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On the crossing number of join of the wheel on six vertices with the discrete graph

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ABSTRACT. The main aim of the paper is to give the crossing number of join product $W_5 + D_n$ for the wheel W_5 on six vertices, and D_n consisting of n isolated vertices. In the proofs, it will be extend the idea of the minimum numbers of crossings between two different subgraphs from the family of subgraphs which do not cross the edges of the graph W_5 onto the family of subgraphs that cross the edges of W_5 at least twice. Further, we give a conjecture that the crossing number of $W_m + D_n$ is equal to $Z(m+1)Z(n) + (Z(m) - 1)\lfloor \frac{n}{2} \rfloor + n$ for m at least three, and where the Zarankiewicz's number $Z(n) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is defined for $n \geq 1$. Recently, our conjecture was proved for the graphs $W_m + D_n$, for any $n = 3, 4, 5$, by Klešč *et al.*, and also for $W_3 + D_n$ and $W_4 + D_n$ due to the result by Klešč, Schrötter and by Staš, respectively. Clearly, the main result of the paper confirms the validity of this conjecture for the graph $W_5 + D_n$.

1. INTRODUCTION

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane. (For the definition of a *drawing* see [8].) It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let D ($D(G)$) be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

The investigation on the crossing number of graphs is a classical and very difficult problem. Garey and Johnson [6] proved that this problem is NP-complete. Recall that the exact values of the crossing numbers are known for only a few families of graphs. The purpose of this article is to extend the known results concerning this topic. In this article will be used definitions and notations of the crossing numbers of graphs presented by Klešč in [9]. Kulli and Muddebihal [15] described the characterization for all pairs of graphs for which their join product is planar graph. In the paper, some parts of proofs are also based on Kleitman's result [7] on the crossing numbers for some complete bipartite graphs. More precisely, he showed that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{for} \quad m \leq 6.$$

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Again by Kleitman's result [7], the crossing numbers for join of two different paths, join of two different cycles, and also for join of path and cycle were established in [9]. Further, the exact values for crossing numbers of $G + D_n$ and of $G + P_n$ for all graphs G on less than five vertices are determined in [13]. At present, the crossing numbers of the graphs $G + D_n$ are known only for few graphs G of order six, see e.g. [2, 3, 8, 10, 14, 17, 18, 19]. In all these cases, the graph G is usually connected and contains at least one cycle.

The methods in the paper will mostly use the combinatorial properties of cyclic permutations. If we place the graph W_5 on the surface of the sphere, from the topological point of view, the resulting number of crossings of $W_5 + D_n$ does not matter which of the regions in the subdrawing of $W_5 \cup T^i$ is unbounded, but on how the subgraph T^i crosses or does not cross the edges of W_5 (the description of T^i will be justified in Section 2). This representation of T^i can best be described by the idea of a configuration utilizing some cyclic permutation on the pre-numbered vertices of the graph W_5. For the first time, the idea of configurations is converted from the family of subgraphs which do not cross the edges of the graph W_5 of order six onto the family of subgraphs whose edges cross the edges of W_5 at least twice. Due to this algebraic topological approach, we can extend known results for the crossing numbers of new graphs. Some of the ideas and methods were used for the first time in [5].

Based on the ability to generalize the optimal drawing for $W_5 + D_n$ in Fig. 3 onto the drawings of the graphs $W_m + D_n$, we are able to postulate that the crossing numbers of the graphs $W_m + D_n$ are equal to $Z(m+1)Z(n) + (Z(m) - 1)\lfloor \frac{n}{2} \rfloor + n$ for $m \geq 3$. To determine this conjecture the Zarankiewicz's number defined by $Z(n) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is also used. In [11], Klešč *et al.* were established the exact values of the crossings numbers of the graphs $S_3 + C_m$, $S_4 + C_m$, and $S_5 + C_m$. Since the graph $S_n + C_m$ is isomorphic with the graph $W_m + D_n$ for all integers $n \geq 1$ and $m \geq 3$, it is not difficult to verify that their results confirm our conjecture in all cases $W_m + D_n$ for each $n = 3, 4, 5$. Moreover, the results of $W_3 + D_n$ by Klešč and Schrötter [13], and of $W_4 + D_n$ by Staš [20] also establish the validity of this conjecture. The main purpose of this article is to extend these results concerning this topic of the join of the wheel W_5 with the discrete graph D_n. Also in this article, some parts of proofs can be simplified by utilizing the work of the software COGA that generates all cyclic permutations by Berežný and Buša [1]. Its C++ version is located also on the website `http://web.tuke.sk/fei-km/coga/`, and the list with all short names of 120 cyclic permutations of six elements have already been collected in Table 1 of [18] or using COGA.

