CARPATHIAN J. MATH. **36** (2020), No. 3, 341 - 349
Online version at https://www.carpathian.cunbm.utcluj.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Quasilinear differential equations with strongly unpredictable solutions

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ABSTRACT. The authors discuss the existence and uniqueness of asymptotically stable unpredictable solutions for some quasilinear differential equations. Two principal novelties are in the basis of this research. The first one is that all coordinates of the solution are unpredictable functions. That is, solutions are *strongly unpredictable*. Secondly, perturbations are *strongly unpredictable functions in the time variable*. Examples with numerical simulations are presented to illustrate the theoretical results.

1. INTRODUCTION AND PRELIMINARIES

The major part of the theory of differential equations is devoted to oscillations, since of demands of science and industry. Therefore, periodic, quasiperiodic and almost periodic solutions are in the main focus of researchers [6] - [9], [11] .

A new type of oscillations, unpredictable functions, has been introduced in the paper [1]. It was shown, in papers [1] and [4], that existence of an unpredictable solution implies Poincaré chaos for a special dynamics in a functional space. Consequently, study of unpredictable solutions is as much beneficial as is chaos. In the present paper we consider oscillations with all coordinates unpredictable. Since of the chaos, irregularity is observable in the oscillations. Another significant novelty is that perturbations are assumed to be nonlinear functions of the time and space variables, and they are strongly unpredictable in the time variable.

Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ will stand for the set of real and natural numbers, respectively. Additionally, the norm $\|u\|_1 = \sup_{t\in\mathbb{R}} \|u(t)\|$, where $\|u\| = \max_{1\le i\le p} |u_i|$, $u = (u_1,\dots,u_p), u_i \in \mathbb{R}, i = 1,2,...,p$, will be used. Correspondingly, for a square matrice $A = \{a_{ij}\}, i,j = 1,2,...,p$, the norm $\|A\| = \max_{i=1,\dots,p} \sum_{j=1}^{p} |a_{ij}|$, will be utilized.

The following definition is the starting point of our research.

Definition 1.1. [1] A uniformly continuous and bounded function $v : \mathbb{R} \to \mathbb{R}^p$ is unpredictable if there exist positive numbers ϵ_0, σ and sequences t_n, u_n both of which diverge to infinity such that $v(t+t_n) \to v(t)$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|v(t+t_n) - v(t)\| \ge \epsilon_0$ for each $t \in [u_n - \sigma, u_n + \sigma]$ and $n \in \mathbb{N}$.

In papers [3] - [5], an example of an unpredictable function was provided and it was shown that properties of unpredictable functions are convenient to be verified and they are easy for numerical simulations. Thus, existence of unpredictable solutions for the differential equation

$$x'(t) = Ax(t) + f(x) + g(t), \tag{1.1}$$

Received: 18.06.2019. In revised form: 08.09.2019. Accepted: 15.09.2019
2010 *Mathematics Subject Classification.* 34A34, 34C28, 34D20.
Key words and phrases. *oscillations, strongly unpredictable perturbations, strongly unpredictable solutions, unpredictable solutions, asymptotical stability.*
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where $g(t)$ is unpredictable function, was investigated.

In the present investigation we extend Definition 1.1 to the class of functions with several independent variables. The following new definition will be of use.

Definition 1.2. A continuous and bounded function $f(t,x) : \mathbb{R}\times G \to \mathbb{R}^p$, $f = (f_1, f_2, ..., f_p)$, $G \subset \mathbb{R}^p$ is a bounded domain, is unpredictable in t if it is uniformly continuous in t and there exist positive numbers ϵ_0, σ and sequences t_n, u_n both of which diverge to infinity such that $\sup\limits_G \|f(t + t_n, x) - f(t,x)\| \to 0$ as $n \to \infty$ uniformly on compact sets in $\mathbb{R}$ and $\inf\limits_G \|f(t + t_n, x) - f(t,x)\| \geq \epsilon_0$ for $t \in [u_n - \sigma, u_n - \sigma]$ and $n \in \mathbb{N}$.

The present study contains two principal novelties. The first one is that strongly unpredictable solutions are considered instead of unpredictable ones. Secondly, we consider nonlinear perturbations, which are functions unpredictable in the time variable. Thus, in the present paper we have significantly enlarged the set of systems, which can be investigated for unpredictable solutions. To this end, we shall need the following two new notions, which are analogues to the last two definitions.

