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Uniformly supported sets and fixed points properties

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ABSTRACT. The theory of finitely supported algebraic structures is a reformulation of Zermelo-Fraenkel set theory in which every set-based construction is finitely supported according to a canonical action of a group of permutations of some basic elements named atoms. In this paper we study the properties of finitely supported sets that contain infinite uniformly supported subsets, as well as the properties of finitely supported sets that do not contain infinite uniformly supported subsets. Particularly, we focus on fixed points properties.

1. INTRODUCTION

Finitely supported structures are related to the recent development of the Fraenkel-Mostowski (FM) axiomatic set theory that represents an axiomatization of the Fraenkel Basic Model of the Zermelo-Fraenkel set theory with atoms (ZFA). Its axioms are the ZFA axioms together with a new axiom of finite support claiming that any set-theoretical construction has to be finitely supported modulo a canonical hierarchically defined permutation action. Therefore, FM sets are actually hereditary finitely supported ZFA sets. Nominal sets [6] represent a Zermelo-Fraenkel set theory (ZF) alternative to the non-standard FM set theory since a nominal set is defined as a usual ZF set endowed with a group action of the group of (finitary) permutations over a certain fixed countable ZF set A (formed by elements whose internal structure is ignored, and called the set of atoms by analogy with the FM approach) satisfying a finite support requirement. This finite support requirement states that for any element in a nominal set there should exist a finite set of atoms such that any permutation fixing pointwise this set of atoms also leaves the element invariant under the related group action. Nominal sets are used to study the binding, scope, freshness and renaming in programming languages and related formal systems. Furthermore, this framework admits a notion of structural recursion for defining syntax-manipulating functions and a notion of proof by structural induction. Certain generalizations of nominal sets are involved in the study of automata, programming languages or Turing machines over infinite alphabets; for this, a relaxed notion of finiteness called 'orbit finiteness' was defined; it means 'having a finite number of orbits (equivalence classes) under a certain group action' [5]. Actually, the theory of finitely supported sets (that are finitely supported elements in the powerset of a nominal set) allows the computational study of structures which are possibly infinite, but contain enough symmetries such that they can be clearly/concisely represented and manipulated.

Finitely supported mathematics (shortly, FSM) is focused on the foundations of structures with finite supports (rather than on applications in computer science). In order to describe FSM as a theory of finitely supported algebraic structures, we refer to the theory of nominal sets (with the mention that the requirement regarding the countability of A is irrelevant). We call these sets *invariant sets*, using the motivation of Tarski regarding logicality (more precisely, a logical notion is defined by Tarski as one that is invariant under

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the permutations of the universe of discourse). FSM is actually represented by finitely supported subsets of invariant sets together with finitely supported internal algebraic operations or with finitely supported relations (that should be finitely supported as subsets in the Cartesian product of two invariant sets). Formally, FSM contains both the family of 'non-atomic' (i.e., ordinary) ZF sets which are proved to be trivial FSM sets (i.e., their elements are left unchanged under the effect of the canonical permutation action) and the family of 'atomic' sets (i.e., sets that contain at least an element of A somewhere in their structure) with finite supports (hierarchically constructed from the empty set and the fixed ZF set A of atoms). Our purpose is to analyze whether a classical ZF result (obtained in the framework of non-atomic sets) can be adequately reformulated by replacing 'non-atomic ZF element/set/structure' with 'atomic and finitely supported element/set/structure' in order to be valid also for atomic sets with finite supports. The translation of the results from a non-atomic framework into an atomic framework (such as ZFA) is not an easy task. Results from ZF may lose their validity when reformulating them in ZFA. For example, it is known that multiple choice principle and Kurepa's maximal antichain principle are both equivalent to the axiom of choice in ZF. However, multiple choice principle is valid in the Fraenkel Second Model, while the axiom of choice fails in this model. Furthermore, Kurepa's maximal antichain principle is valid in the Fraenkel Basic Model, while the axiom of choice fails in this model. This means that the following two statements that are valid in ZF, namely *'Kurepa's principle implies axiom of choice'* and *'Multiple choice principle implies axiom of choice'* fail in ZFA. Similarly, there are examples of ZF results that cannot be reformulated into FSM; we particularly mention choice principles (that are proved to be independent from ZF axioms, but inconsistent in FSM) and Stone duality.

A proof of an FSM result should be internally consistent in FSM and not retrieved from ZF, that means it should involve only finitely supported constructions (even in the intermediate steps). The meta-theoretical techniques for the translation of a result from non-atomic structures to atomic structures are based on a refinement of the finite support principle from [6] called 'S-finite supports principle' claiming that *for any finite set S of atoms, anything that is definable in higher-order logic from S-supported structures by using S-supported constructions is also S-supported*. The formal involvement of the S-finite support principles actually implies a hierarchical constructive method for defining the support of a structure by employing, step-by-step, the supports of the substructures of a related structure. This method was used, for example, to study in a discrete manner the (atomic, finitely supported) fuzzy sets over possibly infinite alphabets [2].

Uniformly supported sets are particularly of interest because they involve boundedness properties of supports, meaning that the support of each element in an uniformly supported set is contained in *the same* finite set of atoms. In this way, all the individuals in an infinite uniformly supported family can be characterized by involving only finitely many characteristics. In this paper we characterize finitely supported sets containing infinite uniformly supported subsets. FSM sets that do not contain infinite uniformly supported subsets are also of interest because they are related to surprising fixed point properties. For particular finitely supported (injective, surjective and order preserving) mappings on the finite powersets of A, we have specific properties that allow to prove the existence of infinitely many fixed points. This paper represents the revised extended version of the conference paper (extended abstract) [3] that continues the approach in [4].

2. Preliminary Results

A finite set is referred to a set of the form $\{x_1, \ldots, x_n\}$. Consider a fixed ZF infinite (not finite) set A of entities whose internal structure is ignored. The elements of A are called 'atoms'. A *transposition* is a function $(a\,b) : A \to A$ that interchanges only a and b. A *permutation* of A in FSM is a bijection of A generated by composing finitely many transpositions. We denote by S_A the group of all permutations of A. According to Prop. 2.6 in [1], an arbitrary bijection on A is finitely supported if and only if it is a permutation.

Definition 2.1. (1) Let X be a ZF set. An S_A-*action* on X is a mapping $\cdot : S_A \times X \to X$ having the properties $Id \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $\pi, \pi' \in S_A$ and $x \in X$. An S_A-*set* is a pair $(X, \cdot)$, where X is a ZF set, and $\cdot$ is an S_A-action on X.

(2) Let $(X, \cdot)$ be an S_A-set. We say that $S \subset A$ *supports* x whenever for each $\pi \in Fix(S)$ we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \,|\, \pi(a) = a, \forall a \in S\}$. The least finite set (w.r.t. the inclusion relation) supporting x (which exists according to [6]) is called *the support of* x and is denoted by $supp(x)$. An empty supported element is called *equivariant*.

(3) Let $(X, \cdot)$ be an S_A-set. We say that X is an *invariant set* if for each $x \in X$ there exists a finite set $S_x \subset A$ which supports x.

Proposition 2.1. [1, 6] *Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets.*

(1) The set A of atoms is an invariant set with the S_A-action $\cdot : S_A \times A \to A$ defined by $\pi \cdot a := \pi(a)$ for all $\pi \in S_A$ and $a \in A$. Furthermore, $supp(a) = \{a\}$ for each $a \in A$.

(2) Let $\pi \in S_A$. If $x \in X$ is finitely supported, then $\pi \cdot x$ is finitely supported and $supp(\pi \cdot x) = \{\pi(u) \,|\, u \in supp(x)\} := \pi(supp(x))$.

(3) The Cartesian product $X \times Y$ is an S_A-set with the S_A-action $\otimes$ defined by $\pi \otimes (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X \times Y, \otimes)$ is also an invariant set.

(4) The powerset $\wp(X) = \{Z \,|\, Z \subseteq X\}$ is also an S_A-set with the S_A-action $\star$ defined by $\pi \star Z := \{\pi \cdot z \,|\, z \in Z\}$ for all $\pi \in S_A$, and all $Z \subseteq X$. For each invariant set $(X, \cdot)$, we denote by $\wp_{fs}(X)$ the set of elements in $\wp(X)$ which are finitely supported according to the action $\star$. $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is an invariant set. The finite powerset of X denoted by $\wp_{fin}(X) = \{Y \subseteq X \,|\, Y finite\}$ and the cofinite powerset of X denoted by $\wp_{cofin}(X) = \{Y \subseteq X \,|\, X \setminus Y finite\}$ are both S_A-sets with the S_A-action $\star$. If X is an invariant set, then both $\wp_{fin}(X)$ and $\wp_{cofin}(X)$ are invariant sets.

(5) We have $\wp_{fs}(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$. If $X \in \wp_{fin}(A)$, then $supp(X) = X$. If $X \in \wp_{cofin}(A)$, then $supp(X) = A \setminus X$.

(6) The disjoint union of X and Y defined by $X + Y = \{(0, x) \,|\, x \in X\} \cup \{(1, y) \,|\, y \in Y\}$ is an S_A-set with the S_A-action $\star$ defined by $\pi \star z = (0, \pi \cdot x)$ if $z = (0, x)$ and $\pi \star z = (1, \pi \diamond y)$ if $z = (1, y)$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X + Y, \star)$ is also an invariant set.

(7) Any ordinary (non-atomic) ZF-set X is an invariant set with the single possible S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$.