2. Cyclic permutations and possible drawings of W_5

Let W_5 be the wheel on six vertices. We consider the join product of W_5 with the discrete graph on n vertices denoted by D_n. The graph $W_5 + D_n$ consists of one copy of the graph W_5 and of n vertices $t_1, t_2, \dots, t_n$, where each vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of W_5. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the six edges incident with the vertex t_i. This means that the graph $T^1 \cup \dots \cup T^n$ is isomorphic with the complete bipartite graph $K_{6,n}$ and

$$W_5 + D_n = W_5 \cup K_{6,n} = W_5 \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2.1}$$

In the paper, it will be used the definitions and notation of the cyclic permutations for a good drawing D of the graph $W_5 + D_n$ as in [18]. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counterclockwise order in which the edges leave t_i, as defined by Hernández-Vélez *et al.* [5]. We use the notation (123456) if the counter-clockwise order the edges incident with the vertex t_i is $t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_iv_5$, and t_iv_6. Recall that a rotation is a cyclic permutation. We separate all subgraphs T^i, $i = 1, \dots, n$, of the graph $W_5 + D_n$ into five mutually-disjoint subsets depending on how many times the considered subgraph T^i crosses the edges of W_5 in D. For $i = 1, \dots, n$, let $R_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 0\}$, $S_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 1\}$, $T_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 2\}$, and $U_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 3\}$. Every other subgraph T^i crosses the edges of W_5 at least four times in D. For $T^i \in R_D \cup S_D \cup T_D \cup U_D$, let F^i denote the subgraph $W_5 \cup T^i$, $i \in \{1, 2, \dots, n\}$, of $W_5 + D_n$ and let $D(W_5 \cup T^i)$ be its subdrawing induced by D.

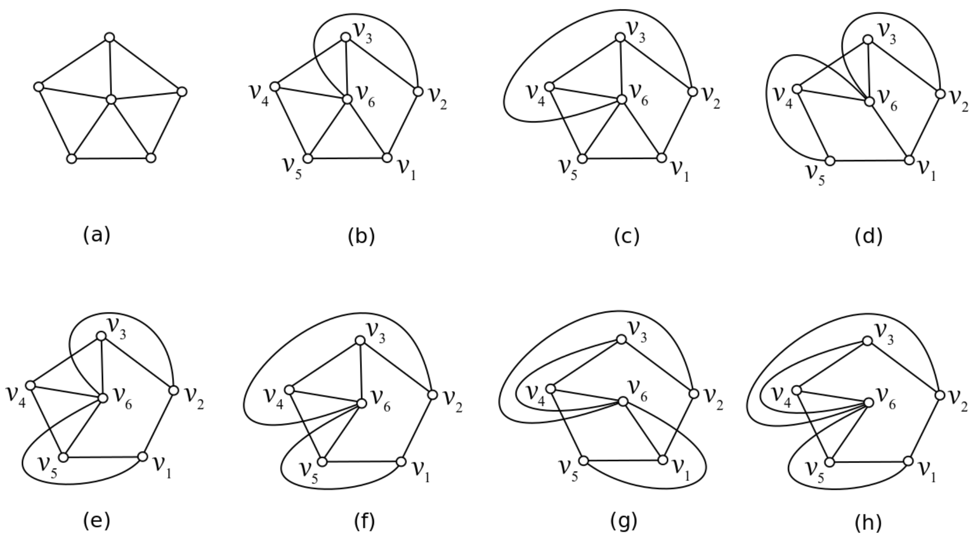

FIGURE 1. Eight possible non isomorphic drawings of the graph W_5 with no crossing among edges of $C_5(W_5)$.

According to the arguments in the proof of the main Theorem 3.1, if we would like to obtain an optimal drawing of $W_5 + D_n$, then the set $R_D \cup S_D \cup T_D$ must be nonempty. Thus, we will only consider drawings of the graph W_5 for which there is the possibility of obtaining a subgraph T^i whose edges cross the edges of W_5 at most twice. Since the graph W_5 consists of one dominating vertex of degree five and of five vertices of degree three which form the subgraph isomorphic with the cycle C_5 (for brevity, we write $C_5(W_5)$), we only need to consider possibilities of crossings between subdrawings of $C_5(W_5)$ and five edges incident with the dominating vertex. Further, due to Lemma 3.1, we obtain at least the considered crossing number of the graph $W_5 + D_n$ if the edges of the cycle $C_5(W_5)$ cross itself in the considered subdrawings of W_5. Let us first consider a good subdrawing of W_5 in which there is no crossing on the edges of $C_5(W_5)$. In this case, we obtain one planar drawing shown in Fig. 1(a). If we consider a good subdrawing of W_5 in which the edges of $C_5(W_5)$ are crossed once, then we obtain two possibilities that are shown in Fig. 1(b) and (c). The drawings of W_5 with two and three crossings are shown in Fig. 1(d), (e), and (f), and in Fig. 1(g) and (h), respectively. The vertex notation of the graph W_5 in Fig. 1 will be justified later.