Definition 1.3. A uniformly continuous and bounded function $v : \mathbb{R} \to \mathbb{R}^p, v = (v_1, \ldots, v_p)$, is strongly unpredictable if there exist positive numbers ϵ_0, σ and sequences t_n, u_n both of which diverge to infinity such that $v(t + t_n) \to v(t)$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $|v_i(t+t_n) - v_i(t)| \geq \epsilon_0$ for each $t \in [u_n - \sigma, u_n + \sigma]$, $i = 1, \ldots, p$ and $n \in \mathbb{N}$.

Definition 1.4. A continuous and bounded function $f(t,x) : \mathbb{R}\times G \to \mathbb{R}^p$, $f = (f_1, f_2, ..., f_p)$, $G \subset \mathbb{R}^p$ is a bounded domain, is strongly unpredictable in t if it is uniformly continuous in t and there exist positive numbers ϵ_0, σ and sequences t_n, u_n both of which diverge to infinity such that $\sup\limits_G \|f(t + t_n, x) - f(t,x)\| \to 0$ as $n \to \infty$ uniformly on compact sets in $\mathbb{R}$ and $\inf\limits_{[u_n-\sigma, u_n-\sigma]\times G} |f_i(t + t_n, x) - f_i(t,x)| > \epsilon_0$ for all $i = 1, \ldots, p$ and $n \in \mathbb{N}$.

Comparing the Definitions 1.1 and 1.3 as well as Definitions 1.2 and 1.4 one can find that an unpredictable function may admits some of coordinates which are not unpredictable scalar valued functions. While, each coordinate of a strongly unpredictable functon is an unpredictable function. That is, the set of all strongly unpredictable functions is a subclass of unpredictable functions.

The main object of the present paper is the system of quasilinear differential equations

$$x'(t) = Ax(t) + f(t,x), \tag{1.2}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^p$, p is a fixed natural number, all eigenvalues of the constant matrix $A \in \mathbb{R}^{p\times p}$ have negative real parts, $f : \mathbb{R} \times G \to \mathbb{R}^p, f = (f_1, \ldots, f_p)$, $G = \{x \in \mathbb{R}^p, \|x\| < H\}$, where H is a positive number. It is true that there exist two real numbers $K \geq 1$ and $\gamma < 0$ such that $\|e^{At}\| \leq Ke^{\gamma t}$ for all $t \geq 0$. Definition 1.2 implies that there exists a positive number M such that $\sup\limits_{\mathbb{R}\times G} \|f(t,x)\| = M < \infty$.

One can see that the main difference between system (1.1) and system (1.2) is that the perturbation in the former one is less general than that of the latter one.

The following conditions will be needed in the paper:

(C1) the function *f(t, x)* is strongly unpredictable in the sence of Definition 1.2;

(C2) there exists a positive constant L such that the function $f(t,x)$ satisfies the inequality $\|f(t,x_1) - f(t,x_2)\| \leq L\,\|x_1 - x_2\|$ for all $t \in \mathbb{R}, x_1, x_2 \in G$;

(C3) $2H(L + \|A\|) < \epsilon_0$;

(C4) $\gamma < -\dfrac{KM}{H}$;

(C5) $\gamma < -KL$.

Our purpose is to prove that system (1.2) possesses a unique strongly unpredictable solution, provided that the function $f(t,x)$ is strongly unpredictable in t. Moreover, we prove that the solution is uniformly globally exponentially stable. Additionally, existence of an unpredictable solution, which is not strongly unpredictable, is considered for the system.

2. Main results

Let U be the set of all uniformly continuous functions $\psi(t) = (\psi_1, \psi_2, ..., \psi_p)$, $\psi_i \in \mathbb{R}$, $i = 1, 2, ..., p$, such that $\|\psi\|_1 < H$, and $\psi(t+t_n) \to \psi(t)$ as $n \to \infty$ uniformly on each closed and bounded interval of the real axis, where sequence t_n is the same as for function $f(t,x)$ in system (1.2).

