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New univalence criteria for an integral operator with Mocanu's and Şerb's lemma

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ABSTRACT. In this paper we consider a general integral operator for analytic functions in the open unit disk $\mathbb{U}$ and we obtain sufficient conditions for univalence of this integral operator, using Mocanu's and Şerb's Lemma. This integral operator was considered in a recent work Bărbatu, C. and Breaz, D., [*The univalence conditions for a general integral operator*, Acta Univ. Apulensis Math. Inform., **51** (2019), 75–87]. The results derived in this paper are shown to follow upon specializing the parameters involved in our results. Several corollaries of the main results are also considered.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of the functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} :| z |< 1\}$$

and satisfy the following usual normalization conditions:

$$f(0) = f^{'}(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

We consider the integral operator:

$$\mathcal{M}_n(f_i, g_i)(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} (g_i{}'(t))^{\beta_i} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] \mathrm{dt} \right\}^{\frac{1}{\delta}}, \tag{1.2}$$

where f_i, g_i are analytic in $\mathbb{U}$, and $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ for all $i = \overline{1, n}$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\mathrm{Re}\delta > 0$.

Remark 1.1. The integral operator $\mathcal{M}_n(f_i, g_i)$ defined by (1.2), introduced by Brbatu and Breaz in the paper [1], is a general integral operator of Pfaltzgraff, Kim-Merkes and Oversea types which extends also the other operators as follows:

i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$ and $\beta_1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [7]

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\alpha \mathrm{dt}.$$

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ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [20]

$$\mathcal{G}_\alpha(z) = \int_0^z \left(f'(t)\right)^\alpha \mathrm{dt}.$$

iii) For $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [2]

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [16].

iv) For $\alpha_i - 1 = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [5]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left[f_i'(t)\right]^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [19] .

v) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = 0$ we obtain the integral operator which was defined and studied by Pescar in [17]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left({f_i}'(t)\right)^{\beta_i} \mathrm{dt}\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Frasin in [6] and by Oversea in [12].

vi) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = 0$ we obtain the integral operator which was studied by Ularu in [21]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left({g_i}'(t)\right)^{\beta_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

Thus, the integral operator $\mathcal{M}_n(f_i, g_i)$, introduced here by the formula (1.2), can be considered as an extension and a generalization of these operators above mentioned.

The following two univalence conditions were derived by Pascu [14] and [15].

Theorem 1.1. (Pascu [14]) *Let $f \in \mathcal{A}$ and $\gamma \in \mathbb{C}$. If $Re\gamma > 0$ and*

$$\frac{1 - |z|^{2Re\gamma}}{Re\gamma} \left|\frac{zf''(z))}{f'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)dt\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Theorem 1.2. (Pascu [15]) *Let $\delta \in \mathbb{C}$ with $Re\delta > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1 - |z|^{2Re\delta}}{Re\delta} \left|\frac{zf''(z))}{f'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, then, for any complex γ with $Re\gamma \geq Re\delta$, the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)dt\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Mocanu and erb, on the other hand, proved the next Theorem in [10].

Theorem 1.3. (Mocanu - Şerb [10]) *Let $M_0 = 1,5936...$ the positive solution of equation*

$$(2 - M)\, e^M = 2. \tag{1.3}$$

If $f \in \mathcal{A}$ and

$$\left|\frac{f''(z))}{f'(z)}\right| \leq M_0,$$

for $z \in \mathbb{U}$, then

$$\left|\frac{zf'(z))}{f(z)} - 1\right| \leq 1, \qquad (z \in \mathbb{U})$$

The bound M_0 is sharp.

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma [8].

Lemma 1.1. (General Schwarz Lemma [8]) *Let f be the function regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R, R > 0\}$ with $|f(z)| < M$ for a fixed number $\mathrm{M} > 0$ fixed. If $f(z)$ has one zero with multiplicity order bigger than a positive integer m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} z^m, \quad z \in \mathbb{U}_R.$$

The equality for $z \neq 0$ can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

2. The main univalence criterion

In this paper, we obtain new conditions for the univalence of the general integral operator $\mathcal{M}_n(f_i, g_i)$, by applying the improvement of Becker univalence criteria, obtained by Pascu in the paper [14]. Also, a lemma given by Mocanu and Şerb in the paper [10], will be used to get some part of results.