3. THE CROSSING NUMBER OF $W_5 + D_n$

In the proof of the main theorem, the following Lemma 3.1 and Lemma 3.2 related to some restricted subdrawings of the graph $W_5 + D_n$ will be helpful.

Lemma 3.1. *In any optimal drawing of the join product $W_5 + D_n$, $n \geq 1$, the edges of $C_5(W_5)$ do not cross each other.*

*P*roof. Assume an optimal drawing of the graph $W_5 + D_n$ in which two edges of $C_5(W_5)$ cross. Let x be the point of the plane in which two edges, say $\{c_i, c_{i+1}\}$ and $\{c_j, c_{j+1}\}$, of $C_5(W_5)$ cross. In the rest of paper, let c_1 be neither of the four vertices. Since the plane is a normal space, in the plane there is an open set A_x such that A_x contains x together with the corresponding segments of the crossed edges. Clearly, we can also assume that the dominating vertex of W_5 is not contained in A_x. Thus, all remaining edges of the drawing are disjoint with A_x, see Fig. 2(a). Fig. 2(b) shows that the edges $\{c_i, c_{i+1}\}$ and $\{c_j, c_{j+1}\}$ can be redrawn into new edges $\{c_i, c_j\}$ and $\{c_{i+1}, c_{j+1}\}$ which do not cross. The vertices either $c_1, c_i, c_j, c_{i+1}, c_{j+1}, c_1$ or $c_1, c_j, c_i, c_{j+1}, c_{i+1}, c_1$ form the 5-cycle again. Since each vertex of the cycle $C_5(W_5)$ is adjacent to the dominating vertex of degree five of W_5, the new drawing of the graph $W_5 + D_n$ with less number of crossings is obtained. This contradiction completes the proof. □

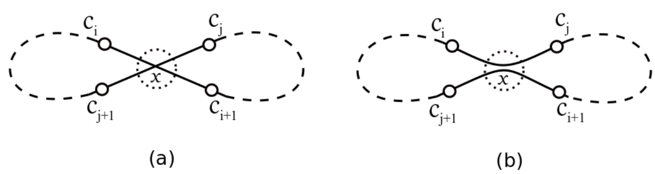

FIGURE 2. Elimination of a crossing in $C_5(W_5)$.

Lemma 3.2. *Let D be a good drawing of $W_5 + D_n$, $n \geq 1$. If the edges of $C_5(W_5)$ are crossed at least $\lceil \frac{n}{2} \rceil$ times, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor$ crossings in D.*

*P*roof. The wheel W_5 consists of two edge-disjoint subgraphs $C_5(W_5)$ and $S_5(W_5)$. Based on the assumption of the number of crossings on the edges of $C_5(W_5)$, let us consider that $\mathrm{cr}_D(C_5(W_5)) + \mathrm{cr}_D(C_5(W_5), S_5(W_5) + D_n) \geq \lceil \frac{n}{2} \rceil$ is fulfilling in the good drawing D of $W_5 + D_n$. The star $S_5(W_5)$ is isomorphic with the complete bipartite graph $K_{1,5}$ and the exact value for the crossing number of the graph $K_{1,5} + D_n$ is given by Mei and Huang [17], i.e., $\mathrm{cr}(K_{1,5,n}) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$. This enforces that the edges of $S_5(W_5) + D_n$ must be crossed at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$ times in D. Consequently, we have

$$\begin{aligned}\mathrm{cr}_D(W_5 + D_n) &= \mathrm{cr}_D(S_5(W_5) + D_n) + \mathrm{cr}_D(C_5(W_5)) + \mathrm{cr}_D(C_5(W_5), S_5(W_5) + D_n) \\ &\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3\left\lfloor \frac{n}{2} \right\rfloor.\end{aligned}$$

□

Now we are able to prove the main result of the paper concerning the crossing number of the join of the wheel W_5 with the discrete graph D_n.

Lemma 3.3. $\mathrm{cr}(W_5 + D_1) = 1$ *and* $\mathrm{cr}(W_5 + D_2) = 5$.

