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On an improved convergence analysis of a two-step Gauss-Newton type method under generalized Lipschitz conditions

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ABSTRACT. We present a local convergence analysis of a two-step Gauss-Newton method under the generalized and classical Lipschitz conditions for the first- and second-order derivatives. In contrast to earlier works, we use our new idea using a center average Lipschitz conditions through which, we define a subset of the original domain that also contains the iterates. Then, the remaining average Lipschitz conditions are at least as tight as the corresponding ones in earlier works. This way, we obtain: weaker sufficient convergence criteria, larger radius of convergence, tighter error estimates and more precise information on the location of the solution. These advantages are obtained under the same computational effort, since the new Lipschitz functions are special cases of the ones in earlier works. Finally, we give a numerical example that confirms the theoretical results, and compares favorably to the results from previous works.

1. INTRODUCTION

Let us consider the nonlinear least squares problem [8]:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}F(x)^T F(x), \tag{1.1}$$

where F is a Fréchet differentiable operator defined on $\mathbb{R}^n$ with its values on $\mathbb{R}^m$, $m \geq n$. In case when $m = n$, this problem reduces to system of nonlinear equations. The basic method for numerical solving the problem (1.1) is the Gauss-Newton method, which is defined as

$$x_{k+1} = x_k - [F'(x_k)^T F'(x_k)]^{-1} F'(x_k)^T F(x_k),\ k = 0, 1, 2, \dots. \tag{1.2}$$

Derivative free iterative methods are used to approximate a solution of nonlinear least squares problems. The convergence analysis of the these methods in the case of zero as well as nonzero residuals was conducted in [2, 6, 7, 12].

We consider a two-step modification of the Gauss-Newton method [3, 5, 9, 10] for solving the problem (1.1)

$$\begin{cases} z_k = (x_k + y_k)/2, \\ x_{k+1} = x_k - [F'(z_k)^T F'(z_k)]^{-1} F'(z_k)^T F(x_k), \\ y_{k+1} = x_{k+1} - [F'(z_k)^T F'(z_k)]^{-1} F'(z_k)^T F(x_{k+1}),\ k = 0, 1, 2, ..., \end{cases} \tag{1.3}$$

where initial approximations x_0 and y_0 are given. In case when $m = n$, this method reduces to the method proposed by Bartish [4] and Werner [16]. The main feature of multistep methods for solving problem (1.1) is that the matrix of derivatives is calculated once in a few steps. Therefore, the computational cost per iteration increases insignificantly.

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In this paper, we study the local convergence of the method (1.3) for the problem (1.1) with zero as well as non-zero residuals. Furthermore, we compute a radius of convergence, convergence order for this method, and a radius of uniqueness ball for the solution of the problem (1.1).

2. Local Convergence Analysis of Method (1.3)

For our study, we present some definitions of the Lipschitz conditions and lemmas. Let $D \subseteq \mathbb{R}^n$. Let us denote $B(x_*, R) = \{x \in D \subseteq \mathbb{R}^n : \|x - x_*\| < R\}$ as an open ball in $\mathbb{R}^n$. Set $R = \sup\{t \geq 0 : B(x_*, t) \subseteq D\}$ and $\rho(x) = \|x - x_*\|$.

Definition 2.1. We say that the Fréchet derivative F' satisfies the center-Lipschitz condition with L_0 average on $B(x_*, R)$, if

$$\|F'(x) - F'(x_*)\| \leq \int_0^{\|x-x_*\|} L_0(u)du, \tag{2.4}$$

where L_0 is an integrable, positive and non-decreasing function on the interval $[0, \frac{3R}{2}]$.

Suppose that equation

$$\beta L_0(t)t = 1,$$

where $\beta = \|(F'(x_*)^T F'(x_*))^{-1} F'(x_*)^T\|$, has at least one positive solution. Denote by d_0 the smallest such solution. Set $B_0 = B(x_*, R) \cap B(x_*, d_0)$ and $d = \min\{d_0, R\}$.