Theorem 2.4. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$, $i = \overline{1,n}$, M_0 the positive solution of the equation (1.3), $M_0 = 1,5936...$ and $f_i, g_i \in \mathcal{A}$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq M_0, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq M_0, \tag{2.4}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{1}{c}\sum_{i=1}^n |\alpha_i - 1| + \frac{2M_0}{(2c+1)^{\frac{2c+1}{2c}}} \sum_{i=1}^n |\beta_i| + \frac{1}{c}\sum_{i=1}^n |\gamma_i| \leq 1. \tag{2.5}$$

Then, for all δ complex numbers, $Re\delta \geq Re\gamma$, the integral operator $\mathcal{M}_n(f_i, g_i)$, given by (1.2) is in the class $\mathcal{S}$.

Proof. Let us define the function

$$M_n(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \left(g_i{}'(t)\right)^{\beta_i} \left(\frac{g_i(t)}{t}\right)^{\gamma_i} \mathrm{dt},$$

for $f_i, g_i \in \mathcal{A}$, $i = \overline{1,n}$.

The function M_n is regular in $\mathbb{U}$ and satisfy the following usual normalization conditions $M_n(0) = M_n^{'}(0) - 1 = 0$.

After some calculus we have

$$\frac{zM_n''(z)}{M_n'(z)} = \sum_{i=1}^{n} \left[(\alpha_i - 1)\left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} + \gamma_i \left(\frac{zg_i'(z)}{g_i(z)} - 1\right)\right],$$

for all $z \in \mathbb{U}$.

Applying the module and multiplying both members with $\frac{1-|z|^{2c}}{c}$, we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zM_n''(z)}{M_n'(z)}\right| \leq$$

$$\leq \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| \left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + |\beta_i| \, |z| \left|\frac{g_i''(z)}{g_i'(z)}\right| + |\gamma_i| \left|\frac{zg_i'(z)}{g_i(z)} - 1\right|\right], \tag{2.6}$$

for all $z \in \mathbb{U}$.

Using (2.4) and Lemma Mocanu and erb, from (2.6) we get

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < 1, \quad \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and hence, we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zM_n''(z)}{M_n'(z)}\right| \leq$$

$$\leq \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} |\alpha_i - 1| + \frac{1-|z|^{2c}}{c} |z| \, M_0 \sum_{i=1}^{n} |\beta_i| + \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} |\gamma_i|, \tag{2.7}$$

for all $z \in \mathbb{U}$.

Since

$$\max_{|z| \leq 1} \frac{\left(1-|z|^{2c}\right)|z|}{c} = \frac{2}{(2c+1)^{\frac{2c+1}{2c}}}, \tag{2.8}$$

from (2.7) and (2.8) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zM_n''(z)}{M_n'(z)}\right| \leq \frac{1}{c}\sum_{i=1}^{n} |\alpha_i - 1| + \frac{2M_0}{(2c+1)^{\frac{2c+1}{2c}}} \sum_{i=1}^{n} |\beta_i| + \frac{1}{c}\sum_{i=1}^{n} |\gamma_i|, \tag{2.9}$$

for all $z \in \mathbb{U}$.

Using (2.5), from (2.9) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zM_n''(z)}{M_n'(z)}\right| \leq 1, \tag{2.10}$$

for all $z \in \mathbb{U}$.

By Theorem 1.2 it results that $\mathcal{M}_n(f_i, g_i) \in \mathcal{S}$.

□

3. Corollaries and consequences

First of all, upon setting $\delta = 1$ in Theorem 2.4, we obtain the following corollary:

Corollary 3.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_0 the positive solution of the equation (1.3), $M_0 = 1,5936...$ and $f_i, g_i \in \mathcal{A}$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq M_0, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq M_0,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{1}{c}\sum_{i=1}^{n}|\alpha_i - 1| + \frac{2M_0}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}|\beta_i| + \frac{1}{c}\sum_{i=1}^{n}|\gamma_i| \leq 1$$

then, the integral operator $\mathcal{M}_n^*$, defined by

$$\mathcal{M}_n^*(z) = \int_0^z \prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1}(g_i{}'(t))^{\beta_i}\left(\frac{g_i(t))}{t}\right)^{\gamma_i}\right]dt \tag{3.11}$$

is in the class $\mathcal{S}$.