Proof. The graphs $W_5 + D_1$ and $W_5 + D_2$ are isomorphic with the graphs $P_2 + C_5$ and $P_3 + C_5$, respectively. The exact values for the crossing numbers of the graphs $P_m + C_n$ are given in [9], that is, $\mathrm{cr}(P_m + C_n) = Z(m)Z(n) + 1$ for any $m \geq 2$, $n \geq 3$ with $\min\{m, n\} \leq 6$. So, $\mathrm{cr}(W_5 + D_1) = \mathrm{cr}(P_2 + C_5) = 1$ and $\mathrm{cr}(W_5 + D_2) = \mathrm{cr}(P_3 + C_5) = 5$. □

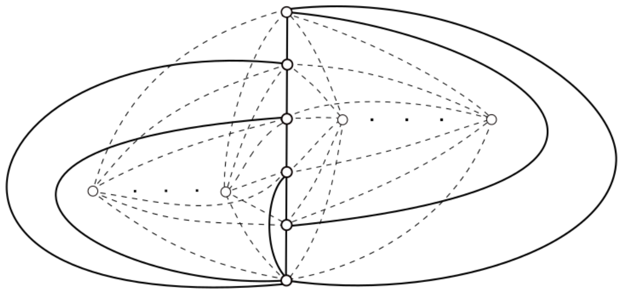

FIGURE 3. The good drawing of $W_5 + D_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor$ crossings.

Theorem 3.1. $\mathrm{cr}(W_5 + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3\left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$.

Proof. In Fig. 3 there is the drawing of $W_5 + D_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor$ crossings. Thus, $\mathrm{cr}(W_5 + D_n) \leq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor$. By Lemma 3.2, the result is true for $n = 1$ and $n = 2$. We prove the reverse inequality by induction on n. Suppose now that, for some $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(W_5 + D_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3\left\lfloor \frac{n}{2} \right\rfloor, \tag{3.2}$$

and let

$$\mathrm{cr}(W_5 + D_m) \geq 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + m + 3\left\lfloor \frac{m}{2} \right\rfloor \qquad \text{for any integer } m < n. \tag{3.3}$$

If $r = |R_D|$, $s = |S_D|$, $t = |T_D|$ and $u = |U_D|$, then the assumption (3.3) together with the well-known fact $\mathrm{cr}(K_{6,n}) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ imply that, in D, there is at least one subgraph T^i by which the edges of W_5 are crossed at most twice. More precisely:

$$\mathrm{cr}_D(W_5) + \mathrm{cr}_D(W_5, K_{6,n}) \leq \mathrm{cr}_D(W_5) + 0r + 1s + 2t + 3u + 4(n - r - s - t - u) < n + 3\left\lfloor \frac{n}{2} \right\rfloor,$$

i.e.,

$$1s + 2t + 3u + 4(n - r - s - t - u) < n + 3\left\lfloor \frac{n}{2} \right\rfloor, \tag{3.4}$$

or easier

$$1s + 2t + 3(n - r - s - t) < n + 3\left\lfloor \frac{n}{2} \right\rfloor. \tag{3.5}$$

This forces that $3r + 2s + t > \lceil \frac{n}{2} \rceil$, and if $r = s = 0$ then $t > \lceil \frac{n}{2} \rceil$. By Lemma 3.1, there is no crossing among edges of $C_5(W_5)$ in all contemplated subdrawings of the graph W_5. Now, we will deal with the possibilities of obtaining a subgraph $T^i \in R_D \cup S_D \cup T_D$ in the drawing D and we show that in all cases the contradiction with the assumption (3.2) is obtained.

Case 1: $\mathrm{cr}_D(W_5) = 0$. The drawing of W_5 is uniquely determined in such a way as shown in Fig. 1(a). It is obvious that the sets R_D and T_D are empty. Further, for $r = 0$ and $t = 0$, the condition (3.4) enforces $s + u > \lfloor \frac{n}{2} \rfloor$. Since each subgraph $T^i \in S_D \cup U_D$ crosses some edge of $C_5(W_5)$ at least once, the edges of the cycle $C_5(W_5)$ must be crossed at least $\lceil \frac{n}{2} \rceil$ times. Lemma 3.2 forces a contradiction with (3.2) in D.

Case 2: $\mathrm{cr}_D(W_5) = 1$. At first, without loss of generality, we can choose the drawing with the vertex notation of W_5 in such a way as shown in Fig. 1(b). Since the sets R_D and S_D are empty, there are at least $\lceil \frac{n}{2} \rceil$ subgraphs T^i whose edges cross the edges of W_5

exactly twice. Further, it is not difficult to verify that one edge of $C_5(W_5)$ is crossed by edges of each such subgraph $T^i \in T_D$ at least once. Again, Lemma 3.2 contradicts the assumption of D.