According to the theory of differential equations [10], a function $\omega(t) = (\omega_1, \omega_2, ..., \omega_p)$ bounded on the whole real axis is a solution of system (1.2) if and only if the integral equation

$$\omega(t) = \int_{-\infty}^{t} e^{A(t-s)} f(t, \omega(s)) ds, \ t \in \mathbb{R}, \tag{2.3}$$

is satisfied.

Lemma 2.1. *Suppose that conditions $(C1) - (C5)$ are valid, then the system (1.2) possesses a unique solution $\omega(t) \in U$.*

Proof. Define an operator Π on U such that

$$\Pi\psi(t) = \int_{-\infty}^{t} e^{A(t-s)} f(s, \psi(s)) ds, \ t \in \mathbb{R}. \tag{2.4}$$

Fix an arbitrary function $\psi(t)$ that belongs to U. We have that

$$\|\Pi\psi(t)\| \leq \int_{-\infty}^{t} \|e^{A(t-s)}\| \|f(s, \psi(s))\| ds \leq \frac{KM}{|\gamma|}$$

for all $t \in \mathbb{R}$. Therefore, by condition (C4) it is true that $\|\Pi\psi\|_1 < H$.

Fix an arbitrary positive number ϵ and a closed interval $[a,b]$, $-\infty < a < b < \infty$, of the real axis. We will show that for sufficiently large n it is true that $\|\Pi\psi(t+t_n) - \Pi\psi(t)\| < \epsilon$ on $[a,b]$. Let us choose two numbers $c < a$, and $\xi > 0$ satisfying

$$\frac{2KM}{|\gamma|} e^{\gamma(a-c)} < \frac{\epsilon}{2} \tag{2.5}$$

and

$$\frac{K}{|\gamma|} \xi (L+1)[1 - e^{\gamma(b-c)}] < \frac{\epsilon}{2}. \tag{2.6}$$

Consider n sufficiently large such that $\|f(t+t_n, x) - f(t,x)\| < \xi$ and $\|\psi(t+t_n) - \psi(t)\| < \xi$ for $t \in [c,b]$ and $x \in G$. Then, the inequality

$$\|\Pi\psi(t+t_n) - \Pi\psi(t)\| \leq \int_{-\infty}^{c} \|e^{A(t-s)}\| \|f(s+t_n, \psi(s+t_n)) - f(s, \psi(s))\| ds +$$
$$\int_{c}^{t} \|e^{A(t-s)}\| \|f(s+t_n, \psi(s+t_n)) - f(s, \psi(s))\| ds \leq \int_{-\infty}^{c} 2KMe^{\gamma(t-s)} ds +$$
$$\int_{c}^{t} \xi(L+1)Ke^{\gamma(t-s)} ds \leq \frac{2}{|\gamma|} KMe^{\gamma(a-c)} + \frac{K}{|\gamma|} \xi (L+1)[1 - e^{\gamma(b-c)}] \tag{2.7}$$

is correct for all $t \in [a,b]$.

By inequalities (2.5) and (2.6) it is true that $\|\Pi\psi(t+t_n) - \Pi\psi(t)\| < \epsilon$ for $t \in [a,b]$, and, therefore, $\Pi\psi(t+t_n) \to \Pi\psi(t)$ uniformly as $n \to \infty$ on each closed and bounded interval of $\mathbb{R}$.

It is easy to verify that $\Pi\psi(t)$ is a uniformly continuous function. Thus, the set U is invariant for the operator Π.

We proceed to show that the operator $\Pi : U \to U$ is contractive. Let $u(t)$ and $v(t)$ be members of U. Then, we obtain that

$$\|\Pi u(t) - \Pi v(t)\| \leq \int_{-\infty}^{t} \|e^{A(t-s)}\| \|f(s,u(s)) - f(s,v(s))\| ds \leq$$

$$\int_{-\infty}^{t} K e^{\gamma(t-s)} L \|u(s) - v(s)\| ds \leq \frac{KL}{|\gamma|} \|u(t) - v(t)\|_1$$

for all $t \in \mathbb{R}$. Therefore, the inequality $\|\Pi u - \Pi v\|_1 \leq \dfrac{KL}{|\gamma|} \|u - v\|_1$ holds, and according to condition $(C5)$ the operator $\Pi : U \to U$ is contractive. The next task is to show that the space U is complete. Consider a Cauchy sequence $\phi_k(t)$ in U, which converges to a limit function $\phi(t)$ on $\mathbb{R}$. Fix a closed and bounded interval $I \subset \mathbb{R}$. We have that