From Definition 2.1, a subset Y of an S_A-set $(X, \cdot)$ is finitely supported by a set $S \subseteq A$ if and only if it is supported as an element in the S_A-set $\wp(X)$, i.e. if and only if $\pi \star Y = Y$ for all $\pi \in Fix(S)$, where $\star$ is the canonical action on the powerset of X described as in Proposition 2.1(4). For a fixed π, the relation $\pi \star Y = Y$ is equivalent with $\pi \star Y \subseteq Y$ (i.e. with $\pi \cdot y \in Y$ for all $y \in Y$). This is because any permutation of atoms has a finite order.

An *FSM set* Y is defined as a finitely supported subset Y of an invariant set X. Its FSM powerset $\wp_{fs}(Y)$ is formed by those finitely supported subsets of X that are contained in Y, and it is a finitely supported subset of $\wp(X)$ (supported by $supp(Y)$) according to Proposition 2.1(2).

A subset Z of an FSM set is called *uniformly supported* if all the elements of Z are supported by the same set S (and so Z is itself supported by S). Due to Proposition 2.1(2), whenever Y is a finitely supported subset of an invariant set X, the uniform powerset of Y denoted by $\wp_{us}(Y) = \{Z \subseteq Y \,|\, Z \text{ uniformly supported}\}$ is a subset of $\wp_{fs}(X)$ supported by $supp(Y)$. This is because, whenever $Z \subseteq Y$ is uniformly supported by S and $\pi \in Fix(supp(Y))$, we have $\pi \star Z \subseteq \pi \star Y = Y$ and $\pi \star Z$ is uniformly supported by $\pi(S)$. Similarly, $\wp_{fin}(Y)$ and $\wp_{cofin}(Y)$ are subsets of $\wp_{fs}(X)$ supported by $supp(Y)$. We consider that $\emptyset$, being a finite subset of X, belongs to $\wp_{us}(Y)$.

Definition 2.2. Let X and Y be invariant sets. Let Z be a finitely supported subset of X and T a finitely supported subset of Y. A function $f : Z \to T$ is *finitely supported* if $f \in \wp_{fs}(X \times Y)$. The set of all finitely supported functions from Z to T is denoted by T^Z_{fs}.

Proposition 2.2. [1] *Let $(X, \cdot)$ and $(Y, \diamond)$ be two invariant sets.*

(1) Y^X (i.e. the set of all functions from X to Y) is an S_A-set with the S_A-action $\widetilde{\star}$ defined by $(\pi \widetilde{\star} f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. A function $f : X \to Y$ is finitely supported w.r.t. Def. 2.2 if and only if it is finitely supported with respect to $\widetilde{\star}$.

(2) Let Z be a finitely supported subset of X and T a finitely supported subset of Y. A function $f : Z \to T$ is supported by a finite set $S \subseteq A$ if and only if for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$, $\pi \diamond f(x) \in T$ and $f(\pi \cdot x) = \pi \diamond f(x)$.

3. Uniform Supports and Fixed Points

Lemma 3.1. *Let X be a uniformly supported subset of an invariant set $(Y, \cdot)$. Then X is finitely supported and $supp(X) = \cup\{supp(x) \,|\, x \in X\}$.*

Proof. Since X is uniformly supported, there exists a finite subset of atoms T such that T supports every $x \in X$, i.e. $supp(x) \subseteq T$ for all $x \in X$. Thus, $\cup\{supp(x) \,|\, x \in X\} \subseteq T$. Clearly, $supp(X) \subseteq \cup\{supp(x) \,|\, x \in X\}$. Conversely, let $a \in \cup\{supp(x) \,|\, x \in X\}$. Thus, there exists $x_0 \in X$ such that $a \in supp(x_0)$. Let b be an atom such that $b \notin supp(X)$ and $b \notin T$. Such an atom exists because A is infinite, while $supp(X)$ and T are both finite. We prove by contradiction that $(b\ a) \cdot x_0 \notin X$. Indeed, suppose that $(b\ a) \cdot x_0 = y \in X$. Since $a \in supp(x_0)$, by Proposition 2.1(2), we have $b = (b\ a)(a) \in (b\ a)(supp(x_0)) = supp((b\ a) \cdot x_0) = supp(y)$. Since $supp(y) \subseteq T$, we get $b \in T$: a contradiction! Therefore, $(b\ a) \star X \neq X$, where $\star$ is the canonical S_A-action on $\wp(Y)$. Since $b \notin supp(X)$, we prove by contradiction that $a \in supp(X)$. Indeed, suppose that $a \notin supp(X)$. We have that $(b\ a) \in Fix(supp(X))$. Since $supp(X)$ supports X, it follows that $(b\ a) \star X = X$ which is a contradiction. Thus, $a \in supp(X)$ and the result follows. □

Lemma 3.2. *Let X be a finitely supported subset of an invariant set $(Y, \cdot)$ such that X does not contain an infinite uniformly supported subset. Then the set $\wp_{us}(X)$ does not contain an infinite uniformly supported subset.*

Proof. Suppose by contradiction that the set $\wp_{us}(X)$ contains an infinite subset $\mathcal{F}$ such that all the elements of $\mathcal{F}$ are different and supported by the same finite set S. By convention, without assuming that $i \mapsto X_i$ is finitely supported, we understand $\mathcal{F}$ as $\mathcal{F} = (X_i)_{i \in I}$ with the properties that $X_i \neq X_j$ whenever $i \neq j$ and $supp(X_i) \subseteq S$ for all $i \in I$. Let us fix an arbitrary $j \in I$. From Lemma 3.1, because X_j is an uniformly supported subset of the invariant set Y, we have $supp(X_j) = \underset{x \in X_j}{\cup} supp(x)$. Therefore, X_j has the property that $supp(x) \subseteq S$ for all $x \in X_j$, and so $\underset{i \in I}{\cup} X_i$ is an uniformly supported subset of X (all its elements being supported by S). Furthermore, $\underset{i \in I}{\cup} X_i$ is infinite since the family $(X_i)_{i \in I}$ is infinite and $X_i \neq X_j$ whenever $i \neq j$. This contradicts the hypothesis. □

Lemma 3.3. *Let X be a finitely supported subset of an invariant set $(Y, \cdot)$ such that X does not contain an infinite uniformly supported subset. Then the set $\wp_{fin}(X)$ does not contain an infinite uniformly supported subset.*

Proof. We always have that $\wp_{fin}(X) \subseteq \wp_{us}(X)$ because any finite subset of X of form $\{x_1, \dots, x_n\}$ is uniformly supported by $supp(x_1) \cup \dots \cup supp(x_n)$. Since $\wp_{us}(X)$ does not contain an infinite uniformly supported subset, it follows that neither $\wp_{fin}(X)$ contains an infinite uniformly supported subset. □

A *finitely supported partially ordered set* $(X, \sqsubseteq, \cdot)$ is a finitely supported subset X of an invariant set $(Y, \cdot)$, equipped with a partial order relation $\sqsubseteq$ that is finitely supported as a subset of $X \times X$. If $\sqsubseteq$ is a lattice order, then $(X, \sqsubseteq, \cdot)$ is called *finitely supported lattice*.

Invariant (empty-supported) partially ordered sets and lattices were treated by the authors in [4], where we focused on the construction of some invariant complete lattices such as the lattice of finitely supported L-fuzzy sets or the lattice of finitely supported (fuzzy) subgroups. Here we extend the framework by replacing 'invariant (empty-supported)' with the more general 'finitely supported', and by adding new fixed point properties for mappings on FSM sets having specific properties related to uniformly supported subsets. Studying FSM infinite (non-)uniformly supported sets is our main goal.

Theorem 3.1. *Let $(X, \sqsubseteq, \cdot, 0)$ be a finitely supported partially ordered set, with a least element 0, containing no infinite uniformly supported subset. Each finitely supported order preserving function $f : X \to X$ possesses a least fixed point lfp(f) which has the properties that $supp(lfp(f)) \subseteq supp(f) \cup supp(X) \cup supp(\sqsubseteq)$ and lfp$(f) = f^n(0)$ for some $n \in \mathbb{N}$.*

Proof. Since $0 \sqsubseteq f(0)$ and f is order preserving, we have that $(f^n(0))_{n\in\mathbb{N}}$ is an ascending chain. By definition, we have $0 \sqsubseteq \pi \cdot 0$ and $0 \sqsubseteq \pi^{-1} \cdot 0$ for each $\pi \in Fix(supp(X))$, which means $0 = \pi \cdot 0$ when π additionally fixes $supp(\sqsubseteq)$ pointwise, and so $supp(0) \subseteq supp(X) \cup supp(\sqsubseteq)$. By induction, we prove $supp(f^n(0)) \subseteq supp(f) \cup supp(X) \cup supp(\sqsubseteq)$ for all $n \in \mathbb{N}$. Clearly, $supp(f^0(0)) = supp(0) \subseteq supp(f) \cup supp(X) \cup supp(\sqsubseteq)$. Let us suppose that $supp(f^k(0)) \subseteq supp(f) \cup supp(X) \cup supp(\sqsubseteq)$ for some $k \in \mathbb{N}$. We have to prove that $supp(f^{k+1}(0)) \subseteq supp(f) \cup supp(X) \cup supp(\sqsubseteq)$. So, we have to prove that each permutation π which fixes $supp(f) \cup supp(X) \cup supp(\sqsubseteq)$ pointwise also fixes $f^{k+1}(0)$. Let $\pi \in Fix(supp(f) \cup supp(X) \cup supp(\sqsubseteq))$. From the inductive hypothesis, we have that $supp(f^k(0)) \subseteq supp(f) \cup supp(X) \cup supp(\sqsubseteq)$, and so $\pi \cdot f^k(0) = f^k(0)$. According to Proposition 2.2, since π fixes $supp(f)$ pointwise we have $\pi \cdot f^{k+1}(0) = \pi \cdot f(f^k(0)) = f(\pi \cdot f^k(0)) = f(f^k(0)) = f^{k+1}(0)$. Therefore, $(f^n(0))_{n\in\mathbb{N}}$ is uniformly supported, and so it has to be finite according to theorem's hypothesis. Since it is an ascending chain it follows that there exists $n_0 \in \mathbb{N}$ such that $f^n(0) = f^{n_0}(0)$ for all $n \geq n_0$. Thus, $f(f^{n_0}(0)) = f^{n_0+1}(0) = f^{n_0}(0)$, and so $f^{n_0}(0)$ is a fixed point of f, and, furthermore, it is supported by $supp(f)$. If x is another fixed point of f, it follows from the monotony of f and from the relation $0 \sqsubseteq x$ that $f^{n_0}(0) = \bigsqcup_{n\in\mathbb{N}} f^n(0) \sqsubseteq x$, and so $f^{n_0}(0) = lfp(f)$. □