Definition 2.2. We say that the Fréchet derivative F' satisfies the restricted Lipschitz condition with L average on B_0, if

$$\|F'(y) - F'(x)\| \leq \int_0^{\|y-x\|} L(u)du, \tag{2.5}$$

where L is an integrable, positive and non-decreasing function on the interval $[0, \frac{3d}{2}]$.

Definition 2.3. We say that the Fréchet derivative F' satisfies the Lipschitz condition with L_1 average on $B(x_*, R)$, if

$$\|F'(y) - F'(x)\| \leq \int_0^{\|y-x\|} L_1(u)du, \tag{2.6}$$

where L_1 is an integrable, positive and non-decreasing function on the interval $[0, \frac{3R}{2}]$.

Remark 2.1. We have that $L_0(t) \leq L_1(t)$ for each $t \in [0, \frac{3R}{2}]$, $L(t) \leq L_1(t)$ for each $t \in [0, \frac{3d}{2}]$. The full rank of F' is shown in [9] using L_1 to obtain

$$\|(F'(x_*)^T F'(x_*))^{-1} F'(x_*)^T\| \|F'(x) - F'(x_*)\| \leq \beta \int_0^{\rho(x)} L_1(u)du, \tag{2.7}$$

$$\|(F'(x)^T F'(x))^{-1} F'(x)^T\| \leq \frac{\beta}{1 - \beta \int_0^{\rho(x)} L_1(u)du} \tag{2.8}$$

and

$$\|(F'(x)^T F'(x))^{-1} F'(x)^T - (F'(x_*)^T F'(x_*))^{-1} F'(x_*)^T\| \leq \frac{\sqrt{2}\beta^2 \int_0^{\rho(x)} L_1(u)du}{1 - \beta \int_0^{\rho(x)} L_1(u)du}. \tag{2.9}$$

However, by using condition (2.4), we obtain instead of (2.7)–(2.9) the more precise estimates, respectively:

$$\|(F'(x_*)^T F'(x_*))^{-1} F'(x_*)^T\| \|F'(x) - F'(x_*)\| \leq \beta \int_0^{\rho(x)} L_0(u)du, \tag{2.10}$$

$$\|(F'(x)^T F'(x))^{-1} F'(x)^T\| \leq \frac{\beta}{1 - \beta \int_0^{\rho(x)} L_0(u)du} \tag{2.11}$$

and

$$\|(F'(x)^T F'(x))^{-1} F'(x)^T - (F'(x_*)^T F'(x_*))^{-1} F'(x_*)^T\| \leq \frac{\sqrt{2}\beta^2 \int_0^{\rho(x)} L_0(u)du}{1 - \beta \int_0^{\rho(x)} L_0(u)du}. \tag{2.12}$$

Definition 2.4. We say that F'' satisfies the restricted Lipschitz condition with N average on B_0, if

$$\|F''(y) - F''(x)\| \leq \int_0^{\|y-x\|} N(u)du, \tag{2.13}$$

where N is an integrable, positive and non-decreasing function on the interval $[0, \frac{3d}{2}]$.

Definition 2.5. We say that F'' satisfies the Lipschitz condition with N_1 average on $B(x_*, R)$, if

$$\|F''(y) - F''(x)\| \leq \int_0^{\|y-x\|} N_1(u)du,$$

where L_1 is an integrable, positive and non-decreasing function on the interval $[0, \frac{3R}{2}]$.

Remark 2.2. It also follows that $N(t) \leq N_1(t)$ for each $t \in [0, \frac{3d}{2}]$. The new convergence analysis will be finer than the old one in [9]. We assume that $L_0(t) \leq L(t)$ for each $t \in [0, \frac{3d}{2}]$. If $L_0(t) \leq L(t)$ for each $t \in [0, \frac{3d}{2}]$, then the results in this paper hold with L_0 replacing L.