In addition, without loss of generality, we can choose the vertex notation of the graph W_5 in such a way as shown in Fig. 1(c). Clearly, the sets R_D and S_D are also empty, that is, $t > \lceil \frac{n}{2} \rceil$. Our aim is to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i cross the edges of W_5 exactly twice. Since there is only one subdrawing of $F^i \setminus \{v_4, v_5\}$ represented by the rotation (1236), there are four ways to obtain the subdrawing of F^i depending on which two edges of W_5 are crossed by the edges t_iv_4 and t_iv_5. Namely, the rotations (123465), (124365), (152346), and (152436). The reader can easily verify that a subgraph $T^i \in T_D$ does not cross the edges of $C_5(W_5)$ only if $\mathrm{rot}_D(t_i) = (123465)$. Assume now the set $T_D^* = \{T^i \in T_D : \mathrm{rot}_D(t_i) = (123465)\}$ and let $t_1 = |T_D^*|$. Note that T_D^* is a subset of T_D and therefore, $t_1 \leq t$. Further, we denote by γ the number of all subgraphs which cross the edges of W_5 at least trice but at most four times, and also cross the edges of $C_5(W_5)$ at least once. Hence, there are two subcases to consider:

a) If $\gamma \geq t_1$, then $\lceil \frac{n}{2} \rceil < t \leq \gamma + t - t_1$, which yields that the edges of $C_5(W_5)$ are crossed by at least $\lceil \frac{n}{2} \rceil$ different subgraphs. Consequently, Lemma 3.2 also confirms a contradiction with (3.2) in D.

b) Let $\gamma < t_1$ and let us also assume the subgraph $W_5 \cup T^i$ of $W_5 + D_n$, for some T^i from the nonempty set T_D^*. Then $\mathrm{cr}_D(W_5 \cup T^i, T^j) \geq 2 + 6 = 8$ holds for any $T^j \in T_D^*$ with $j \neq i$ provided that $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$, for more see [21], and $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 2 + 4 = 6$ is fulfilling for any $T^k \in T_D \setminus T_D^*$ again using the properties of cyclic permutations. Thus, by fixing the subgraph $W_5 \cup T^i$, we have

$$\mathrm{cr}_D(W_5 + D_n) \geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 8(t_1 - 1) + 6(t - t_1) + 3\gamma + 5(n - \gamma - t) + 3$$

$$= 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5n + t + 2(t_1 - \gamma) - 5 \geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5n + \left\lceil \frac{n}{2} \right\rceil - 5$$

$$\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3\left\lfloor \frac{n}{2} \right\rfloor.$$

Case 3: $\mathrm{cr}_D(W_5) = 2$. At first, without loss of generality, we can choose the vertex notation of the graph W_5 in such a way as shown in Fig. 1(d). Our aim is also to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if $T^i \in T_D$. The vertex t_i must be placed in the quadrangular region with four vertices either v_1, v_2, v_3, and v_6 or v_1, v_6, v_4, and v_5 of W_5 on its boundary. Thus, the subgraph F^i can be represented by (123645), (136452), or (152364). If we denote by $T_D^* = \{T^i \in T_D : \mathrm{rot}_D(t_i) = (123645)\}$, the same process as in the previous case can be applied.

In addition, without loss of generality, we can consider the drawing of W_5 with the vertex notation in such a way as shown in Fig. 1(e). Clearly, the set R_D is empty, but the set S_D can be nonempty. So, two possible subcases may occur:

a) Let S_D be the nonempty set, that is, $2s + t > \lceil \frac{n}{2} \rceil$. Now, for a $T^i \in S_D$, the subgraph $F^i = W_5 \cup T^i$ is uniquely represented by $\mathrm{rot}_D(t_i) = (142365)$ and $\mathrm{cr}_D(W_5 \cup T^i, T^j) \geq 1 + 6 = 7$ holds for any $T^j \in S_D$ with $j \neq i$ provided that $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$. Moreover, it is not difficult to verify in possible regions of $D(W_5 \cup T^i)$ that $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 6$ is true for any subgraph $T^k \in T_D$, and $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 5$ is also fulfilling for any $T^k \notin S_D \cup T_D$. Thus, by fixing the subgraph $W_5 \cup T^i$, we have

$$\mathrm{cr}_D(W_5+D_n) \geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+7(s-1)+6t+5(n-s-t)+3 = 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor$$

$$+5n+(2s+t)-4 \geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+5n+\left\lceil\frac{n}{2}\right\rceil-4 \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+3\left\lfloor\frac{n}{2}\right\rfloor.$$