(2.8) $$\|\phi(t+t_n) - \phi(t)\| < \|\phi(t+t_n) - \phi_k(t+t_n)\| + \|\phi_k(t+t_n) - \phi_k(t)\| + \|\phi_k(t) - \phi(t)\|.$$

Now, one can take sufficiently large p and k such that each term on right hand-side of (2.8) is smaller than $\frac{\epsilon}{3}$ for an arbitrary small ϵ and $t \in I$. The inequality implies that $\|\phi(t+t_n) - \phi(t)\| < \epsilon$ on I. That is the sequence $\phi(t+t_p)$ uniformly converges to $\phi(t)$ on I. Likewise, one can check that the limit function is bounded and uniformly continuous [10]. The completeness of U is proved. By contraction mapping theorem there exists the unique fixed point, $\omega(t) \in U$, of the operator Π, which is the unique bounded solution of the system (1.2). The lemma is proved. □

Theorem 2.1. *If conditions $(C1)-(C5)$ are fulfilled, then system (1.2) admits a unique uniformly globally exponentially stable strongly unpredictable solution.*

Proof. According to Lemma 2.1, system (1.2) has a unique solution $\omega(t) \in U$. What is left is to show that there exist a positive integer number l and positive number κ such that $|\omega_i(t+t_n) - \omega_i(t_n)| \geq \frac{\epsilon_0}{2l}$ for each $i = 1, ..., p$, $t \in [u_n - \kappa, u_n + \kappa]$ and all $n \in \mathbb{N}$.

One can show that there exist a positive number κ_1 and a positive integer l such that the following inequalities are valid,

(2.9) $$\kappa_1 < \sigma,$$

(2.10) $$\kappa_1 \Big(\epsilon_0 - 2H(L + \|A\|)\Big) > \frac{3\epsilon_0}{2l}.$$

Let the numbers κ_1 and l as well as a number $n \in \mathbb{N}$, and $i = 1, 2, ..., p$, be fixed.

Using the relations

$$\omega_i(t) = \omega_i(u_n) + \int_{u_n}^{t} \sum_{j=1}^{p} a_{ij}\omega_j(s) ds + \int_{u_n}^{t} f_i(s, \omega(s)) ds$$

and

$$\omega_i(t+t_n) = \omega_i(t_n + u_n) + \int_{u_n}^{t} \sum_{j=1}^{p} a_{ij}\omega_j(s+t_n) ds + \int_{u_n}^{t} f_i(s+t_n, \omega(s+t_n)) ds$$

we obtain that

$$\omega_i(t+t_n) - \omega_i(t) = \omega_i(u_n+t_n) - \omega_i(u_n) + \int_{u_n}^{t} \sum_{j=1}^{p} a_{ij}[\omega_j(s+t_n) - \omega_j(s)]ds +$$

$$\int_{u_n}^{t} [f_i(s+t_n, \omega(s+t_n)) - f_i(s, \omega(s))]ds. \tag{2.11}$$

Denote $\Delta = |\omega_i(u_n+t_n) - \omega_i(u_n)|$ and consider two cases (i) $\Delta \geq \epsilon_0/l$, (ii) $\Delta < \epsilon_0/l$ such that the remaining proof naturally falls into two parts.

(i) There exists a positive number $\kappa \leq \kappa_1$ such that

$$|\omega_i(t+s) - \omega_i(t)| < \frac{\epsilon_0}{4l}, \quad t \in \mathbb{R}, |s| < \kappa.$$

Therefore, the relation

$$|\omega_i(t_n+t) - \omega_i(t)| \geq |\omega_i(t_n+u_n) - \omega_i(u_n)| - |\omega_i(u_n) - \omega_i(t)| -$$
$$|\omega_i(t_n+t) - \omega_i(t_n+u_n)| \geq \frac{\epsilon_0}{l} - \frac{\epsilon_0}{4l} - \frac{\epsilon_0}{4l} = \frac{\epsilon}{2l},$$

is valid if $t \in [u_n - \kappa, u_n + \kappa]$.