From Theorem 3.1, Lemma 3.2 and Lemma 3.3, and using the fact that the inclusion relation on the powerset of a finitely supported set X is supported by $supp(X)$, we have:

Corollary 3.1. *Let X be a finitely supported subset of an invariant set $(Y, \cdot)$ having the property that it does not contain an infinite uniformly supported subset. Then:*

(1) Any finitely supported order preserving function $f : \wp_{us}(X) \to \wp_{us}(X)$ has a least fixed point supported by $supp(f) \cup supp(X)$.
(2) Any finitely supported order preserving function $f : \wp_{fin}(X) \to \wp_{fin}(X)$ has a least fixed point supported by $supp(f) \cup supp(X)$.

Theorem 3.2. *Let $(L, \sqsubseteq, \cdot)$ be a finitely supported lattice having the property that every finitely supported subset $X \subseteq L$ has a least upper bound $\sqcup X$ with respect to the order relation $\sqsubseteq$. Then each finitely supported order preserving function $f : L \to L$ possesses a least fixed point lfp(f) and a greatest fixed point gfp(f) which are both supported by $supp(f) \cup supp(\sqsubseteq) \cup supp(L)$.*

Proof. Firstly, we remark that every finitely supported subset of L also has a greatest lower bound w.r.t. $\sqsubseteq$. Let $U \in \wp_{fs}(L)$. Let $V = \cap\{\downarrow x \,|\, x \in U\}$, where $\downarrow x = \{y \in L \,|\, y \sqsubseteq x\}$. We claim that V is finitely supported by $supp(U) \cup supp(\sqsubseteq) \cup supp(L)$. Let $\pi \in Fix(supp(U) \cup supp(\sqsubseteq) \cup supp(L))$. Let $v \in V$, that is, $v \sqsubseteq x$ for all $x \in U$. We claim that $\pi \cdot v \in V$. Indeed, let $y \in U$ be an arbitrary element from U. Since $\pi \star U = U$, for our $y \in U$ there exists $x \in U$ such that $\pi \cdot x = y$. However, $v \sqsubseteq x$, and because π fixes $supp(\sqsubseteq)$ pointwise we also have $\pi \cdot v \sqsubseteq \pi \cdot x = y$. Hence $\pi \cdot v \in V$, and so $\pi \star V \subseteq V$. We prove by contradiction that $\pi \star V = V$. Let us suppose that $\pi \star V \subsetneq V$. By induction, we get $\pi^n \star V \subsetneq V$ for all $n \geq 1$. However, π is a finite permutation, and so it has a finite order. We obtain $V \subsetneq V$, a contradiction. Clearly, $\sqcup V$ is the greatest lower bound of U.

Let $Z = \{z \in L \,|\, z \sqsubseteq f(z)\}$. Firstly, we remark that Z is non-empty because the least element of L belongs to Z. We claim that $supp(f) \cup supp(\sqsubseteq)$ supports Z. Let $\pi \in Fix(supp(f) \cup supp(\sqsubseteq))$ and $z \in Z$ be arbitrarily chosen. Then $z \sqsubseteq f(z)$ (or, equivalently $(z, f(z)) \in\sqsubseteq$), and because $\sqsubseteq$ is supported by $supp(\sqsubseteq)$, we also have $\pi \otimes (z, f(z)) \in\sqsubseteq$, that is $\pi \cdot z \sqsubseteq \pi \cdot f(z)$, where $\otimes$ represents the canonical action on $Z \times Z$ defined as in Proposition 2.1. Since $\pi \in Fix(supp(f))$ and $supp(f)$ supports f, according to Proposition 2.2, we have $\pi \cdot z \sqsubseteq \pi \cdot f(z) = f(\pi \cdot z)$, and so $\pi \cdot z \in Z$. Thus, $\pi \star Z \subseteq Z$, and so $\pi \star Z = Z$. Therefore, $supp(f) \cup supp(\sqsubseteq)$ supports Z, and so there exists the least upper bound of Z, namely $z_0 = \sqcup Z$. As in ZF, we get $f(z_0) = z_0$ and z_0 is the greatest fixed point of f.

We prove that z_0 is supported by $supp(f) \cup supp(\sqsubseteq) \cup supp(L)$. Let $\pi \in Fix(supp(f) \cup supp(\sqsubseteq) \cup supp(L))$ and $X \in \wp_{fs}(L)$. According to Proposition 2.1(2), we have that $\pi \star X$ is finitely supported, and since $\pi \star X \subseteq \pi \star L = L$, there exists $\sqcup(\pi \star X)$. Let $x \in X$. We have $x \sqsubseteq \sqcup X$, and so $\pi \cdot x \sqsubseteq \pi \cdot \sqcup X$ because π fixes $supp(\sqsubseteq)$ pointwise. Thus, we have $\sqcup(\pi \star X) \sqsubseteq \pi \cdot \sqcup X$ (1). We can apply (1) firstly for π and Z, and, secondly, for $\pi^{-1} \in Fix(supp(f) \cup supp(\sqsubseteq) \cup supp(L))$ and $\pi \star Z$ from which we obtain $\sqcup Z \sqsubseteq \pi^{-1} \cdot \sqcup(\pi \star Z)$. Since π fixes $supp(\sqsubseteq)$ pointwise, we get $\pi \cdot \sqcup Z = \sqcup(\pi \star Z)$. Since $supp(f) \cup supp(\sqsubseteq) \cup supp(L)$ supports Z, we have $\pi \star Z = Z$, and so $\pi \cdot z_0 = z_0$. Similarly, f has a least fixed point which is the greatest lower bound of the finitely supported set $\{z \in L \,|\, f(z) \sqsubseteq z\}$. □

Corollary 3.2. *Let X be a finitely supported subset of an invariant set $(Y, \cdot)$. Any finitely supported order preserving (w.r.t. the inclusion relation) function $f : \wp_{fs}(X) \to \wp_{fs}(X)$ has a least fixed point and a greatest fixed point supported by $supp(f) \cup supp(X)$.*

Proof. Let $\mathcal{F} = (X_i)_{i \in I}$ be a finitely supported family of finitely supported subsets of X. We have to prove that $\bigcup_{i \in I} X_i \in \wp_{fs}(X)$. We claim that $supp(\mathcal{F})$ supports $\bigcup_{i \in I} X_i$. Let us consider $\pi \in Fix(supp(\mathcal{F}))$. Let $x \in \bigcup_{i \in I} X_i$. There exists $j \in I$ such that $x \in X_j$. Since $\pi \in Fix(supp(\mathcal{F}))$, we have $\pi \star X_j \in \mathcal{F}$, that there exists $k \in I$ such that $\pi \star X_j = X_k$. Therefore, $\pi \cdot x \in \pi \star X_j = X_k$, and so $\pi \cdot x \in \bigcup_{i \in I} X_i$. We obtain $\pi \star \bigcup_{i \in I} X_i = \bigcup_{i \in I} X_i$. Since $\subseteq$ is supported by $supp(X)$, the result follows from Theorem 3.2. □

Related to Theorem 3.2, the existence of fixed points can be proved even when relaxing the requirement "there exists a least upper bound for each finitely supported subset $X \subseteq L$", but in this case we cannot prove the existence of least or greatest fixed points.