b) Let S_D be the empty set, that is, each subgraph T^i crosses the edges of W_5 at least twice. Our aim is again to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i cross the edges of W_5 exactly twice. Since there is only one subdrawing of $F^i \setminus \{v_4\}$ represented by the rotation (12365), there are four ways how to obtain the subdrawing of F^i depending on which two edges of W_5 are crossed by the edge t_iv_4. Namely, the rotations (123465), (123645), (123654), and (124365). One can easily show that the subgraph $T^i \in T_D$ only with either $\mathrm{rot}_D(t_i) = (123465)$ or $\mathrm{rot}_D(t_i) = (123645)$ does not cross the edges of $C_5(W_5)$. Assume now the set $T_D^* = \{T^i \in T_D : \mathrm{rot}_D(t_i) = (123465) \text{ or } \mathrm{rot}_D(t_i) = (123645)\}$ and let $t_1 = |T_D^*|$. Therewith, we define γ by the same way as in the cases above, and if $\gamma \geq t_1$ then we can observe the same arguments.

In the next part, let us suppose that $\gamma < t_1$ and let us also assume the subgraph $W_5 \cup T^i$ of $W_5 + D_n$ with $T^i \in T_D^*$. Now, for this $T^i \in T_D^*$, we will discuss the possibility of obtaining a subdrawing of $W_5 \cup T^i \cup T^k$ in D with just three crossings on edges of the graph $W_5 \cup T^i$ by one subgraph $T^k \in U_D$:

(1) Let the edges of $W_5 \cup T^i$ be crossed by each subgraph $T^k \in U_D$ at least four times. Then $\mathrm{cr}_D(W_5 \cup T^i, T^j) \geq 2+5 = 7$ holds for any $T^j \in T_D^*$ with $j \neq i$ and $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 2+4 = 6$ is fulfilling for any $T^k \in T_D \setminus T_D^*$. Thus, by fixing the subgraph $W_5 \cup T^i$, we have

$$\mathrm{cr}_D(W_5+D_n) \geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+7(t_1-1)+6(t-t_1)+4\gamma+5(n-\gamma-t)+4$$

$$= 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+5n+t+(t_1-\gamma)-3 \geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+5n+\left\lceil\frac{n}{2}\right\rceil-3$$

$$\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+3\left\lfloor\frac{n}{2}\right\rfloor.$$

(2) If there is a $T^k \in U_D$ such that $\mathrm{cr}_D(W_5 \cup T^i, T^k) = 3$, i.e, $\mathrm{cr}_D(T^i, T^k) = 0$, then, by fixing the subgraph $T^i \cup T^k$, we have

$$\mathrm{cr}_D(W_5+D_n) = \mathrm{cr}_D(W_5+D_{n-2}) + \mathrm{cr}_D(T^i \cup T^k) + \mathrm{cr}_D(K_{6,n-2}, T^i \cup T^k)$$

$$+\mathrm{cr}_D(W_5, T^i \cup T^k) \geq 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+n-2+3\left\lfloor\frac{n-2}{2}\right\rfloor+0$$

$$+6(n-2)+2+3 = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+3\left\lfloor\frac{n}{2}\right\rfloor,$$

where the edges of the subgraph $T^i \cup T^k$ are crossed by any T^j, $j \neq i,k$, at least six times due to the well-known fact that $\mathrm{cr}(K_{6,3}) = 6$.

Finally, without loss of generality, we assume the drawing of W_5 with the vertex notation in such a way as shown in Fig. 1(f). Clearly, the set R_D is empty, but the set S_D can be nonempty. So two possible subcases may occur:

a) Let S_D be the nonempty set, that is, there is a subgraph $T^i \in S_D$. Now, for some $T^i \in S_D$, the subgraph $F^i = W_5 \cup T^i$ can be represented by either (124365) or (123465) if either v_2v_3 or v_3v_6 is crossed by the edge t_iv_4, respectively. If there is a $T^i \in S_D$ with $\mathrm{rot}_D(t_i) = (124365)$, then we can easily verify in possible regions of $D(W_5 \cup T^i)$ that $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 6$ for any subgraph $T^k \in S_D \cup T_D \cup U_D$, and $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 5$ for any $T^k \notin S_D \cup T_D \cup U_D$. Thus, by fixing the subgraph $W_5 \cup T^i$, we have

$$\mathrm{cr}_D(W_5+D_n) \geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 6(s+t+u-1) + 5(n-s-t-u) + 3$$

$$= 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 5n + (s+t+u) - 3 \geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor$$

$$+5n + \Big\lceil\frac{n}{2}\Big\rceil - 3 \geq 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + n + 3\Big\lfloor\frac{n}{2}\Big\rfloor,$$