(ii) It is true that $\inf_G |f_i(t+u_n, x) - f_i(u_n, x)| \geq \epsilon_0$, for all $\|x\| < H, t \in [u_n - \kappa_1, u_n + \kappa_1]$. Then, using condition $(C3)$ and relation (2.10) we get that

$$|\omega_i(t_n+t) - \omega_i(t)| \geq \int_{u_n}^{t} |f_i(t_n+s, \omega(t_n+s)) - f_i(s, \omega(t_n+s))|ds -$$
$$\int_{u_n}^{t} |f_i(s, \omega(t_n+s)) - f_i(s, \omega(s))|ds - \int_{u_n}^{t} |\sum_{j=1}^{p} a_{ij}[\omega_j(s) - \omega_j(t_n+s)]|ds -$$
$$|\omega_i(t_n+u_n) - \omega_i(u_n)| \geq \kappa_1\epsilon_0 - 2\kappa_1 LH - 2\kappa_1\|A\|H - \frac{\epsilon_0}{l} =$$
$$\kappa_1(\epsilon_0 - 2H(L + \|A\|)) - \frac{\epsilon_0}{l} \geq \frac{\epsilon_0}{2l},$$

for $t \in [u_n - \kappa_1, u_n + \kappa_1]$.

Thus, one can conclude that $\omega(t)$ is a strongly unpredictable function with positive numbers $\frac{\epsilon_0}{2l}, \kappa$ and sequences t_n and u_n.

The asymptotical stability of the solution $\omega(t)$ can be verified as stability of a bounded solution in [10]. The theorem is proved. □

We have considered the problem of existence and uniqueness of strongly unpredictabe solutions. This requires a special condition, $(C3)$. In what follows, we will search for quasilinear systems with unpredictable solutions, which are not strongly unpredictable. For this reason, assume that the following condition is valid.

(C6) The function *f(t, x)* is unpredictable in the sense of Definition 1.4.

Theorem 2.2. *Suppose that the conditions $(C2) - (C6)$ hold. Then the system (1.2) admits a unique uniformly globally exponentially stable unpredictable solution.*

Proof. One can easily see, proceeding in the way of the last theorem, that under the conditions $(C2) - (C6)$, there exists a unique solution $\omega(t) \in U$ for system (1.2). The solution is globally uniformly asymptotically stable. What is left is to show that there exist a natural number l and positive number κ such that $\|\omega(t+t_n) - \omega(t_n)\| \geq \frac{\epsilon_0}{2l}$ for each $t \in [u_n - \kappa, u_n + \kappa]$ and all $n \in \mathbb{N}$.

We have that

$$\omega(t+t_n)-\omega(t)=\omega(u_n+t_n)-\omega(u_n)+\int_{u_n}^{t} A[\omega(s+t_n)-\omega(s)]ds$$
$$+\int_{u_n}^{t}[f(t_n+s,\omega(s+t_n))-f(s,\omega(s))]ds,\ t\in\mathbb{R}. \tag{2.12}$$

One can find a positive number κ_1 and natural numbers l and k such that

$$\kappa_1<\sigma, \tag{2.13}$$

$$\|\omega(t+s)-\omega(t)\|<\frac{\epsilon_0}{k},\quad t\in\mathbb{R},|s|<\kappa_1, \tag{2.14}$$

$$\kappa_1\Big(1-(\frac{1}{l}+\frac{2}{k})(L+\|A\|)\Big)>\frac{3}{2l}. \tag{2.15}$$

Denote $\Delta=\|\omega(t_n+u_n)-\omega(u_n)\|$ and consider two alternative cases (i) $\Delta\geq\epsilon_0/l$ and (ii) $\Delta<\epsilon_0/l$.

(i) For the case $\Delta\geq\epsilon_0/l$, fix additionally a positive number $\kappa\leq\kappa_1$ sufficiently small for

$$\|\omega(t+s)-\omega(t)\|<\frac{\epsilon_0}{4l},\quad t\in\mathbb{R},|s|<\kappa.$$

Therefore,

$$\|\omega(t_n+t)-\omega(t)\|\geq\|\omega(t_n+u_n)-\omega(u_n)\|-\|\omega(u_n)-\omega(t)\|-$$
$$\|\omega(t_n+t)-\omega(t_n+u_n)\|\geq\frac{\epsilon_0}{l}-\frac{\epsilon_0}{4l}-\frac{\epsilon_0}{4l}=\frac{\epsilon}{2l}, \tag{2.16}$$

if $t\in[u_n-\kappa,u_n+\kappa]$ and $n\in\mathbb{N}$.

