $\{x_n\} \subset X$ be a sequence that converges to $x_0 \in X$ and $\bar{y} \in R_{SPV}(x_0)$. Then it is easy to have

$$\langle G(x_0, \bar{y}), y - \bar{y}\rangle \leq 0, \quad \forall y \in Y. \tag{3.2}$$

Note that $R_{SPV}(x_0)$ is a closed set in Y. For the fact that $\bar{y} \in R_{SPV}(x_0)$, there is a sequence $\{y_n\} \subset R_{SPV}(x_0)$ such that $\{y_n\}$ converges to $\bar{y}$. By the continuity of $G(x, y)$ and the compactness of Y, if follows from (3.2) that there exists a positive integer N such that, for any $n > N$,

$$\langle G(x_n, y_n), y - y_n\rangle \leq 0, \quad \forall y \in Y.$$

This shows that $\{y_n\} \subset R_{SPV}(x_n)$. Thus, by Lemma 2.1, we know that $R_{SPV}(x)$ is l.s.c. and so Lemma 2.2 implies that $R_{SPV}(x)$ has a continuous selection $r(x)$. □

Theorem 3.6. *Let $F(x, y)$ and $G(x, y)$ be continuous functions. Moreover, assume that $-G(x, y)$ is pseudomonotone with respect to $y \in Y$. Then $S_{SPE} \neq \emptyset$.*

Proof. By Theorem 3.5, we know that the set-valued mapping $R_{SPE}(x)$ has a continuous selection $r(x)$. Thus, for any given $\beta > 0$, we can introduce the mapping $B_\beta : X \to X$ by setting

$$B_\beta(x) = P_X\left[x + \beta F(x, r(x))\right], \quad \forall x \in X.$$

Since $F(x, y)$ and $r(x)$ are continuous and P_X is non-expensive, we know that the mapping B_β is continuous. By Brouwer's fixed point theorem, $B_\beta(x)$ has a fixed point $x^* \in X$. It is easy see that $x^* \in S_{SPV}$ and so Theorem 3.2 implies that $x^* \in S_{SPE}$. Thus, $S_{SPE} \neq \emptyset$. □

Theorem 3.7. *Under the assumptions of Theorem 3.6, the Stackelberg-population competition model has a Nash equilibrium.*

Proof. Clearly, Theorem 3.6 shows that there is $x^* \in X$ such that $x^* \in S_{SPE}$. Let $y^* = r(x^*)$. Then it is easy to see that (x^*, y^*) is a Nash equilibrium of the Stackelberg-population competition model. □

At the end of this section, we give a realistic example to comprehend how the Stackelberg-population competition model works in economics. In order to describe an industry structure, Cournot competition model is a valid economic model in which firms compete on the quantities of output they will produce at the same time. Sandholm turn the model to the population case (see Example 3.1.3 in [28]), following from this we can also use our model to describe the production competition in a two-level society. There is a leader population of firms that choose production quantities from the set $S^0 = \{1, \cdots, m\}$ and a follower population of firms choose production quantities from $S^1 = \{1, \cdots, n\}$. The firms' aggregate production is denoted as

$$a(x, y) = \sum_{i \in S^0} i x_i + \sum_{j \in S^1} j y_j.$$

The inverse demand function of aggregate production $p : \mathbb{R}^+ \to \mathbb{R}^+$ is decreasing. What's more, let $c_0 : S^0 \to \mathbb{R}$ and $c_1 : S^1 \to \mathbb{R}$ be the leader and follower firms' production cost functions respectively. Then the payoff to a firm of leader population producing quantity $i \in S^0$ at product state $X \times Y$ is

$$F_i(x, y) = ip(a(x, y)) - c_0(i),$$

and the payoff to a follower firm producing quantity $j \in S^1$ is

$$G_j(x, y) = jp(a(x, y)) - c_1(j).$$

Assume that the inverse demand function p and two cost functions c_0 and c_1 are continuous. Now we check that $G(x, y)$ is monotone with respect to y. In fact, for any given $y, z \in Y$, one has

$$\langle G(x, y) - G(x, z), y - z\rangle = [p(a(x, y)) - p(a(x, z))] \sum_{j \in S^1} j(y_j - z_j).$$

We consider two cases. In the first case, assume that

$$\sum_{j \in S^1} j(y_j - z_j) \geq 0.$$

Then

$$a(x, y) - a(x, z) = \sum_{j \in S^1} j y_j - \sum_{j \in S^1} j z_j \geq 0.$$

Notice that the inverse demand function p is decreasing. We know that

$$p(a(x, y)) - p(a(x, z)) \leq 0$$

and so

$$\langle G(x, y) - G(x, z), y - z\rangle \leq 0.$$

The same result we will get in the other case $\sum_{j \in S^1} j(y_j - z_j) < 0$. Therefore, $-G(x, y)$ is monotone with respect to y. By Theorem 3.7, we can obtain the existence of the Nash equilibrium for the Stackelberg-population competition model. In other words, the firms in both leader population and follower population have the optimal production.

4. Conclusions

This paper is devoted to investigate Nash equilibrium for a new Stackelberg-population competition model by employing the variational inequality technique and Brouwer's fixed point theorem. In order to describe optimal behavior of the leader and follower populations, the Stackelberg-population equilibrium response set and Stackelberg-population equilibrium leader set are introduced. The main contributions of this paper are as follows: (i) the traditional population game is generalized to Stackelberg-population competition model which captures the desired features of both population games and Stackelberg competition model within the same framework; (ii) some characterization results are given for the Stackelberg-population equilibrium response set and the Stackelberg-population equilibrium leader set by using the variational inequality technique and the Brouwer fixed point theorem; (iii) An existence theorem of Nash equilibria for Stackelberg-population competition model is proved under some mild conditions.