Let $\mathbb{R}^{m\times n}$, $m \geq n$, denote a set of all $m \times n$ matrices. Then, for a full rank matrix $A \in \mathbb{R}^{m\times n}$, its Moore-Penrose pseudo-inverse [8] is defined as $A^\dagger = (A^T A)^{-1} A^T$.

Next, we present the local convergence analysis of method (1.3) utilizing the preceding notation.

Theorem 2.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$, $m \geq n$, be a twice Fréchet differentiable operator on subset D. Assume that the problem (1.1) has a solution $x_* \in D$ and a Fréchet derivative $F'(x_*)$ has full rank. Suppose that Fréchet derivatives $F'(x)$ and $F''(x)$ satisfy the Lipschitz conditions with L_0, L and N average, respectively*

Furthermore, assume function

$$\begin{aligned} h_0(p) &= (\beta/8)\int_0^p N(u)(p-u)^2 du + \beta p\Big(\int_0^{(3/2)p} L(u)du + \int_0^p L_0(u)du\Big) \\ &+ \sqrt{2}\alpha\beta^2 \int_0^p L_0(u)du - p \end{aligned} \tag{2.14}$$

has a minimal zero r on $[0, d]$, which also satisfies

$$\beta \int_0^r L_0(u)du < 1. \tag{2.15}$$

Then, for each $x_0, y_0 \in B(x_*, r)$ the sequences $\{x_k\}$ and $\{y_k\}$, which are generated by the method (1.3), are well defined, remain in $B(x_*, r)$ for all $k \geq 0$, and converge to x_* such that

$$\rho(x_{k+1}) \leq \gamma\rho(x_k)^3 + \eta\rho(x_k)\rho(y_k) + \theta\rho(z_k), \tag{2.16}$$

$$\rho(y_{k+1}) \leq \gamma\rho(x_{k+1})^3 + (\eta/3)(\rho(x_k) + \rho(y_k) + \rho(x_{k+1}))\rho(x_{k+1}) + \theta\rho(z_k), \tag{2.17}$$

$$r_{k+1} = \max\{\rho(x_{k+1}), \rho(y_{k+1})\} \leq q r_k \leq \cdots \leq q^{k+1} r_0, \tag{2.18}$$

where $r_0 = \max\{\rho(x_0), \rho(y_0)\}$,

$$(2.19) \quad q = \gamma\rho(x_0)^2 + \theta + \eta,$$

$$(2.20) \quad \gamma = \frac{\beta \int_0^{\rho(x_0)} N(u)(\rho(x_0)-u)^2 du}{8\rho(x_0)^3\left(1-\beta\int_0^{\rho(z_0)} L_0(u)du\right)}, \ \theta = \frac{\sqrt{2}\alpha\beta^2\int_0^{\rho(z_0)} L_0(u)du}{\rho(z_0)\left(1-\beta\int_0^{\rho(z_0)} L_0(u)du\right)},$$

$$(2.21) \quad \eta = \frac{\beta\int_0^{\rho(x_0)+\rho(y_0)/2} L(u)du}{(2\rho(x_0)+\rho(y_0))/3\left(1-\beta\int_0^{\rho(z_0)} L_0(u)du\right)},$$

$$(2.22) \quad \alpha = \|F(x_*)\|, \quad \beta = \|(F'(x_*)^T F'(x_*))^{-1} F'(x_*)^T\|.$$