(ii) One can find that from (2.14) it follows that

$$\|\omega(t_n+t)-\omega(t)\|<\frac{\epsilon_0}{l}+\frac{\epsilon_0}{k}+\frac{\epsilon_0}{k}=\epsilon_0\Big(\frac{1}{l}+\frac{2}{k}\Big) \tag{2.17}$$

if $t\in[u_n-\kappa_1,u_n+\kappa_1]$.

It is true that $\inf_G\|f(t_n+u_n,x)-f(u_n,x)\|\geq\epsilon_0, t\in[u_n-\kappa_1,u_n+\kappa_1]$ and $n\in\mathbb{N}$.

Now, using condition $(C3)$, relations (2.15) and (2.17) we get that

$$\|\omega(t_n+t)-\omega(t)\|\geq\int_{u_n}^{t}\|f(t_n+s,\omega(t_n+s))-f(s,\omega(t_n+s))\|ds-$$
$$\int_{u_n}^{t}\|f(s,\omega(t_n+s))-f(s,\omega(s))\|ds-\int_{u_n}^{t}\|A\|\|\omega(s)-\omega(t_n+s)\|ds-$$
$$\|\omega(t_n+u_n)-\omega(u_n)\|\geq\kappa_1\epsilon_0-\kappa_1L\epsilon_0\Big(\frac{1}{l}+\frac{2}{k}\Big)-\kappa_1\|A\|\epsilon_0\Big(\frac{1}{l}+\frac{2}{k}\Big)-\frac{\epsilon_0}{l}\geq\frac{\epsilon_0}{2l}$$

for $t\in[u_n-\kappa_1,u_n+\kappa_1]$.

Thus, the solution $\omega(t)$ is unpredictable with positive numbers $\frac{\epsilon_0}{2l},\kappa$ and sequences t_n,u_n.

The theorem is proved. □

3. Examples

First, we will construct two examples of unpredictable functions.

Example 1. Let $\psi_i, i \in \mathbb{Z}$, be an unpredictable solution [1] of the logistic discrete equation

$$\lambda_{i+1} = \mu\lambda_i(1-\lambda_i) \tag{3.18}$$

with $\mu = 3.92$. The sequence belongs to the unit interval $[0,1]$. There exist a positive number ϵ_0 and sequences ζ_n, η_n both of which diverge to infinity such that $|\psi_{i+\zeta_n} - \psi_i| \to 0$ as $n \to \infty$ for each i in bounded intervals of integers and $|\psi_{\zeta_n+\eta_n} - \psi_{\eta_n}| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

Consider the following integral

$$\Theta(t) = \int_{-\infty}^{t} e^{-2(t-s)}\Omega(s)ds, t \in \mathbb{R}, \tag{3.19}$$

where $\Omega(t)$ is a piecewise constant function defined on the real axis through the equation $\Omega(t) = \psi_i$ for $t \in [i, i+1), i \in \mathbb{Z}$.

It is worth noting that $\Theta(t)$ is bounded on the whole real axis such that $\sup_{t\in\mathbb{R}} |\Theta(t)| \leq 1/2$.

Next, we will show that $\Theta(t)$ is an unpreditable scalar function.

Consider a fixed bounded and closed interval $[\alpha, \beta]$, of the axis and a positive number ϵ. Without loss of generality one can assume that α and β are integers. Let us fix a positive number ξ and an integer $\gamma < \alpha$, which satisfy the following inequalities $e^{-2(\alpha-\gamma)} < \frac{\epsilon}{2}$ and $\xi[1 - e^{-2(\beta-\gamma)}] < \epsilon$. Let n be a large natural number such that $|\Omega(t+\zeta_n) - \Omega(t)| < \xi$ on $[\gamma, \beta]$. Then for all $t \in [\alpha, \beta]$ we obtain that