where the condition (3.4) enforces $s+t+u \geq \lceil\frac{n}{2}\rceil$, for $r=0$. If there is a $T^i \in S_D$ only with $\mathrm{rot}_D(t_i) = (123465)$, then $\mathrm{cr}_D(T^i, T^j) \geq 6$ holds for any $T^j \in S_D$ with $j \neq i$ provided that $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$. Further, one can easily verify in possible regions of $D(W_5 \cup T^i)$ that $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 6$ for any subgraph $T^k \in T_D$, and $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 5$ for any $T^k \in U_D$. Thus, by fixing the subgraph $W_5 \cup T^i$, we have

$$\mathrm{cr}_D(W_5+D_n) \geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 7(s-1) + 6t + 5u + 4(n-s-t-u) + 3$$

$$= 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 4n + (3s+2t+u) - 4 \geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor$$

$$+4n + \Big(3n - 3\Big\lfloor\frac{n}{2}\Big\rfloor + 1\Big) - 4 \geq 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + n + 3\Big\lfloor\frac{n}{2}\Big\rfloor,$$

where the condition (3.4) also enforces $3s+2t+u > 3n - 3\lfloor\frac{n}{2}\rfloor$, for $r=0$.

b) Let S_D be the empty set, that is, each subgraph T^i crosses the edges of W_5 at least twice. Further, each subgraph $T^i \in T_D$ is uniquely represented by $\mathrm{rot}_D(t_i) = (142365)$, that is, the edge v_1v_2 of $C_5(W_5)$ is crossed by the edge t_iv_4. As $t > \lceil\frac{n}{2}\rceil$, we can apply Lemma 3.2.

Case 4: $\mathrm{cr}_D(W_5) = 3$. At first, without loss of generality, we can consider the drawing of W_5 with the vertex notation in such a way as shown in Fig. 1(g). In this case, by applying the same process as for the drawing in Fig. 1(b), we obtain at least $\lceil\frac{n}{2}\rceil$ subgraphs $T^i \in T_D$ whose edges cross the edges of $C_5(W_5)$. Hence, by Lemma 3.2, the discussed drawing contradicts the assumption of D again.

Finally, without loss of generality, we assume the drawing of W_5 with the vertex notation in such a way as shown in Fig. 1(h). Clearly, the set S_D is empty, but the set R_D can be nonempty. So, two possible subcases may occur:

a) Let R_D be the nonempty set, that is, there is a subgraph $T^i \in R_D$. Now, for some $T^i \in R_D$, the subgraph $F^i = W_5 \cup T^i$ is uniquely represented by $\mathrm{rot}_D(t_i) = (123465)$ and one can easily verify by a discussion in possible regions of $D(W_5 \cup T^i)$ that $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 6$ holds for any subgraph T^k, $k \neq i$. Thus, by fixing the subgraph $W_5 \cup T^i$, we have

$$\mathrm{cr}_D(W_5+D_n) \geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 6(n-1) + 3 \geq 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + n + 3\Big\lfloor\frac{n}{2}\Big\rfloor.$$

b) Let R_D be the empty set, that is, each subgraph T^i crosses the edges of W_5 at least twice. Since some edges of any such subgraph $T^i \in T_D$ cross also the edges of $C_5(W_5)$, Lemma 3.2 contradicts the assumption (3.2) in D.

Thus, it was shown in all mentioned cases that there is no good drawing D of the graph $W_5 + D_n$ with fewer than $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + 3\lfloor\frac{n}{2}\rfloor$ crossings. This completes the proof of the main theorem. □

4. Conclusions

Let W_n and S_n denote the wheel and the star on $n+1$ vertices, respectively. In general, the graph $S_n + C_m$ is isomorphic with the graph $W_m + D_n$ for all integers $n \geq 1$ and $m \geq 3$. Using the results of Klešč *et al.* [11] and by the aforementioned isomorphisms, the crossing numbers of the graphs $W_m + D_n$ for $n = 3, 4, 5$, and $m \geq 3$ were established. The crossing number of $W_4 + D_n$ for any $n \geq 1$ was recently determined by Staš [20]. Theorem 3.1 extends this result for the graphs $W_5 + D_n$ for any $n \geq 1$. The result in Theorem 3.1 has already been claimed by Ma and Cai [16] (see [4]). Since this paper does not seem to be available in English, we have not been able to verify the results.

As we partially mentioned in the proof of Lemma 3.3, the graphs $W_m + D_1$ and $W_m + D_2$ are isomorphic with the graphs $P_2 + C_m$ and $P_3 + C_m$, respectively. The exact values for the crossing numbers of the graphs $P_m + C_n$ are given by Klešč [9], that is, $\mathrm{cr}(P_m + C_n) = Z(m)Z(n) + 1$ for any $m \geq 2$, $n \geq 3$ with $\min\{m, n\} \leq 6$. This fact allow us to determine another results for the join product of the wheels W_m with the discrete graph on one and two vertices.