Proof. The proof of this theorem is carried out by induction and similar to the one in [9] but there are some crucial differences, where $L_0,\ L,\ N$ replace $L_1,\ L_1,\ N_1$, respectively. These differences bring in the finer local convergence analysis. Indeed, see also the numerical example. Let choose arbitrary $x_0, y_0 \in B(x_*, r)$. For x_1, y_1 that are generated by (1.3), we have

$$\begin{aligned}
x_1 - x_* &= x_0 - x_* - \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T F(x_0)\\
&= \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T \left[F'(z_0)(x_0 - x_*) - F(x_0) + F(x_*)\right]\\
&+ \left[F'(x_*)^T F'(x_*)\right]^{-1} F'(x_*)^T F(x_*) - \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T F(x_*)\\
&= \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T \left[\left(F'\left(\frac{x_0+x_*}{2}\right)(x_0 - x_*) - F(x_0) + F(x_*)\right)\right.\\
&\left. + \left(F'(z_0) - F'\left(\frac{x_0+x_*}{2}\right)\right)(x_0 - x_*)\right]\\
&+ \left[F'(x_*)^T F'(x_*)\right]^{-1} F'(x_*)^T F(x_*) - \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T F(x_*);\\
y_1 - x_* &= x_1 - x_* - \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T F(x_1)\\
&= \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T \left[F'(z_0)(x_1 - x_*) - F(x_1) + F(x_*)\right]\\
&+ \left[F'(x_*)^T F'(x_*)\right]^{-1} F'(x_*)^T F(x_*) - \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T F(x_*)\\
&= \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T \left[\left(F'\left(\frac{x_1+x_*}{2}\right)(x_1 - x_*) - F(x_1) + F(x_*)\right)\right.\\
&\left. + \left(F'(z_0) - F'\left(\frac{x_1+x_*}{2}\right)\right)(x_1 - x_*)\right]\\
&+ \left[F'(x_*)^T F'(x_*)\right]^{-1} F'(x_*)^T F(x_*) - \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T F(x_*).
\end{aligned}$$

We can write according to Lemma 1 from [16] with the value $\omega = 1/2$

$$\begin{aligned}
&F(x) - F(y) - F'\left(\frac{x+y}{2}\right)(x-y)\\
&= \frac{1}{4}\int_0^1 (1-t)\left[F''\left(\frac{x+y}{2} + \frac{t}{2}(x-y)\right) - F''\left(\frac{x+y}{2} + \frac{t}{2}(y-x)\right)\right](x-y)^2 dt.
\end{aligned}$$

We get by setting $x = x_*$ and $y = x_0$ in the equality above

$$\begin{aligned}
&\left\|F(x_*) - F(x_0) - F'\left(\frac{x_0+x_*}{2}\right)(x_* - x_0)\right\|\\
&\le \frac{1}{4}\int_0^1 (1-t)\int_0^{t\|x_0-x_*\|} N(u)du\|x_0 - x_*\|^2 dt\\
&= \frac{1}{8}\int_0^{\rho(x_0)} N(u)\left(1 - \frac{u}{\rho(x_0)}\right)^2 du\rho(x_0)^2 = \frac{1}{8}\int_0^{\rho(x_0)} N(u)(\rho(x_0)-u)^2 du,\\
&\left\|F'\left(\frac{x_0+y_0}{2}\right) - F'\left(\frac{x_0+x_*}{2}\right)\right\| \le \int_0^{\rho(y_0)/2} L(u)du.
\end{aligned}$$

By the monotonicity of $L_0(u)$, $L(u)$ and $N(u)$ with Lemmas 2.3 [14], 2.4 [11] functions $\frac{1}{t}\int_0^t L_0(u)du$, $\frac{1}{t}\int_0^t L(u)du$ and $\frac{1}{t^3}\int_0^t N(u)(t-u)^2du$ are non-decreasing by t. Hence, from (2.14) and (2.15) it follows that

$$q \le \frac{1}{r_0}\left[\frac{\beta\int_0^{r_0} N(u)(r_0-u)^2du}{8\left(1-\beta\int_0^{r_0} L_0(u)du\right)} + \frac{\beta r_0\int_0^{(3/2)r_0} L(u)du + \sqrt{2}\alpha\beta^2\int_0^{r_0} L_0(u)du}{1-\beta\int_0^{r_0} L_0(u)du}\right]$$
$$< \frac{1}{r}\left[\frac{\beta\int_0^{r} N(u)(r-u)^2du}{8\left(1-\beta\int_0^{r} L_0(u)du\right)} + \frac{\beta r\int_0^{(3/2)r} L(u)du}{1-\beta\int_0^{r} L_0(u)du} + \frac{\sqrt{2}\alpha\beta^2\int_0^{r} L_0(u)du}{1-\beta\int_0^{r} L_0(u)du}\right] \le 1.$$