Theorem 3.3. *Let $(X, \sqsubseteq, \cdot)$ be a finitely supported partially ordered set having the property that every* uniformly supported *subset has a least upper bound. Then each finitely supported, order preserving function $f : X \to X$ for which there is $x_0 \in X$ such that $x_0 \sqsubseteq f(x_0)$ has a fixed point.*

Proof. Let $Z = \{z \in X \mid z \sqsubseteq f(z) \text{ and } supp(z) \subseteq supp(x_0) \cup supp(f) \cup supp(\sqsubseteq) \cup supp(X)\}$. We remark that Z is non-empty since $x_0 \in Z$. By definition, Z is uniformly supported and there is $z_0 = \sqcup Z$. We claim z_0 is supported by $supp(x_0) \cup supp(f) \cup supp(\sqsubseteq) \cup supp(X)$. Let $\pi \in Fix(supp(x_0) \cup supp(f) \cup supp(\sqsubseteq) \cup supp(X))$ and $Y \in \wp_{us}(X)$. We have that $\pi \star Y \in \wp_{us}(X)$, and so there exists $\sqcup(\pi \star Y)$. Let $y \in Y$. We have $y \sqsubseteq \sqcup Y$, and so $\pi \cdot y \sqsubseteq \pi \cdot \sqcup Y$ because π fixes $supp(\sqsubseteq)$ pointwise. Thus, we have $\sqcup(\pi \star Y) \sqsubseteq \pi \cdot \sqcup Y$ (1). We can apply (1) firstly for π and Z, and, secondly, for $\pi^{-1} \in Fix(supp(f) \cup supp(x_0) \cup supp(\sqsubseteq) \cup supp(X))$ and $\pi \star Z \in \wp_{us}(X)$ from which we obtain $\sqcup Z \sqsubseteq \pi^{-1} \cdot \sqcup(\pi \star Z)$. Since π fixes $supp(\sqsubseteq)$ pointwise, we finally get $\pi \cdot \sqcup Z = \sqcup(\pi \star Z)$. Since $supp(x_0) \cup supp(f) \cup supp(\sqsubseteq) \cup supp(X)$ supports Z, we have $\pi \star Z = Z$, and so $\pi \cdot z_0 = z_0$. For each $z \in Z$ we have $z \sqsubseteq z_0$, and so $z \sqsubseteq f(z) \sqsubseteq f(z_0)$, from which $z_0 \sqsubseteq f(z_0)$, which means that $z_0 \in Z$. However, because f is order-preserving and $supp(f(z)) \subseteq supp(f) \cup supp(z) \subseteq supp(f) \cup supp(x_0) \cup supp(f) \cup supp(\sqsubseteq) \cup supp(X)$ for all $z \in Z$, we have $f(z) \in Z$ for each $z \in Z$, and so, $f(z_0) \sqsubseteq z_0$. □

Theorem 3.4. *Let $(X, \sqsubseteq, \cdot)$ be a finitely supported partially ordered set containing no infinite uniformly supported subset. Let $f : X \to X$ be a finitely supported function with the property that $x \sqsubseteq f(x)$ for all $x \in X$. Then for each $x \in X$, there exists some $m \in \mathbb{N}$ such that $f^m(x)$ is a fixed point of f.*

Proof. Let us fix an arbitrary element $x \in X$. We consider the ascending sequence $(x_n)_{n\in\mathbb{N}}$ which has the first term $x_0 = x$ and the general term $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$. We prove by induction that $supp(x_n) \subseteq supp(f) \cup supp(x)$ for all $n \in \mathbb{N}$. Clearly, $supp(x_0) = supp(x) \subseteq supp(f) \cup supp(x)$. Assume that $supp(x_k) \subseteq supp(f) \cup supp(x)$. Let $\pi \in Fix(supp(f) \cup supp(x))$. Thus, $\pi \cdot x_k = x_k$ according to the inductive hypothesis. According to Proposition 2.2, because π fixes $supp(f)$ pointwise and $supp(f)$ supports f, we get $\pi \cdot x_{k+1} = \pi \cdot f(x_k) = f(\pi \cdot x_k) = f(x_k) = x_{k+1}$. Since $supp(x_{k+1})$ is the least set supporting x_{k+1}, we obtain $supp(x_{k+1}) \subseteq supp(f) \cup supp(x)$. Thus, $(x_n)_{n\in\mathbb{N}} \subseteq X$ is uniformly supported, and so $(x_n)_{n\in\mathbb{N}}$ must be finite. Since, by hypothesis we have $x_0 \sqsubseteq x_1 \sqsubseteq \ldots \sqsubseteq x_n \sqsubseteq \ldots$, there should exist $m \in \mathbb{N}$ such that $x_m = x_{m+1}$, i.e. $f^m(x) = f^{m+1}(x) = f(f^m(x))$, and so the result follows. □

Theorem 3.5. *Let $(X, \sqsubseteq, \cdot)$ be a finitely supported partially ordered set with the property that every uniformly supported subset has a least upper bound. If $f : X \to X$ is a finitely supported function having the properties that $f(\sqcup Y) = \sqcup f(Y)$ for every uniformly supported subset Y of X and there exist $x_0 \in X$ and $k \in \mathbb{N}^*$ such that $x_0 \sqsubseteq f^k(x_0)$, then f has a fixed point.*

Proof. As in the proof of Theorem 3.4, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is uniformly supported by $supp(f) \cup supp(x_0)$. Thus, there exists $\underset{n\in\mathbb{N}}{\sqcup} f^n(x_0)$ supported by $supp(f) \cup supp(x_0) \cup supp(\sqsubseteq)$. Since $x_0 \sqsubseteq f^k(x_0)$, we get $\underset{n\in\mathbb{N}}{\sqcup} f^{n+1}(x_0) = \underset{n\in\mathbb{N}}{\sqcup} f^n(x_0)$, and so $f(\underset{n\in\mathbb{N}}{\sqcup} f^n(x_0)) = \underset{n\in\mathbb{N}}{\sqcup} f(f^n(x_0)) = \underset{n\in\mathbb{N}}{\sqcup} f^{n+1}(x_0) = \underset{n\in\mathbb{N}}{\sqcup} f^n(x_0)$, which means $\underset{n\in\mathbb{N}}{\sqcup} f^n(x_0)$ is a fixed point. □

Lemma 3.4. *Let $(X_1, \cdot), \ldots, (X_n, \cdot)$ be invariant sets (equipped with possibly different S_A-actions). Then $supp((x_1, \ldots, x_n)) = supp(x_1) \cup \ldots \cup supp(x_n)$ for all $x_i \in X_i$, $i \in \{1, \ldots, n\}$.*

Proof. Let $U = (x_1, \ldots, x_n)$ and $S = supp(x_1) \cup \ldots \cup supp(x_n)$. Obviously, S supports U. Indeed, let us consider $\pi \in Fix(S)$. We have that $\pi \in Fix(supp(x_i))$ for all $i \in \{1, \ldots, n\}$. Therefore, $\pi \cdot x_i = x_i$ for all $i \in \{1, \ldots, n\}$, and so $\pi \otimes (x_1, \ldots, x_n) = (\pi \cdot x_1, \ldots, \pi \cdot x_n) = (x_1, \ldots, x_n)$. Thus, $supp(U) \subseteq S$. It remains to prove that $S \subseteq supp(U)$. Fix $\pi \in Fix(supp(U))$. Since $supp(U)$ supports U, we have $\pi \otimes (x_1, \ldots, x_n) = (x_1, \ldots, x_n)$, and so $(\pi \cdot x_1, \ldots, \pi \cdot x_n) = (x_1, \ldots, x_n)$, from which we get $\pi \cdot x_i = x_i$ for all $i \in \{1, \ldots, n\}$. Thus, $supp(U)$ supports x_i for all $i \in \{1, \ldots, n\}$, and so $supp(x_i) \subseteq supp(U)$ for all $i \in \{1, \ldots, n\}$. Therefore, $S = supp(x_1) \cup \ldots \cup supp(x_n) \subseteq supp(U)$. □

Theorem 3.6. *(1) Let X be an infinite, finitely supported subset of an invariant set Y. Then the sets $\wp_{fs}(\wp_{fin}(X))$ and $\wp_{fs}(\wp_{fs}(X))$ contain infinite uniformly supported subsets.*

(2) Let X and Y be two finitely supported subsets of an invariant set Z. If neither X nor Y contain infinite uniformly supported subsets, then $X \times Y$ does not contain an infinite uniformly supported subset.

(3) Let X and Y be two finitely supported subsets of an invariant set Z. If neither X nor Y contain infinite uniformly supported subsets, then $X + Y$ does not contain an infinite uniformly supported subset.

Proof. 1. Obviously, $\wp_{fin}(X)$ is a finitely supported subset of the invariant set $\wp_{fs}(Y)$, supported by $supp(X)$. This is because whenever $Z \in \wp_{fin}(X)$ and $\pi \in Fix(supp(X))$, we have that $\pi \star Z \in \wp_{fin}(X)$. The family $\wp_{fs}(\wp_{fin}(X))$ represents the family of those subsets of $\wp_{fin}(X)$ which are finitely supported as subsets of the invariant set $\wp_{fs}(Y)$. As above, according to Proposition 2.1, we have that $\wp_{fs}(\wp_{fin}(X))$ is a finitely supported subset of the invariant set $\wp_{fs}(\wp_{fs}(Y))$, supported by $supp(\wp_{fin}(X)) \subseteq supp(X)$. Let X_i be the set of all i-sized subsets from X, i.e. $X_i = \{Z \subseteq X \,|\, |Z| = i\}$. Since X is infinite, it follows that each $X_i, i \geq 1$ is non-empty. Obviously, we have that any i-sized subset $\{x_1, \ldots, x_i\}$ of X is finitely supported (as a subset of Y) by $supp(x_1) \cup \ldots \cup supp(x_i)$. Therefore, $X_i \subseteq \wp_{fin}(X)$ and $X_i \subseteq \wp_{fs}(Y)$ for all $i \in \mathbb{N}$. Since $\cdot$ is a group action, the image of an i-sized subset of X under an arbitrary permutation is an i-sized subset of Y. However, any permutation of atoms that fixes $supp(X)$ pointwise also leaves X invariant, and so for any permutation $\pi \in Fix(supp(X))$ we have that $\pi \star Z$ is an i-sized subset of X whenever Z is an i-sized subset of X. Thus, each X_i is a subset of $\wp_{fin}(X)$ finitely supported by $supp(X)$, and so the family $(X_i)_{i \in \mathbb{N}}$ is uniformly supported and infinite.