We would like to mention that the variational inequality technique and the fixed point theorem play important roles for obtaining our main results. Thus, it is would be interesting to propose some algorithms for computing Nash equilibrium of Stackelberg-population competition model by applying some known algorithms for solving variational inequalities and fixed point problems [9, 10, 11]. Moreover, turning our eyes to the social states and the payoff functions, it would be important to study Stackelberg-population quasi-equilibrium problems, nonsmooth Stackelberg-population equilibrium problems and stochastic Stackelberg-population equilibrium problems.

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References

[1] Amir, R. and Grilo, I., *Stackelberg versus Cournot Equilibrium*, Games Econ. Behav., **26**(1999), 1–21

[2] Arieli, I. and Young, H. P., *Fast convergence in population games*, Department of economics Discussion Paper Series, University of Oxford, 2011

[3] Askar, S. S., *Tripoly Stackelberg game model: One leader versus two followers*, Appl. Math. Compu., **328** (2018), 301–311

[4] Aubin, J. P. and Ekeland, I., *Applied Nonlinear Analysis*, Wiley, New York, 1984

[5] Balder, E. J., *A unifying pair of Cournot-Nash equilibrium existence results*, J. Econ. Th., **102** (2002), 437–470

[6] Blume, L. E., *Population Games*, In: Arthur, W.B., Durlauf, S.N. and Lane, D. A. (eds), The Economy as an Evolving Complex System II, pp. 425–460, Santa Fe Institute, Santa Fe, 1997

[7] D'Amato, E., Daniele, E., Mallozzi, L., Petrone, G. and Tancredi, S., *A hierarchical multiModal hybrid Stackelberg-Nash GA for a leader with multiple followers game*, In: Dynamics of Information Systems: Mathematical Foundations, pp. 267–280, Springer, New York, 2012

[8] Eshel, I. and Sansone, E., *Evolutionary and dynamic stability in continuous population games*, J. Math. Biol, **46** (2003), 445–459

[9] Facchinei, F. and Pang, J.S., *Finite-Dimensional Variational Inequalities and Complementarity Problems*, I and II, Springer, Berlin, 2003

[10] Glowinski, R., Lions, J. L. and Tremolieres, R., *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981

[11] Gürsoy, F., Karakaya, V. and Rhoades, B. E., *Data dependence results of new multi-step and S-iterative schemes for contractive-like operators*, Fixed Point Theory Appl **2013** (2013): 76, https://doi.org/10.1186/1687-1812-2013-76

[12] Han, Y. and Huang, N. J., *Existence and stability of solutions for a class of generalized vector equilibrium problems*, Positivity, **20** (2016), 829–846

[13] Julien, L. A., *A note on Stackelberg competition*, J. Econ., **103** (2011), 171–187

[14] Kicsiny, R., Varga, Z. and Scarelli, A., *Backward induction algorithm for a class of closed-loop Stackelberg games*, Eur. J. Oper. Res., **237** (2014), 1021–1036

[15] Lanni, A., *Learning correlated equilibria in population games*, Math. Social Sci., **42** (2001), 271–294

[16] Lahkar, R., *Large population aggregative potential games*, Dyn. Games Appl., **7** (2017), 443–467

[17] Lee, S. and Yi, Y., *Distributed sharing of base stations for greening: a population game approach*, In: Cheng J., Hossain E., Zhang H., Saad W., Chatterjee M. (eds) Game Theory for Networks, GameNets 2016, Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunications Engineering, vol 174. Springer, Cham, pp. 79–89

[18] Lu, J., Xiao, Y. B. and Huang, N. J., *A Stackelberg quasi-equilibrium problem via quasi-variational inequalities*, Carpathian J. Math., **34** (2018), 355–362

[19] Lu, J., Zhou, L. W., Xiao, Y. B. and Huang, N. J., *A nonsmooth Stackelberg equilibrium problem via mixed variational inequalities*, Carpathian J. Math., **35** (2019), 339–347

[20] Miachael, E., *Continuous selections*, Annas of Math., **63** (1956), 361–382

[21] Moskovitz, D. and Dines, L. L., *Convexity in a linear space with an inner product*, Duke Math. J., **5** (1939), 520–534

[22] Nagy, S., *Stackelberg equilibria via variational inequalities and projections*, J. Global Optim., **57** (2013), 821–828

[23] Nash, J., *Non-cooperative games*, Annals of Math., **54** (1951), 286–295

[24] Novak, A. J., Feichtinger, G. and Leitmann, G., *A differential game related to terrorism: nash and stackelberg strategies*, J. Optim. Theory Appl., **144** (2010), 533–555

[25] Reluga, T. C. and Galvani, A. P., *A general approach for population games with application to vaccination*, Math. Biosci, **230** (2011), 67–78

[26] Sandholm, W. H., *Negative externalities and evolutionary implementation*, Rev. Eco. Stud., **72** (2005), 885–915

[27] Sandholm, W. H., *Large population potential games*, J. Econ. Th., **144** (2009), 1710–1725

[28] Sandholm, W. H., *Population Games and Evolutionary Dynamicss*, The MIT Press, Cambridge, Massachusetts, London, England, 2010

[29] Yang, G. H., Yang, H. and Song, Q. Q., *Stability of weighted Nash equilibria for multiobjective population games*, J. Nonlinear Sci. Appl., **9** (2016), 4167–4176

[30] Yang, G. H. and Yang, H., *Stability of weakly Pareto-Nash equilibria and Pareto-Nash equilibria for multiobjective population games*, Set-Valued Var. Anal., **25** (2017), 427–439

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