$$|\Theta(t+\zeta_n) - \Theta(t)| = |\int_{-\infty}^{t} e^{-2(t-s)}(\Omega(s+\zeta_n) - \Omega(s))ds| =$$
$$|\int_{-\infty}^{\gamma} e^{-2(t-s)}(\Omega(s+\zeta_n) - \Omega(s))ds + \int_{\gamma}^{\beta} e^{-2(t-s)}(\Omega(s+\zeta_n) - \Omega(s))ds| \leq$$
$$\int_{-\infty}^{\gamma} e^{-2(t-s)}2ds + \int_{\gamma}^{\beta} e^{-2(t-s)}\xi ds \leq e^{-2(\alpha-\gamma)} + \frac{\xi}{2}[1 - e^{-2(\beta-\gamma)}] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $|\Theta(t+\zeta_n) - \Theta(t)| \to 0$ as $n \to \infty$ uniformly on the interval $[\alpha, \beta]$.

There exists a positive number $\kappa < 1$ such that $1 - e^{-2\kappa} \leq \frac{\epsilon_0}{6}$. Moreover, the following inequalities are valid, $|\Omega(t+\zeta_n) - \Omega(t)| = |\psi_{\zeta_n+\eta_n} - \psi_{\eta_n}| \geq \epsilon_0, t \in [\eta_p, \eta_p + 1), p \in \mathbb{N}$.

Let us fix the numbers κ and $p = n$ and consider two alternative cases: (i) $|\Theta(\eta_n + \zeta_n) - \Theta(\eta_n)| < \frac{\epsilon_0}{3}$ and (ii) $|\Theta(\eta_n + \zeta_n) - \Theta(\eta_n)| \geq \frac{\epsilon_0}{3}$.

(i) Using the relation

$$\Theta(t+\zeta_n) - \Theta(t) = \Theta(\eta_n + \zeta_n) - \Theta(\eta_n) + \int_{\eta_n}^{t} e^{-2(t-s)}(\Omega(s+\zeta_n) - \Omega(s))ds \tag{3.20}$$

we obtain that

$$|\Theta(t+\zeta_n) - \Theta(t)| \geq |\int_{\eta_n}^{t} e^{-2(t-s)}(\Omega(s+\zeta_n) - \Omega(s))ds| - |\Theta(\eta_n+\zeta_n) - \Theta(\eta_n)| \geq$$
$$\int_{\eta_n}^{t} e^{-2(t-s)}\epsilon_0 ds - \frac{\epsilon_0}{3} \geq \frac{\epsilon_0}{2} - \frac{\epsilon_0}{3} = \frac{\epsilon_0}{6}$$

for $t \in [\eta_n, \eta_n + 1)$.

(ii) From the relation (3.20) we get

$$|\Theta(t+\zeta_n)-\Theta(t)| \geq |\Theta(\eta_n+\zeta_n)-\Theta(\eta_n)| - |\int_{\eta_n}^{t} e^{-2(t-s)}(\Omega(s+\zeta_n)-\Omega(s))ds| \geq$$

$$\frac{\epsilon_0}{3} - \int_{\eta_n}^{t} e^{-2(t-s)}2ds \geq \frac{\epsilon_0}{3} - [1-e^{-2\kappa}] \geq \frac{\epsilon_0}{6}$$

for $t \in [\eta_n, \eta_n+\kappa)$.

Thus, the function $\Theta(t)$ is unpredictable (strongly) with positive numbers $\bar{\epsilon}_0 = \frac{\epsilon_0}{6}, \frac{\kappa}{3}$ and sequences $t_n = \zeta_n$ and $u_n = \eta_n + \frac{\kappa}{2}$.

Example 2. Consider the function $g(t,x) = (arctan(x)+2)\Theta(t)$ of two variables t and x, where $\Theta(t)$ is the function from the last example. It is easy to see that function $g(t,x)$ is continuously differentable if $t \neq i, i \in \mathbb{Z}$, and bounded such that $\sup_{\mathbb{R}\times G} |g(t,x)| = \dfrac{\pi}{4}+1$.

Moreover, $\sup_{\mathbb{R}\times G} |\dfrac{\partial g(t,x)}{\partial x}| = 1/2, t \neq i, i \in \mathbb{Z}$.

Let us fix arbitrary compact interval $I \subset \mathbb{R}$ and positive number ϵ. We have that $|\Theta(t+t_n)-\Theta(t)| < \epsilon$ for $t \in I$ sufficiently large n. Consequently,

$$|g(t+t_n,x)-g(t,x)| \leq |arctan(x)+2||\Theta(t+t_n)-\Theta(t)| < (\frac{\pi}{2}+2)\epsilon.$$

That is $g(t+t_n,x) \to g(t,x)$ as $n \to \infty$ uniformly for $(t,x) \in I \times G$.