2. Suppose by contradiction that there is an infinite injective family $((x_i, y_i))_{i \in I} \subseteq X \times Y$ and a finite $S \subseteq A$ with the property that $supp((x_i, y_i)) \subseteq S$ for all $i \in I$. According to Lemma 3.4, we obtain $supp(x_i) \cup supp(y_i) \subseteq S$ for all $i \in I$. Thus, $supp(x_i) \subseteq S$ for all $i \in I$ and $supp(y_i) \subseteq S$ for all $i \in I$. Since the family $((x_i, y_i))_{i \in I}$ is infinite and injective, then at least one of the uniformly supported families $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ is infinite, a contradiction.

3. Suppose by contradiction that $X+Y$ contains an infinite uniformly supported subset. Thus, there exists an infinite injective family $(z_i)_{i \in I} \subseteq X \times Y$ and a finite $S \subseteq A$ such that $supp(z_i) \subseteq S$ for all $i \in I$. According to the construction of the disjoint union of two S_A-sets (see Proposition 2.1), there should exist an infinite family of $(z_i)_i$ of form $((0, x_j))_{x_j \in X}$ which is uniformly supported by S, or an infinite family of form $((1, y_k))_{y_k \in Y}$ which is uniformly supported by S. Since 0 and 1 are constants, this means there should exist at least an infinite uniformly supported family of elements from X, or an infinite uniformly supported family of elements from Y, a contradiction. □

Example 3.1. The following invariant sets do not contain infinite uniformly supported subsets:

(1) The set A of atoms, because there are at most $|S|$ atoms supported by a certain finite set $S \subseteq A$, namely the elements of S.
(2) The set $\wp_{fs}(A)$, because there are at most $2^{|S|+1}$ subsets of A supported by a certain finite set $S \subseteq A$, namely the subsets of S and the supersets of $A \setminus S$.

Proposition 3.3. *The set $\wp_{fs}(A)^A_{fs}$ does not contain infinite uniformly supported subsets.*

Proof. The result follows by involving the proof of Corollary 58 in [4]. □

Corollary 3.3. *(1) The set A^A_{fs} does not contain an infinite uniformly supported subset.*

(2) The set $(A^n)^A_{fs}$ does not contain an infinite uniformly supported subset.

Proof. The first item follows directly from Proposition 3.3. The second item follows from item 1 and Theorem 3.6(2) since there exists an equivariant bijection between $(A^n)^A_{fs}$ and $(A^A_{fs})^n$ defined as below. If $f : A \to A^n$ is finitely supported with $f(a) = (a_1, \dots, a_n)$, we associate to f the Cartesian pair $(f_1, \dots, f_n)$ where for each $i \in \mathbb{N}$, $f_i : A \to A$ is a finitely supported function (supported by $supp(f)$) defined by $f_i(a) = a_i$ for all $a \in A$. □

Theorem 3.7. *(1) Let X be a finitely supported subset of an invariant set. If X does not contain an infinite uniformly supported subset, then each finitely supported injective mapping $f : X \to X$ should be surjective.*

(2) Let X be a finitely supported subset of an invariant set. If $\wp_{fs}(X)$ does not contain an infinite uniformly supported subset, then each finitely supported surjective mapping $f : X \to X$ should be injective. The converse does not hold.

Proof. 1. Assume by contradiction that $f : X \to X$ is a finitely supported injection with the property that $Im(f) \subsetneq X$. This means that there exists $x_0 \in X$ such that $x_0 \notin Im(f)$. We can form a sequence of elements from X which has the first term x_0 and the general term $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$. Since $x_0 \notin Im(f)$ it follows that $x_0 \neq f(x_0)$. Since f is injective and $x_0 \notin Im(f)$, by induction we obtain that $f^n(x_0) \neq f^m(x_0)$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Furthermore, x_{n+1} is supported by $supp(f) \cup supp(x_n)$ for all $n \in \mathbb{N}$. Indeed, let $\pi \in Fix(supp(f) \cup supp(x_n))$. According to Proposition 2.2, $\pi \cdot x_{n+1} = \pi \cdot f(x_n) = f(\pi \cdot x_n) = f(x_n) = x_{n+1}$. Since $supp(x_{n+1})$ is the least set supporting x_{n+1}, we obtain $supp(x_{n+1}) \subseteq supp(f) \cup supp(x_n)$ for all $n \in \mathbb{N}$. By induction, we have $supp(x_n) \subseteq supp(f) \cup supp(x_0)$ for all $n \in \mathbb{N}$. Thus, all x_n are supported by the same set of atoms $supp(f) \cup supp(x_0)$, which means the family $(x_n)_{n\in\mathbb{N}}$ is infinite and uniformly supported, contradicting the hypothesis.

2. Let $f : X \to X$ be a finitely supported surjection. Since f is surjective, we can define the function $g : \wp_{fs}(X) \to \wp_{fs}(X)$ by $g(Y) = f^{-1}(Y)$ for all $Y \in \wp_{fs}(X)$ which is finitely supported by $supp(f) \cup supp(X)$ (according to the S-finite support principle) and injective. Alternatively, we can provide a direct proof that g is finitely supported. Let Y be an arbitrary element from $\wp_{fs}(X)$. We claim that $f^{-1}(Y) \in \wp_{fs}(X)$. Let π fix $supp(f) \cup supp(Y) \cup supp(X)$ pointwise, and $y \in f^{-1}(Y)$. This means $f(y) \in Y$. Since π fixes $supp(f)$ pointwise and $supp(f)$ supports f, we have $f(\pi \cdot y) = \pi \cdot f(y) \in \pi \star Y = Y$, and so $\pi \cdot y \in f^{-1}(Y)$. Therefore, $f^{-1}(Y)$ is finitely supported, and so the function g is well defined. We claim that g is supported by $supp(f) \cup supp(X)$. Let π fix $supp(f) \cup supp(X)$ pointwise. For any arbitrary $Y \in \wp_{fs}(X)$ we get $\pi \star Y \in \wp_{fs}(X)$ and $\pi \star g(Y) \in \wp_{fs}(X)$. Furthermore, π^{-1} fixes $supp(f)$ pointwise, and so $f(\pi^{-1} \cdot x) = \pi^{-1} \cdot f(x)$ for all $x \in X$. For any arbitrary $Y \in \wp_{fs}(X)$, we have that $z \in g(\pi \star Y) = f^{-1}(\pi \star Y) \Leftrightarrow f(z) \in \pi \star Y \Leftrightarrow \pi^{-1} \cdot f(z) \in Y \Leftrightarrow f(\pi^{-1} \cdot z) \in Y \Leftrightarrow \pi^{-1} \cdot z \in f^{-1}(Y) \Leftrightarrow z \in \pi \star f^{-1}(Y) = \pi \star g(Y)$. If follows that $g(\pi \star Y) = \pi \star g(Y)$ for all $Y \in \wp_{fs}(X)$, and so g is finitely supported. Now, since $\wp_{fs}(X)$ does not contain an infinite uniformly supported subset, it follows from item 1 that g is surjective.

Now let us consider two elements $a, b \in X$ such that $f(a) = f(b)$. We prove by contradiction that $a = b$. Suppose that $a \neq b$. Let us consider $Y = \{a\}$ and $Z = \{b\}$. Obviously, $Y, Z \in \wp_{fs}(X)$. Since g is surjective, for Y and Z there is $Y_1, Z_1 \in \wp_{fs}(X)$ such that $f^{-1}(Y_1) = g(Y_1) = Y$ and $f^{-1}(Z_1) = g(Z_1) = Z$. We know that $f(Y) \cap f(Z) = \{f(a)\}$. Thus, $f(a) \in f(Y) = f(f^{-1}(Y_1)) \subseteq Y_1$. Similarly, $f(a) = f(b) \in f(Z) = f(f^{-1}(Z_1)) \subseteq Z_1$, and so $f(a) \in Y_1 \cap Z_1$. Thus, $a \in f^{-1}(Y_1 \cap Z_1) = f^{-1}(Y_1) \cap f^{-1}(Z_1) = Y \cap Z$. However, since we assumed that $a \neq b$, we have that $Y \cap Z = \emptyset$, which represents a contradiction. It follows that $a = b$, and so f is injective.

In order to prove the invalidity of the reverse implication, we prove that any finitely supported surjective mapping $f : \wp_{fin}(A) \to \wp_{fin}(A)$ is also injective, while $\wp_{fs}(\wp_{fin}(A))$

contains an infinite uniformly supported subset (see Theorem 3.6(1)). Let us consider a finitely supported surjection $f : \wp_{fin}(A) \to \wp_{fin}(A)$. Let $X \in \wp_{fin}(A)$. Then $supp(X) = X$ and $supp(f(X)) = f(X)$. Since $supp(f)$ supports f and $supp(X)$ supports X, for any π fixing pointwise $supp(f) \cup supp(X) = supp(f) \cup X$ we have $\pi \star f(X) = f(\pi \star X) = f(X)$, and so $supp(f) \cup X$ supports $f(X)$, that is $f(X) = supp(f(X)) \subseteq supp(f) \cup X$ (claim 1).