Thus, by Lemmas in [6, 11, 13, 14], conditions (2.4), (2.5), (2.13), and the afore-derived estimates (2.10)-(2.12), we obtain

$$\|x_1 - x_*\| \le \left\|\left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T\right\|$$
$$\times\left\|\left(F'\left(\frac{x_0+x_*}{2}\right)(x_0-x_*) - F(x_0) + F(x_*)\right) + \left(F'(z_0) - F'\left(\frac{x_0+x_*}{2}\right)\right)(x_0-x_*)\right\|$$
$$+\left\|\left[F'(x_*)^T F'(x_*)\right]^{-1} F'(x_*)^T F(x_*) - \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T F(x_*)\right\|$$
$$\le \frac{\beta\rho(x_0)^3\int_0^{\rho(x_0)} N(u)(\rho(x_0)-u)^2du}{8\rho(x_0)^3\left(1-\beta\int_0^{\rho(z_0)} L_0(u)du\right)}$$
$$+\frac{\beta\rho(x_0)\rho(y_0)\int_0^{\rho(y_0)/2} L(u)du}{\rho(y_0)\left(1-\beta\int_0^{\rho(z_0)} L_0(u)du\right)} + \frac{\sqrt{2}\alpha\beta^2\rho(z_0)\int_0^{\rho(z_0)} L_0(u)du}{\rho(z_0)\left(1-\beta\int_0^{\rho(z_0)} L_0(u)du\right)}$$
$$< \gamma\rho(x_0)^3 + \eta\rho(x_0)\rho(y_0) + \theta\rho(z_0) < qr_0 < r.$$

Similarly,

$$\|y_1 - x_*\| = \left\|\left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T\right\|$$
$$\times\left\|\left(F'\left(\frac{x_1+x_*}{2}\right)(x_1-x_*) - F(x_1) + F(x_*)\right) + \left(F'(z_0) - F'\left(\frac{x_1+x_*}{2}\right)\right)(x_1-x_*)\right\|$$
$$+\left\|\left[F'(x_*)^T F'(x_*)\right]^{-1} F'(x_*)^T F(x_*) - \left[F'(z_0)^T F'(z_0)\right]^{-1} F'(z_0)^T F(x_*)\right\|$$
$$\le \frac{\beta\rho(x_1)^3\int_0^{\rho(x_1)} N(u)(\rho(x_1)-u)^2du}{8\rho(x_1)^3\left(1-\beta\int_0^{\rho(z_0)} L_0(u)du\right)}$$
$$+\frac{\beta\rho(x_1)\rho(z_0')\int_0^{\rho(z_0')} L(u)du}{\rho(z_0')\left(1-\beta\int_0^{\rho(z_0)} L_0(u)du\right)} + \frac{\sqrt{2}\alpha\beta^2\rho(z_0)\int_0^{\rho(z_0)} L_0(u)du}{\rho(z_0)\left(1-\beta\int_0^{\rho(z_0)} L_0(u)du\right)}$$

$$\le \gamma\rho(x_1)^3 + (\eta/3)\rho(x_1)(\rho(x_0)+\rho(y_0)+\rho(x_1)) + \theta\rho(z_0)$$
$$< \gamma\rho(x_0)^3 + (\eta/3)\rho(x_0)(2\rho(x_0)+\rho(y_0)) + \theta\rho(z_0) < qr_0 < r,$$

where $\rho(z_0') = (\rho(x_0)+\rho(y_0)+\rho(x_1))/2$. Therefore, $x_1, y_1 \in B(x_*, r)$ and both (2.16) and (2.17) follow for $k=0$. Also, (2.18) is satisfied $r_1 = \max\{\|x_1-x_*\|, \|y_1-x_*\|\} \le qr_0$.