On the other hand, it is true that $|\Theta(t+t_n)-\Theta(t)| \geq \bar{\epsilon}_0$ for all $t \in [u_n-\kappa, u_n+\kappa]$ and $n \in \mathbb{N}$. This is why, we obtain that

$$|g(t+t_n,x)-g(t,x)| = |arctan(x)+2||\Theta(t+\zeta_n)-\Theta(t)| \geq (-\frac{\pi}{2}+2)\bar{\epsilon}_0,$$

for $(t,x) \in [u_n-\kappa, u_n+\kappa] \times G, n \in \mathbb{N}$. Thus, $g(t,x)$ is an unpredictable (strongly) in t function.

Example 3. Let us consider the system of differential equations

$$\begin{aligned} x_1' &= -3x_1 - x_2 - x_3 + 0.51g(t,x_3) \\ x_2' &= -x_1 - 3x_2 - x_3 - 0.62g(t,x_1) \\ x_3' &= x_1 + x_2 - x_3 + 0.51g(t,x_2), \end{aligned} \tag{3.21}$$

where $g(t,x)$ is the function from the last example. The eigenvalues of the matrix of coefficients are equal to $-2, -2$ and -3. One can find that conditions $(C3)-(C6)$ are valid for system (3.21) with $\gamma = -2$, $K = 6$ and $L = 0.31$. According to Theorem 2.2, there exists the unique asymptotically stable unpredictable solution of system (3.21). The simulation results for the solution of (3.21) with initial data $\psi_1(0) = 0.05$, $\psi_2(0) = -0.1$, $\psi_3(0) = 0.15$ are seen in Figure 1. They confirm the irregularity of the dynamics.

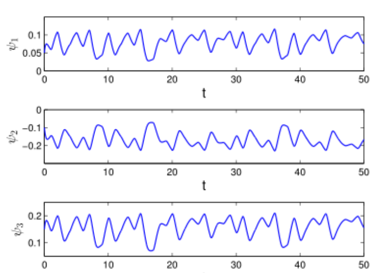

FIGURE 1. The coordinates of the solution, $\psi(t)$, of system (3.21).

Acknowledgements. M. Akhmet has been supported by a grant (118F161) from TÜBİTAK, the Scientific and Technological Research Council of Turkey.

M. Tleubergenova and A. Zhamanshin have been supported in parts by the MES RK grant (AP05132573) "Cellular neural networks with continuous/discrete time and singular perturbations" (2018-2020). Republic of Kazakhstan.

REFERENCES

[1] Akhmet, M. U. and Fen, M. O., *Poincare chaos and unpredictable functions*, Commun. Nonlinear Sci. Numer. Simul., **41** (2017), 85–94

[2] Akhmet, M. U. and Fen, M. O., *Unpredictable points and chaos* Commun. Nonlinear Sci. Numer. Simul., **40** (2016), 1–5

[3] Akhmet, M. U. and Fen, M. O., *Existence of unpredictable solutions and chaos*, Turkish J. Math., **41** (2017), 254–266

[4] Akhmet, M. and Fen, M. O., *Non-autonomous equations with unpredictable solutions*, Commun. Nonlinear Sci. Numer. Simul., **59** (2018), 657–670

[5] Akhmet, M., Fen, M. O., Tleubergenova, M. and Zhamanshin, A., *Poincaré chaos for a hyperbolic quasilinear system*, Miskolc Math. Notes, **20** (2019), 33–44

[6] Bohr, H. A., *Almost Periodic Functions*, Chelsea Publishing Company, 1947

[7] Corduneanu, C., *Almost Periodic Oscillations and Waves*, Springer, 2009

[8] Farkas, M., *Periodic Motion*, Springer-Verlag, New York 1994

[9] Fink, A. M., *Almost periodic differential equations, Springer-Verlag*, New York 1974

[10] Hartman, P., *Ordinary differential equations*, SIAM, 2002

[11] Hino, Y., Naito, T., VanMinh, N. and Jong Son Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces*, CRC Press 2001

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