For a fixed $m \geq 1$, let us fix m (arbitrarily chosen) atoms $b_1, \dots, b_m \in A \setminus supp(f)$. Let us consider $\mathcal{U} = \{\{a_1, \dots, a_n, b_1, \dots, b_m\} \,|\, a_1, \dots, a_n \in supp(f), n \geq 1\} \cup \{\{b_1, \dots, b_m\}\}$. The set $\mathcal{U}$ is finite since $supp(f)$ is finite and $b_1, \dots, b_m \in A \setminus supp(f)$ are fixed. Let us consider $Y \in \mathcal{U}$, that is $Y \setminus supp(f) = \{b_1, \dots, b_m\}$. There exists $Z \in \wp_{fin}(A)$ such that $f(Z) = Y$. According to (claim 1), Z must be either of form $Z = \{c_1, \dots, c_k, b_{i_1}, \dots, b_{i_l}\}$ with $c_1, \dots, c_k \in supp(f)$ and $b_{i_1}, \dots, b_{i_l} \in A \setminus supp(f)$ or of form $Z = \{b_{i_1}, \dots, b_{i_l}\}$ with $b_{i_1}, \dots, b_{i_l} \in A \setminus supp(f)$. In both cases we have $\{b_1, \dots, b_m\} \subseteq \{b_{i_1}, \dots, b_{i_l}\}$. We should prove that $l = m$. Assume by contradiction that there exists b_{i_j} with $j \in \{1, \dots, l\}$ such that $b_{i_j} \notin \{b_1, \dots, b_m\}$. Then $(b_{i_j}\ b_1) \star Z = Z$ since both $b_{i_j}, b_1 \in Z$ and Z is a finite subset of A (b_{i_j} and b_1 are interchanged in Z under the effect of the transposition $(b_{i_j}\ b_1)$, while the other atoms belonging to Z are left unchanged, meaning that the whole Z is left invariant under $\star$). Furthermore, since $b_{i_j}, b_1 \notin supp(f)$ we have that $(b_{i_j}\ b_1)$ fixes $supp(f)$ pointwise, and, because $supp(f)$ supports f, we get $f(Z) = f((b_{i_j}\ b_1) \star Z) = (b_{i_j}\ b_1) \star f(Z)$ which is a contradiction because $b_1 \in f(Z)$ while $b_{i_j} \notin f(Z)$. Thus, $\{b_{i_1}, \dots, b_{i_l}\} = \{b_1, \dots, b_m\}$, and so $Z \in \mathcal{U}$. Therefore, $\mathcal{U} \subseteq f(\mathcal{U})$ which means $|\mathcal{U}| \leq |f(\mathcal{U})|$. However, since f is a function and $\mathcal{U}$ is finite, we get $|f(\mathcal{U})| \leq |\mathcal{U}|$. We obtain $|\mathcal{U}| = |f(\mathcal{U})|$ and, because $\mathcal{U}$ is finite with $\mathcal{U} \subseteq f(\mathcal{U})$, we get $\mathcal{U} = f(\mathcal{U})$ (claim 2) which means that $f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ is surjective. Since $\mathcal{U}$ is finite, $f|_{\mathcal{U}}$ should be injective, i.e. $f(U_1) \neq f(U_2)$ whenever $U_1, U_2 \in \mathcal{U}$ with $U_1 \neq U_2$ (claim 3).

Whenever $d_1, \dots, d_v \in A \setminus supp(f)$ with$\{d_1, \dots, d_v\} \neq \{b_1, \dots, b_m\}$, $v \geq 1$, and considering $\mathcal{V} = \{\{a_1, \dots, a_n, d_1, \dots, d_v\} \,|\, a_1, \dots, a_n \in supp(f), n \geq 1\} \cup \{\{d_1, \dots, d_v\}\}$, we conclude that $\mathcal{U}$ and $\mathcal{V}$ are disjoint. Whenever $U_1 \in \mathcal{U}$ and $V_1 \in \mathcal{V}$ we have $f(U_1) \in \mathcal{U}$ and $f(V_1) \in \mathcal{V}$ by using the same arguments used to prove (claim 2), and so $f(U_1) \neq f(V_1)$ (claim 4). If $\mathcal{T} = \{\{a_1, \dots, a_n\} \,|\, a_1, \dots, a_n \in supp(f)\}$ and $Y \in \mathcal{T}$, then there is $T' \in \wp_{fin}(A)$ such that $Y = f(T')$. Similarly as in (claim 2), we should have $T' \in \mathcal{T}$. Otherwise, if T' belongs to some $\mathcal{V}$ considered above, i.e. if T' contains an element outside $supp(f)$, we get the contradiction $Y = f(T') \in \mathcal{V}$, and so $\mathcal{T} \subseteq f(\mathcal{T})$ from which $\mathcal{T} = f(\mathcal{T})$ since $\mathcal{T}$ is finite (using similar arguments as those involved to prove (claim 3) from $\mathcal{U} \subseteq f(\mathcal{U})$). Thus, $f|_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ is surjective. Since $\mathcal{T}$ is finite, $f|_{\mathcal{T}}$ should be also injective, namely $f(T_1) \neq f(T_2)$ whenever $T_1, T_2 \in \mathcal{T}$ with $T_1 \neq T_2$ (claim 5). The case $supp(f) = \emptyset$ is contained in the above analysis; it leads to $f(\emptyset) = \emptyset$ and $f(X) = X$ for all $X \in \wp_{fin}(A)$. We also have $f(T_1) \neq f(V_1)$ whenever $T_1 \in \mathcal{T}$ and $V_1 \in \mathcal{V}$ since $f(T_1) \in \mathcal{T}$, $f(V_1) \in \mathcal{V}$ and $\mathcal{T}$ and $\mathcal{V}$ are disjoint (claim 6). Since $b_1, \dots, b_m$ and $d_1, \dots, d_v$ are arbitrarily chosen from $A \setminus supp(f)$, the injectivity of f leads from the claims (3), (4), (5), (6). □

Proposition 3.4. *Let X be a finitely supported subset of an invariant set $(Y, \cdot)$. If X contains a finitely supported, totally ordered subset $(Z, \leq)$, then Z is uniformly supported.*

Proof. We claim that Z is uniformly supported by $supp(\leq) \cup supp(Z)$. Let us consider $\pi \in Fix(supp(\leq) \cup supp(Z))$ and let $z \in Z$. Since π fixes $supp(Z)$ pointwise and $supp(Z)$ supports Z, we obtain that $\pi \cdot z \in Z$, and so we should have either $z < \pi \cdot z$, or $z = \pi \cdot z$, or $\pi \cdot z < z$. If $z < \pi \cdot z$, then, since π fixes $supp(\leq)$ pointwise and since the mapping $u \mapsto \pi \cdot u$ is bijective from Z to $\pi \star Z$, we get $z < \pi \cdot z < \pi^2 \cdot z < \ldots < \pi^n \cdot z < \ldots$ for all $n \in \mathbb{N}$. However, there is $m \in \mathbb{N}$ such that $\pi^m = Id$, and so we get $z < z$ which is a contradiction. Similarly, the assumption $\pi \cdot z < z$, leads to the relation $\ldots < \pi^n \cdot z < \ldots < \pi \cdot z < z$ for all $n \in \mathbb{N}$ which is also a contradiction since π has finite order. Therefore, $\pi \cdot z = z$, and because z was arbitrary chosen form Z, Z should be a uniformly supported. □

Theorem 3.8. *Let X be a finitely supported subset of an invariant set $(Y, \cdot)$. If there exists a finitely supported bijection between X and $X + X$, then X contains an infinite uniformly supported subset. The converse does not hold.*

Proof. Let us consider an element y_1 belonging to an invariant set (whose action is also denoted by $\cdot$) with $y_1 \notin X$ (such an element can be a non-empty element in $\wp_{fs}(X) \setminus X$, for instance). Fix $y_2 \in X$. One can define a mapping $f : X \cup \{y_1\} \to X \times \{0, 1\}$ by $f(x) = \begin{cases} (x, 0) & \text{for } x \in X \\ (y_2, 1) & \text{for } x = y_1 \end{cases}$. Clearly, f is injective and it is supported by $S = supp(X) \cup supp(y_1) \cup supp(y_2)$ because for all π fixing S pointwise we have $f(\pi \cdot x) = \pi \cdot f(x)$ for all $x \in X \cup \{y_1\}$. Therefore, since $X + X$ is actually $X \times \{0, 1\}$, there is a finitely supported injection $g : X \cup \{y_1\} \to X$. The mapping $h : X \to X$ defined by $h(x) = g(x)$ is injective, supported by $supp(g) \cup supp(X)$, and $g(y_1) \in X \setminus h(X)$, which means h is not surjective. According to Theorem 3.7(1), X should contain an infinite uniformly supported subset.

Let us denote $Z = A \cup \mathbb{N} \cong A + \mathbb{N}$. Clearly, Z is an invariant set that contains an infinite uniformly supported subset. Assume by contradiction that there is a finitely supported bijection between Z and $Z + Z$ (denote this by $|Z| = |Z + Z|$), that is $|A \cup \mathbb{N}| = |A + A + \mathbb{N}| = |(\{0, 1\} \times A) \cup \mathbb{N}|$ (obviously, $\mathbb{N} + \mathbb{N} \cong \mathbb{N}$). Thus, there is a finitely supported injection $f' : (\{0, 1\} \times A) \cup \mathbb{N} \to A \cup \mathbb{N}$, and so there exists a finitely supported injection $f : (\{0, 1\} \times A) \to A \cup \mathbb{N}$. We prove that whenever $\varphi : A \to A \cup \mathbb{N}$ is finitely supported and injective, we have $\varphi(a) \in A$ for $a \notin supp(\varphi)$. Let us assume by contradiction that there is $a \notin supp(\varphi)$ such that $\varphi(a) \in \mathbb{N}$. Since $supp(\varphi)$ is finite, there is $b \notin supp(\varphi)$, $b \neq a$. Thus, $(a\, b)$ fixes $supp(\varphi)$ pointwise, and so $\varphi(b) = \varphi((a\, b)(a)) = (a\, b) \diamond \varphi(a) = \varphi(a)$ since $(\mathbb{N}, \diamond)$ is a trivial invariant set. This contradicts the injectivity of φ. We can consider the mappings $\varphi_1, \varphi_2 : A \to A \cup \mathbb{N}$ defined by $\varphi_1(a) = f(0, a)$ for all $a \in A$ and $\varphi_2(a) = f(1, a)$ for all $a \in A$, that are injective and supported by $supp(f)$. Therefore, $f(\{0\} \times A) = \varphi_1(A)$ contains at most finitely many element from $\mathbb{N}$, and $f(\{1\} \times A) = \varphi_2(A)$ also contains at most finitely many element from $\mathbb{N}$. Thus, f is an injection from $(\{0, 1\} \times A)$ to $A \cup T$ where T is a finite subset of $\mathbb{N}$. It follows that $f(\{0\} \times A)$ contains an infinite finitely supported subset of atoms U, and $f(\{1\} \times A)$ contains an infinite finitely supported subset of atoms V. Since f is injective, it follows that U and V are infinite disjoint finitely supported subsets of A, contradicting the fact that any subset of A is either finite or cofinite. □