Letting $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.4, we have:

Corollary 3.2. *Let $\gamma, \alpha_i, \beta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_0 the positive solution of the equation (1.3), $M_0 = 1,5936...$ and $f_i, g_i \in \mathcal{A}$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq M_0, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq M_0,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{1}{c}\sum_{i=1}^{n}|\alpha_i - 1| + \frac{2M_0}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}|\beta_i| \leq 1$$

then, the integral operator $\mathcal{F}_n$ defined by

$$\mathcal{F}_n(z) = \int_0^z \prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1}(g_i{}'(t))^{\beta_i}\right]dt \tag{3.12}$$

is in the class $\mathcal{S}$.

Remark 3.2. The integral operator from (3.12) is a known result, proven in [21].

Letting $\delta = 1$ and $\beta_i = 0$ in Theorem 3.1, we have

Corollary 3.3. *Let $\gamma, \alpha_i, \gamma_i$be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_0 the positive solution of the equation (1.3), $M_0 = 1,5936...$ and $f_i, g_i \in \mathcal{A}$, $i = \overline{1,n}$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq M_0, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq M_0,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{1}{c}\sum_{i=1}^{n}|\alpha_i - 1| + \frac{1}{c}\sum_{i=1}^{n}|\gamma_i| \leq 1$$

then, the integral operator $\mathcal{G}_n$ defined by

$$\mathcal{G}_n(z) = \int_0^z \prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1}\left(\frac{g_i(t))}{t}\right)^{\gamma_i}\right]dt \tag{3.13}$$

is in the class $\mathcal{S}$.

Remark 3.3. The integral operator from (3.13) is another known result proven in [11].

Letting $\delta = 1$ and $\alpha_i = 1$ in Theorem 2.4, we obtain the next corollary:

Corollary 3.4. *Let $\gamma, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_0 the positive solution of the equation (1.3), $M_0 = 1,5936...$ and $g_i \in \mathcal{A}$. If*

$$\left| \frac{g_i''(z)}{g_i'(z)} \right| \leq M_0,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{2M_0}{(2c+1)^{\frac{2c+1}{2c}}} \sum_{i=1}^{n} |\beta_i| + \frac{1}{c} \sum_{i=1}^{n} |\gamma_i| \leq 1$$

then, the integral operator $\mathcal{I}_n$ defined by

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left[(g_i{}'(t))^{\beta_i} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] dt \tag{3.14}$$

is in the class $\mathcal{S}$.

Remark 3.4. The integral operator given by (3.14) was proved in [17].

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.4, we obtain:

Corollary 3.5. *Let α be complex numbers, $a = Re\alpha > 0$, M_0 the positive solution of the equation (1.3), $M_0 = 1,5936...$ and $f, g \in \mathcal{A}$. If*

$$\left| \frac{f''(z)}{f'(z)} \right| \leq M_0, \quad \left| \frac{g''(z)}{g'(z)} \right| \leq M_0,$$

for all $z \in \mathbb{U}$, and

$$2(\alpha - 1) \left(\frac{1}{a} + \frac{M_0}{(2a+1)^{\frac{2a+1}{2a}}} \right) \leq 1,$$

then the integral operator $\mathcal{M}$ defined by

$$\mathcal{M}(z) = \left\{ \alpha \int_0^z \left[f(t) g'(t) \frac{g(t)}{t} \right]^{\alpha - 1} dt \right\}^{\frac{1}{\alpha}}, \tag{3.15}$$

is in the class $\mathcal{S}$.

4. Conclusions and further study

The future research will study similar properties using other classes of analytical functions for the integral operators which were defined in this paper, further generalizing the integral operators and getting results related to convexity, starlikeness, univalence conditions, etc.

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