Similarly an induction step is carried out. □

In case of zero residual ($\alpha = \|F(x_*)\| = 0$) the results of Theorem 2.1 are:

Corollary 2.1. *Suppose that x_* satisfies (1.1), $F(x_*) = 0$, $F(x)$ is a twice Fréchet differentiable operator in $B(x_*, R)$, $F'(x_*)$ has full rank, and both $F'(x)$ and $F''(x)$ satisfy the Lipschitz conditions with L_0, L and N average as in (2.4), (2.5) and (2.13), respectively, where L_0, L and N are* *positive non-decreasing functions on $[0, 3R/2]$. Furthermore, assume function h_0 has a minimal zero r on $[0, R]$, which also satisfies:*

$$\beta \int_0^r L_0(u)du < 1,$$

where

$$h_0(p) = (\beta/8)\int_0^p N(u)(p-u)^2 du + \beta p\Big(\int_0^{(3/2)p} L(u)du + \int_0^p L_0(u)du\Big) - p.$$

Then, the Gauss-Newton type method (1.3) is convergent for each $x_0, y_0 \in B(x_*, r)$ such that

$$\begin{aligned} \rho(x_{k+1}) &\leq \gamma\rho(x_k)^3 + \eta\rho(x_k)\rho(y_k), \\ \rho(y_{k+1}) &\leq \gamma\rho(x_{k+1})^3 + (\eta/3)(\rho(x_k) + \rho(y_k) + \rho(x_{k+1}))\rho(x_{k+1}), \\ r_{k+1} &= \max\{\rho(x_{k+1}), \rho(y_{k+1})\} \leq q r_k \leq \cdots \leq q^{k+1} r_0. \end{aligned}$$

Corollary 2.2. *Convergence order of the iterative method (1.3) in case of zero residual is equal to $1+\sqrt{2}$.*

Theorem 2.2. ***(The uniqueness of solution)*** *Suppose x_* satisfies (1.1) and $F(x)$ has a continuous derivative $F'(x)$ in the ball $B(x_*, r)$. Moreover, $F'(x_*)$ has full rank and $F'(x)$ satisfies the Lipschitz condition with L_0 average (2.4). Let $r > 0$ satisfy*

$$\frac{\beta}{r}\int_0^r L_0(u)(r-u)du + \frac{\alpha\beta_0}{r}\int_0^r L_0(u)du \leq 1, \tag{2.23}$$

where α and β are defined in (2.22) and $\beta_0 = \|[F'(x_*)^T F'(x_*)]^{-1}\|$. Then, x_* is a unique solution of the problem (1.1) in $B(x_*, r)$.

The proof of this theorem is analogous to the one in [6].

Remark 2.3. If $L_0(t) = L(t) = L_1(t)$ and $N(t) = N_1(t)$ then our results reduces to the ones in [9].

Otherwise, the new results improve the old ones. We have as an example :

$$h_0(p) \leq h^{old}(p), \text{ for each } p \in [0, \frac{3d}{2}],$$

where $h^{old}(p)$ is defined as $h_0(p)$ but (L_0, L), N are replaced by L_1, N_1, respectively. Estimate shows that

$$r^{old} \leq r. \tag{2.24}$$

Moreover, if preceding inequality is strict, we obtain a wider choice of initial points x_0. The information on the solution is more precise, since by the new and old results we have, in case L_0, L, L_1, N, N_1 are constant functions (see also the numerical example)

$$r_0 = \frac{2(1-\alpha\beta_0 L_0)}{\beta L_0} \leq r_0^{old} = \frac{2(1-\alpha\beta_0 L)}{\beta L}.$$