Theorem 4.2. $\mathrm{cr}(W_m + D_1) = 1$ *and* $\mathrm{cr}(W_m + D_2) = Z(m) + 1$ *for* $m \geq 3$.

One can easily verify that these results also confirm the validity of our conjecture for the graphs $W_m + D_1$ and $W_m + D_2$. Further, determining the crossing number of a graph $G + D_n$ is an essential step in establishing the so far unknown values of the numbers of crossings of graphs $G + P_n$ and $G + C_n$, where P_n and C_n are the path and the cycle on n vertices, respectively. Using the result in Theorem 3.1 and the optimal drawing of $W_5 + D_n$ in Fig. 3, we are able to postulate that $\mathrm{cr}(W_5 + P_n)$ and $\mathrm{cr}(W_5 + C_n)$ are equal to $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ for $n \geq 2$ and $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 5$ for $n \geq 3$, respectively.

References

[1] Berežný, Š. and Buša, Jr., J., *Algorithm of the Cyclic-Order Graph Program (Implementation and Usage)*, J. Math. Model. and Geometry, **7** (2019), No. 3, 1–8

[2] Berežný, Š. and Staš, M., *Cyclic permutations and crossing numbers of join products of symmetric graph of order six*, Carpathian J. Math., **34** (2018), No. 2, 143–155

[3] Berežný, Š. and Staš, M., *Cyclic permutations and crossing numbers of join products of two symmetric graphs of order six*, Carpathian J. Math., **35** (2019), No. 2, 137–146

[4] Clancy, K., Haythorpe, M. and Newcombe, A., *A survey of graphs with known or bounded crossing numbers*, arXiv:1901.05155 (2019), Section 4.4.1

[5] Hernández-Vélez, C., Medina, C. and Salazar, G., *The optimal drawing of $K_{5,n}$*, Electron. J. Combin., **21** (2014), No. 4, ♯ P4.1, 29 pp.

[6] Garey, M. R. and Johnson, D. S. *Crossing number is NP-complete* SIAM J. Algebraic. Discrete Methods, **4** (1983), 312–316

[7] Kleitman, D. J., *The crossing number of $K_{5,n}$*, J. Combinatorial Theory, **9** (1970), 315–323

[8] Klešč, M., *The crossing number of join of the special graph on six vertices with path and cycle*, Discrete Math., **310** (2010), 1475–1481

[9] Klešč, M., *The join of graphs and crossing numbers*, Electron. Notes in Discrete Math., **28** (2007), 349–355

[10] Klešč, M., Kravecová, D. and Petrillová, J., *The crossing numbers of join of special graphs*, Electrical Engineering and Informatics 2: Proceeding of the Faculty of Electrical Engineering and Informatics of the Technical University of Košice, (2011), 522–527

[11] Klešč, M., Petrillová, J. and Valo, M., *On the crossing numbers of Cartesian products of wheels and trees*, Discuss. Math. Graph Theory, **71** (2017), 339–413

[12] Klešč, M. and Schrötter, Š., *The crossing numbers of join of paths and cycles with two graphs of order five*, Combinatorial Algorithms, Sprinder, LNCS, **7125** (2012), 160–167

[13] Klešč, M. and Schrötter, Š., *The crossing numbers of join products of paths with graphs of order four*, Discuss. Math. Graph Theory, **31** (2011), 321–331

[14] Klešč, M. and Valo, M., *Minimum crossings in join of graphs with paths and cycles*, Acta Electrotechnica et Informatica, **12** (2012), No. 3, 32–37

[15] Kulli, V. R. and Muddebihal, M. H. *Characterization of join graphs with crossing number zero* Far East J. Appl. Math., **5** (2001), 87–97

[16] Ma, Z. and Cai, J., *The Crossing Number of* $W_5 \times S_n$, Acta. Math. Appl. Sinica, **31** (2008), No. 4, 615–623

[17] Mei, H. and Huang, Y., *The Crossing Number of* $K_{1,5,n}$, Int. J. Math. Combin., **1** (2007), No. 1, 33–44

[18] Staš, M., *Cyclic permutations: Crossing numbers of the join products of graphs*, Proc. Aplimat 2018: 17^{th} Conference on Applied Mathematics, (2018), 979–987

[19] Staš, M., *Determining crossing numbers of graphs of order six using cyclic permutations*, Bull. Aust. Math. Soc., **98** (2018), 353–362

[20] Staš, M., *On the crossing number of the join of the wheel on five vertices with the discrete graph*, Bull. Aust. Math. Soc., **101** (2020), 353–361

[21] Woodall, D. R., *Cyclic-order graphs and Zarankiewicz's crossing number conjecture*, J. Graph Theory, **17** (1993), 657–671

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