4. Particular Properties of the Finite Powerset of Atoms

Proposition 4.5. *Let $f : A \to A$ be a finitely supported function. If $Im(f) \setminus supp(f) \neq \emptyset$ or $|Im(f|_{A \setminus supp(f)})| > 1$, then f has infinitely many fixed points.*

Proof. Let $f : A \to A$ be a function that is finitely supported. If there exists $a \notin supp(f)$ with $f(a) = a$, then for each $b \notin supp(f)$ we have that $(a\, b) \in Fix(supp(f))$, and from Proposition 2.2, $f(b) = f((a\, b)(a)) = (a\, b)(f(a)) = (a\, b)(a) = b$. Thus, $f|_{A \setminus supp(f)} = Id$, where Id is the identity mapiing on A. If for all $a \notin supp(f)$ we have $f(a) \neq a$, then we prove that $f(a) \in supp(f)$ for all $a \notin supp(f)$. Assume by contradiction that $f(a) = b \in A \setminus supp(f)$ for a certain $a \notin supp(f)$. Then $(a\, b) \in Fix(supp(f))$, and so $f(b) = f((a\, b)(a)) = (a\, b)(f(a)) = (a\, b)(b) = a$. Let us consider $c \in A \setminus supp(f)$, $c \neq a, b$ (which exists since A is infinite, while $supp(f)$ is finite). Thus, $(a\, c) \in Fix(supp(f))$ and so $f(c) = f((a\, c)(a)) = (a\, c)(f(a)) = (a\, c)(b) = b$. Furthermore, $(b\, c) \in Fix(supp(f))$ and so $f(b) = f((b\, c)(c)) = (b\, c)(f(c)) = (b\, c)(b) = c$, contradicting the functionality of f (we also obtained $f(b) = a$). Thus, $f(a) \in supp(f)$ for any $a \notin supp(f)$. Furthermore, if

$u, v \notin supp(f)$, then we have $f(u), f(v) \in supp(f)$, and because $(u\,v) \in Fix(supp(f))$, we get $f(u) = f((u\,v)(v)) = (u\,v)(f(v)) = f(v)$. Thus, there is $u_0 \in supp(f)$ such that $Im(f|_{A \setminus supp(f)}) = \{u_0\}$. We conclude that either $f|_{A\setminus supp(f)} = Id$ or the image of $f|_{A\setminus supp(f)}$ is an one-element subset of $supp(f)$. If $Im(f) \setminus supp(f) \neq \emptyset$, let us assume by contradiction that we are not in the case $f|_{A\setminus supp(f)} = Id$, which means that $Im(f|_{A\setminus supp(f)})$ is an one-element subset of $supp(f)$. Now let us consider an arbitrary element $x \in supp(f)$. Then for each $\pi \in Fix(supp(f) \cup \{x\})$ we have $\pi(f(x)) = f(\pi(x)) = f(x)$ which means $f(x)$ is supported by $supp(f) \cup \{x\}$, and so $f(x) \in supp(f) \cup \{x\} = supp(f)$. This means $Im(f|_{supp(f)}) \subseteq supp(f)$, and so $Im(f) \subseteq supp(f)$ which contradicts the hypothesis that $Im(f) \setminus supp(f) \neq \emptyset$. Thus, $f|_{A\setminus supp(f)} = Id$, and so f has infinitely many fixed points, namely all the elements in $A \setminus supp(f)$. If the image of $f|_{A\setminus supp(f)}$ contains more than one element, then we are necessarily in the case $f|_{A\setminus supp(f)} = Id$. ☐

Theorem 4.9. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be finitely supported and injective. Then for each $X \in \wp_{fin}(A)$ we have $X \setminus supp(f) \neq \emptyset$ if and only if $f(X) \setminus supp(f) \neq \emptyset$. Moreover, $X \setminus supp(f) = f(X) \setminus supp(f)$. Additionally, if f is order preserving, then $X \setminus supp(f) = f(X \setminus supp(f))$ for all $X \in \wp_{fin}(A)$, and $f(supp(f)) = supp(f)$.*

Proof. Let $Y \in \wp_{fin}(A)$. According to Proposition 2.1(5), we have $supp(Y) = Y$ and $supp(f(Y)) = f(Y)$. According to Proposition 2.2, for any permutation $\pi \in Fix(supp(f) \cup supp(Y)) = Fix(supp(f) \cup Y)$ we have $\pi \star f(Y) = f(\pi \star Y) = f(Y)$ which means $supp(f) \cup Y$ supports $f(Y)$, that is $f(Y) = supp(f(Y)) \subseteq supp(f) \cup Y$ (claim 1). If $Y \subseteq supp(f)$, we get $f(Y) \subseteq supp(f)$ (claim 2). Let $X \in \wp_{fin}(X)$ with $X \subseteq supp(f)$. According to (claim 2) we get $f(X) \subseteq supp(f)$.

Conversely, assume $f(X) \subseteq supp(f)$. Applying (claim 2) by induction we get $f^n(X) \subseteq supp(f)$ for all $n \in \mathbb{N}^*$ (claim 3). Since $supp(f)$ is finite, it should exist $m, k \in \mathbb{N}^*$ with $m \neq k$ such that $f^m(X) = f^k(X)$. Assume $m > k$. Due to the injectivity of f, we get $f^{m-k}(X) = X$ which by (claim 3) leads to $X \subseteq supp(f)$. Therefore, $X \subseteq supp(f)$ if and only if $f(X) \subseteq supp(f)$, and so $X \setminus supp(f) \neq \emptyset$ if and only if $f(X) \setminus supp(f) \neq \emptyset$.

Let be $Z \in \wp_{fin}(A)$ such that $f(Z) \setminus supp(f) \neq \emptyset$ or equivalently $Z \setminus supp(f) \neq \emptyset$. Thus, Z has the form $Z = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ with $a_1, \ldots, a_n \in supp(f)$ and $b_1, \ldots, b_m \in A \setminus supp(f)$, $m \geq 1$, or the form $Z = \{b_1, \ldots, b_m\}$ with $b_1, \ldots, b_m \in A \setminus supp(f)$, $m \geq 1$. According to (claim 1), $f(Z)$ should be $f(Z) = \{c_1, \ldots, c_k, b_{i_1}, \ldots, b_{i_l}\}$ with $c_1, \ldots, c_k \in supp(f)$ and $b_{i_1}, \ldots, b_{i_l} \in A \setminus supp(f)$, or $f(Z) = \{b_{i_1}, \ldots, b_{i_l}\}$ with $b_{i_1}, \ldots, b_{i_l} \in A \setminus supp(f)$. In both cases we have the property that $\{b_{i_1}, \ldots, b_{i_l}\}$ is non-empty (i.e. it should contain at least one element, say b_{i_1}) and $\{b_{i_1}, \ldots, b_{i_l}\} \subseteq \{b_1, \ldots, b_m\}$. If $m = 1$, then $l = 1$ and $b_{i_1} = b_1$. Now let consider $m > 1$. Assume by contradiction that there exists $j \in \{1, \ldots, m\}$ such that $b_j \notin \{b_{i_1}, \ldots, b_{i_l}\}$. Then $(b_{i_1}\ b_j) \star Z = Z$ since both $b_{i_1}, b_j \in Z$ and Z is a finite subset of atoms (b_{i_1} and b_j are interchanged in Z under the effect of the transposition $(b_{i_1}\ b_j)$, while the other atoms belonging to Z are left unchanged, meaning that the whole Z is left invariant under $\star$). Furthermore, since $b_{i_1}, b_j \notin supp(f)$ we have $(b_{i_1}\ b_j) \in Fix(supp(f))$, and by Proposition 2.2 we get $f(Z) = f((b_{i_1}\ b_j) \star Z) = (b_{i_1}\ b_j) \star f(Z)$ which is a contradiction because $b_{i_1} \in f(Z)$ while $b_j \notin f(Z)$. Thus, $\{b_{i_1}, \ldots, b_{i_l}\} = \{b_1, \ldots, b_m\}$, and so $Z \setminus supp(f) = f(Z) \setminus supp(f)$. The case $supp(f) = \emptyset$ is included in the above analysis; it leads to $f(\emptyset) = \emptyset$ and $f(X) = X$ for all $X \in \wp_{fin}(A)$.