Furthermore, the error bounds on the distances $\|x_k - x_*\|$ are more precise. Hence, fewer iteration are needed to ahieve a desired error tolerance. The uniqueness of the solution is improved by (2.23) and (2.24). Finally, notice $\beta\int_0^r L_1(u)du < 1$ [9] $\Rightarrow \beta\int_0^r L_0(u)du < 1$ but not necessarily vice versa unless if $L_1 = L_0$. Hence, the new sufficient convergence criteria are weaker than the ones in [6, 9]. It is worth noticing that these advantages are obtained under the same computational cost, since in practice the computation of functions L_1 and N_1 requires the computations of functions (L_0, L) and N as special cases. The same technique has been used by us to the Newton's method [1], and can be used on other methods too for the local as well as the semi-local case.

3. NUMERICAL EXAMPLE

We provide example to confirm the theoretical results. Define function F on $D = B(0,1)$ by $F(x) = (x+\mu, \lambda x^3 + x - \mu)^T$, $\lambda, \mu \in \mathbb{R}$. Then, we have $x_* = 0$, $\alpha = \sqrt{2}|\mu|$, $\beta = \sqrt{2}/2$, $N(t) = N_1(t) = 6|\lambda|$, $L_0(t) = 6|\lambda|t$, $L(t) = 6|\lambda|(t+A)$, $L_1(t) = 6|\lambda|(t+1)$, where $A = \max_{t \in B_0} |t|$, so $L_0(t) < L(t) \leq L_1(t)$, and $N(t) = N_1(t)$ for each $t \geq 0$. Hence, the new aforementioned advantages hold. We get $r^{old} < r$.

TABLE 1. Value of radii.

	$\lambda = 0.3,\ \mu = 0$	$\lambda = 0.2,\ \mu = 0.3$
r	0.3952	0.4717
r^{old}	0.2668	0.2421

TABLE 2. Results for $\lambda = 0.3,\ \mu = 0$.

k	$\rho(x_{k+1})$	RHS of (2.16)	RHS [9]	$\rho(y_{k+1})$	RHS of (2.17)	RHS [9]
0	4.1445e-03	1.2183e-01	1.9742e-01	1.0732e-04	1.4139e-03	2.2912e-03
1	2.2494e-09	9.4029e-07	1.5237e-06	4.5749e-15	6.7379e-12	1.0919e-11
2	0	2.1750e-23	3.5246e-23	0	0	0

TABLE 3. Results for $\lambda = 0.2,\ \mu = 0.3$.

k	$\rho(x_{k+1})$	RHS of (2.16)	RHS [9]	$\rho(y_{k+1})$	RHS of (2.17)	RHS [9]
0	7.7749e-03	9.9377e-02	2.3756e-01	5.1442e-03	1.2579e-02	1.2786e-01
1	3.8056e-06	3.4768e-04	3.4522e-03	3.7553e-06	2.8613e-04	3.3744e-03
2	1.2863e-12	1.6746e-07	1.9749e-06	1.2863e-12	1.6744e-07	1.9748e-06
3	1.1816e-17	5.6971e-14	6.7192e-13	1.1816e-17	5.6971e-14	6.7192e-13

Let us show that estimates (2.16) and (2.17) are fulfilled. We choose $x_0 = 0.24$ and $y_0 = 0.2401$. Obtained results for problem with zero and non-zero residuals are shown in Tables 2 and 3, respectively. The "RHS" columns show the values of the right-hand side of the estimates (2.16), (2.17) and the corresponding ones from [9]. These results confirm that our new technique gives tighter error estimates.