Assume now that f is order preserving. Let us fix $X \in \wp_{fin}(A)$. Consider the case $X \setminus supp(f) \neq \emptyset$, that is either $X = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ with $a_1, \ldots, a_n \in supp(f)$ and $b_1, \ldots, b_m \in A \setminus supp(f)$, $m \geq 1$, or $X = \{b_1, \ldots, b_m\}$ with $b_1, \ldots, b_m \in A \setminus supp(f)$, $m \geq 1$. Therefore, we get $X \setminus supp(f) = \{b_1, \ldots, b_m\}$, and by involving the above arguments we have either $f(X \setminus supp(f)) = \{u_1, \ldots, u_i, b_1, \ldots, b_m\}$ with $u_1, \ldots, u_i \in supp(f)$ or $f(X \setminus supp(f)) = \{b_1, \ldots, b_m\}$. In both cases we have $X \setminus supp(f) \subseteq f(X \setminus supp(f))$, and since f

is order preserving we can construct an ascending chain $X \setminus supp(f) \subseteq f(X \setminus supp(f)) \subseteq \ldots \subseteq f^n(X \setminus supp(f)) \subseteq \ldots$. Since for any $n \in \mathbb{N}$ we have that $f^n(X \setminus supp(f))$ is supported by $supp(f) \cup supp(X \setminus supp(f)) = supp(f) \cup supp(X)$ (this follows by induction on n using Proposition 2.2 and the proving method presented in Theorem 3.1(1)) and $\wp_{fin}(A)$ does not contain an infinite uniformly supported subset (the elements of $\wp_{fin}(A)$ supported by $supp(f) \cup supp(X)$ are precisely the subsets of $supp(f) \cup supp(X)$), the related chain should be stationary, that is there exists $m \in \mathbb{N}$ such that $f^m(X \setminus supp(f)) = f^{m+1}(X \setminus supp(f))$. Due to the injectivity of f, this leads to $X \setminus supp(f) = f(X \setminus supp(f))$.

The remaining case is $X \subseteq supp(f)$. Then $X \setminus supp(f) = \emptyset$ and $f(\emptyset) \subseteq supp(f)$. In the finite set $supp(f)$ we can define the chain $\emptyset \subseteq f(\emptyset) \subseteq f^2(\emptyset) \subseteq \ldots \subseteq f^n(\emptyset) \subseteq \ldots$ which is uniformly supported by $supp(f)$. Therefore, there exists $m \in \mathbb{N}$ such that $f^m(\emptyset) = f^{m+1}(\emptyset)$. According to the injectivity of f, we get $X \setminus supp(f) = \emptyset = f(\emptyset) = f(X \setminus supp(f))$.

According to (claim 2) we have $f(supp(f)) \subseteq supp(f)$, and since f is order preserving we can construct in $supp(f)$ the chain $\ldots \subseteq f^n(supp(f)) \subseteq \ldots \subseteq f(supp(f)) \subseteq supp(f)$. Since $supp(f)$ is finite, the chain should be stationary and $f^{m+1}(supp(f)) = f^m(supp(f))$ for some positive integer m. Since f is injective, this leads to $f(supp(f)) = supp(f)$. □

Corollary 4.4. *Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be finitely supported and surjective. Then for each $X \in \wp_{fin}(A)$ we have $X \setminus supp(f) \neq \emptyset$ if and only if $f(X) \setminus supp(f) \neq \emptyset$. In either of these cases we have $X \setminus supp(f) = f(X) \setminus supp(f)$. Furthermore, if f is order preserving, then $X \setminus supp(f) = f(X \setminus supp(f))$ for all $X \in \wp_{fin}(A)$, and $f(supp(f)) = supp(f)$.*

Proof. According to Theorem 3.7(2) a finitely supported surjective mapping $f : \wp_{fin}(A) \to \wp_{fin}(A)$ should be also injective. Then the results follow according to Theorem 4.9. □

We prove that there exist two incomparable (via injections) atomic FSM sets such that no one of them contains an infinite uniformly supported subset, as well as two incomparable atomic FSM sets such that one of them contains an infinite uniformly supported subset and the other one does not contain an infinite uniformly supported subset.

Theorem 4.10. *(1) The sets $\wp_{fin}(A)$ and $T_{fin}(A)$ are incomparable via finitely supported injections, where $T_{fin}(A)$ is the set of all finite injective tuples of atoms.*

(2) The sets $\wp_{fin}(A)$ and $T^{\delta}_{fin}(A)$ are incomparable via finitely supported injections, where $T^{\delta}_{fin}(A) = \underset{n \in \mathbb{N}}{\cup} A^n$ is the set of all finite tuples of atoms (not necessarily injective).

Proof. Each A^n is an invariant set, and so their union $T^{\delta}_{fin}(A)$ is an invariant set that contains an uniformly supported subset (by $\{a\}$), namely $(a), (a, a), \ldots, (a, a, \ldots, a), \ldots$, with a a fixed atom. $T_{fin}(A)$ is an equivariant subset of $T^{\delta}_{fin}(A)$ that does not contain an infinite uniformly supported subset. Let us assume by contradiction that $f : \wp_{fin}(A) \to T^{\delta}_{fin}(A)$ is finitely supported and injective. Let $X \in \wp_{fin}(A)$. Since the support of a finite subset of atoms coincides with the related subset, and the support of a finite tuple of atoms is represented by the set of atoms forming the related tuple (see the proof of Theorem 3.6(2)), according to Proposition 2.2, for any permutation $\pi \in Fix(supp(f) \cup supp(X)) = Fix(supp(f) \cup X)$ we have $\pi \otimes f(X) = f(\pi \star X) = f(X)$, where $\otimes$ is the canonical action on $T^{\delta}_{fin}(A)$ constructed as in Proposition 2.1. This means $supp(f) \cup X$ supports $f(X)$, that is $supp(f(X)) \subseteq supp(f) \cup X$, and so the atoms forming $f(X)$ are contained in $supp(f) \cup X$ (claim 1). Let us take two distinct atoms $b_1, b_2 \in A \setminus supp(f)$. We consider the cases:

Case 1. The tuple $f(\{b_1, b_2\})$ contains only elements from $supp(f)$. Let $c_1 \in A \setminus supp(f)$ distinct from b_1, b_2. Then c_1 does not appear in the tuple $f(\{b_1, b_2\})$, and so the transposition $(b_2\, c_1)$ fixes the tuple $f(\{b_1, b_2\})$. Since $(b_2\, c_1) \in Fix(supp(f))$ according to Proposition 2.2, we get $f(\{b_1, c_1\}) = f((b_2\, c_1) \star \{b_1, b_2\}) = (b_2\, c_1) \otimes f(\{b_1, b_2\}) = f(\{b_1, b_2\})$, contradicting the injectivity of f.

Case 2. The tuple $f(\{b_1, b_2\})$ contains an element outside $supp(f)$. Connecting this assertion with (claim 1), we have that at least b_1 or b_2 appear (possibly multiple times) in the tuple $f(\{b_1, b_2\})$. Say b_1 is in the tuple $f(\{b_1, b_2\})$. Since $(b_1\, b_2) \in Fix(supp(f))$, from Proposition 2.2 we get $f(\{b_1, b_2\}) = f((b_1\, b_2) \star \{b_1, b_2\}) = (b_1\, b_2) \otimes f(\{b_1, b_2\}) = (b_1\, b_2)(f(\{b_1, b_2\}))$, which is a contradiction because b_2 replaces b_1 in the (ordered) tuple $f(\{b_1, b_2\})$ under the effect of the transposition $(b_1\, b_2)$.

Now, let us assume by contradiction that $f : T_{fin}(A) \to \wp_{fin}(A)$ is finitely supported and injective. Let $X \in T_{fin}(A)$. For any permutation $\pi \in Fix(supp(f) \cup supp(X))$ we have $\pi \star f(X) = f(\pi \otimes X) = f(X)$. This means $supp(f) \cup supp(X)$ supports $f(X)$, that is $f(X) = supp(f(X)) \subseteq supp(f) \cup supp(X)$, and so $f(X)$ is contained in the union between $supp(f)$ and the set of atoms forming X (claim 1). Since f is injective and $supp(f)$ is finite, there exist two distinct atoms $b_1, b_2 \in A \setminus supp(f)$ such that $f((b_1, b_2))$ contains at least one atom outside $supp(f)$. Connecting this assertion with (claim 1), we have that at least b_1 or b_2 belong to $f((b_1, b_2))$. We distinguish two cases.

Case 1. $b_1, b_2 \in f((b_1, b_2))$. This means the transposition $(b_1\, b_2)$ interchanges b_1 and b_2 in the finite set $f((b_1, b_2))$, but leaves the set $f((b_1, b_2))$ unchanged, that is $(b_1\, b_2) \star f((b_1, b_2)) = f((b_1, b_2))$. Then, because $(b_1\, b_2) \in Fix(supp(f))$, from Proposition 2.2 we get $f((b_2, b_1)) = f((b_1\, b_2) \otimes (b_1, b_2)) = (b_1\, b_2) \star f((b_1, b_2)) = f((b_1, b_2))$, contradicting the injectivity of f.

Case 2. $b_1 \in f((b_1, b_2))$ and $b_2 \notin f((b_1, b_2))$. According to (claim 1), all the other elements in $f((b_1, b_2))$ (if they exist) belong to $supp(f)$ (claim 2). Let $c_1 \in A \setminus supp(f)$ distinct from b_1, b_2. According to (claim 2) $c_1 \notin f((b_1, b_2))$, and so the transposition $(b_2\, c_1)$ fixes $f((b_1, b_2))$ pointwise. Since $(b_2\, c_1) \in Fix(supp(f))$ according to Proposition 2.2, we get $f((b_1, c_1)) = f((b_2\, c_1) \otimes (b_1, b_2)) = (b_2\, c_1) \star f((b_1, b_2)) = f((b_1, b_2))$, contradicting the injectivity of f. Analogously we treat the case $b_2 \in f((b_1, b_2))$ and $b_1 \notin f((b_1, b_2))$. $\square$

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