By comparing these tables, we see that our error estimates are more precise. By solving (2.23), we find the uniqueness radius is $\tau = 2.1712$ for $\lambda = 0.3,\ \mu = 0$ and $\tau = 2.2470$ for $\lambda = 0.2,\ \mu = 0.3$. Then by also solving (2.23), but for L_1 replacing L_0 and L, we find $\tau^{old} = 0.1497$ for $\lambda = 0.3,\ \mu = 0$ and $\tau^{old} = 0.7500$ for $\lambda = 0.2,\ \mu = 0.3$. Notice that $\tau^{old} < \tau$.

4. CONCLUSION

The convergence ball of the iterative methods is very small in general. We introduce the center Lipschitz condition to help us find a more precise location, where the iterates lie, which in turn leads to at least as small as Lipschitz constants and functions. This way we obtain a larger radius of convergence (so, a wider choice of initial points), tighter error bounds on the $\rho(x_k)$, $\rho(y_k)$ (so, fewer iterates are needed to obtain a desired error tolerance) and a better information on the uniqueness ball. These advantages are obtained under the same computational cost as before, since in practice the new Lipschitz constants and functions are special cases of the old ones.

REFERENCES

[1] Argyros, I. K. and Hilout, S., *On an improved convergence analysis of Newton's method*, Appl. Math. Comput., **225** (2013), 372–386

[2] Argyros, I. K. and Magrenán, Á. A., *Iterative Methods and Their Dynamics with Applications: A Contemporary Study*, CRC Press, 2017

[3] Bartish, M. Ia. and Chypurko, A. I., *About applications of a modification of the Gauss-Newton method*, Visnyk of Lviv Univ. Ser. Appl. Math. and Infor., **1** (1999), 3–7 (in Ukrainian)

[4] Bartish, M. Ia., *About one iterative method for solving functional equations*, Dopov. AN URSR. Ser. A., **30** (1968), 387–391 (in Ukrainian)

[5] Bartish, M. Ia., Chypurko, A. I. and Shakhno, S. M., *About one modification of the Gauss-Newton method*, Visnyk of Lviv Univ. Ser. Mech. Math., **42** (1995), 35–38 (in Ukrainian)

[6] Chen, J. and Li, W.,*Convergence of Gauss-Newton method's and uniqueness of the solution*, Appl. Math. Comput., **170** (2005), 686–705

[7] Chong, C., Hu, N. and Wang, J., *Convergence behavior of Gauss-Newton's method and extensions of the Smale point estimate theory*, J. Complex., **26** (2010), 268–295

[8] Dennis, J. M. and Schnabel, R. B., *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, New York, 1983

[9] Iakymchuk, R. P., Shakhno, S. M. and Yarmola H. P., *Convergence analysis of a two-step modification of the Gauss-Newton method and its applications*, Journal of Numerical and Applied Mathematics, **126** (2017), 61–74

[10] Iakymchuk, R., and Shakhno, S., *On the Convergence Analysis of a Two-Step Modification of the Gauss-Newton Method*, PAMM, **14** (2014), 813–814

[11] Shakhno, S. M., *Local convergence of a two-step Newton-like method for solving nonlinear equations under the generalized Lipschitz conditions*, Physical and Mathematical Modeling and Information Technology, **16** (2012), 163-172 (in Ukrainian)

[12] Shakhno, S. M. and Gnatyshyn, O. P., *On an iterative algorithm of order* $1.839\ldots$ *for solving the nonlinear least squares problems*, Appl. Math. Comput., **161** (2005), 253–264

[13] Steward, G. W., *On the continuity of the generalized inverse*, SIAM J. Appl. Math., **17** (1960), 33–45

[14] Wang, X. H., *Convergence of Newton's method and uniqueness of the solution of equations in Banach space*, IMA Journal of Numerical Analysis, **20** (2000), 123–134

[15] Wedin, P.-Å., *Perturbation theory for pseudo-inverses*, BIT Numerical Mathematics, **13** (1973), 217–232

[16] Werner, W., *Über ein Verfahren der Ordnung* $1+\sqrt{2}$ *zur Nullstellenbestimmung*, Numer. Math., **32** (1979), 333–342

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