CARPATHIAN J. MATH. 36 (2020), No. 3, 415 - 422
Online version at https://www.carpathian.cunbm.utcluj.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Quantitative approximation by nonlinear convolution operators of Landau-Choquet type

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ABSTRACT. By using the concept of Choquet nonlinear integral with respect to a monotone set function, we introduce the nonlinear convolution operators of Landau-Choquet type, with respect to a family of submodular set functions. Quantitative approximation results in terms of the modulus of continuity are obtained with respect to some particular possibility measures. For some subclasses of functions we prove that these Landau-Choquet type operators can have essentially better approximation properties than their classical correspondents.

1. INTRODUCTION

In 1885, Karl Weierstrass obtained in [18] his famous result regarding the uniform approximation of continuous functions on a compact interval, by polynomial sequences. His proof is heavily based on the the sequence of singular convolution integrals (what we today call as Gauss-Weierstrass operators)

$$W_n(f)(x) = \frac{1}{n\sqrt{\pi}} \int_{\mathbb{R}} f(s) \cdot e^{-((s-x)/n)^2} ds, n \in \mathbb{N}, x \in \mathbb{R}.$$

In 1908, Edmund Landau recaptured in [15] the Weierstrass' result by using the sequence of convolution operators defined by (what we today call as Landau operators)

$$L_n(f)(x) = \frac{\sqrt{n}}{\sqrt{\pi}} \int_0^1 f(s)[1-(s-x)^2]^n ds,\ n \in \mathbb{N}, x \in [0,1].$$

In a very recent paper [7], in essence by replacing in the expression of the Gauss-Weierstrass operators defined just above, the usual linear Lebesgue integral with the nonlinear Choquet integral, the first author has introduced and studied nice approximation properties of the so-called now Gauss-Weierstrass-Choquet operators.

The main goal of the present paper is to use the same idea for the above Landau operators and to introduce and study the approximation properties of the now called Landau-Choquet operators.

It is worth mentioning here that the well-known Feller's probabilistic scheme in constructing linear and positive approximation operators (see [4], Chapter 7), was extended in [5] by replacing the classical linear integral with respect to a measure, with the nonlinear Choquet integral with respect to a monotone set-valued function (capacity). Also, in the papers [5]-[13], approximation results for various nonlinear approximation operators based on the Choquet integral with respect to a family of submodular set functions were obtained.

In the present paper, for the Landau-Choquet integral operators with respect to some particular possibility measures, we obtain quantitative approximation results in terms of

Received: 26.10.2019. In revised form: 30.01.2020. Accepted: 06.02.2020
2010 *Mathematics Subject Classification.* 41A36, 41A25, 28A10, 28A12, 28A25.
Key words and phrases. *monotone set function, possibility measure, nonlinear Choquet integral, nonlinear Landau-Choquet operators, modulus of continuity.*
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the modulus of continuity $\omega_1(f; \cdot)$ and, for some subclasses of functions we get essentially better approximation properties than their classical correspondents.

2. Preliminaries

In order to give the reader a flavor on the topic, firstly in this section we present some concepts and results concerning the Choquet integral.

Definition 2.1. Let $(\Omega, \mathcal{C})$ be a measurable space, i.e. Ω is a nonempty set and $\mathcal{C}$ be a σ-ring (or σ-algebra) of subsets in Ω with $\emptyset \in \mathcal{C}$.

(i) (see, e.g., [17], p. 63) The set function $\mu : \mathcal{C} \to [0, +\infty]$ is called a monotone measure (or capacity) if $\mu(\emptyset) = 0$ and $A, B \in \mathcal{C}$, with $A \subset B$, implies $\mu(A) \leq \mu(B)$. The monotone measure μ is called normalized if $\Omega \in \mathcal{C}$ and $\mu(\Omega) = 1$.

(ii) (see [1], or, e.g., [16], or, e.g., [17], p. 179) Let μ be a normalized monotone measure and consider $\mathcal{G} = \{X : \Omega \to \mathbb{R}_+; X \text{ is measurable on } (\Omega, \mathcal{C})\}$. Recall that $X : \Omega \to \mathbb{R}$ is measurable (or more precisely $\mathcal{C}$-measurable), if for any B, Borelian subset in $\mathbb{R}$, we have $X^{-1}(B) \in \mathcal{C}$.

For $A \in \mathcal{C}$ and $X \in \mathcal{G}$, the Choquet integral of X on A with respect to a monotone measure μ is defined by

$$(C) \int_A X d\mu = \int_0^\infty \mu(F_\alpha(X) \bigcap A) d\alpha,$$

where

$$F_\alpha(X) = \{\omega \in \Omega; X(\omega) \geq \alpha\}.$$

If

$$(C) \int_A X d\mu < +\infty$$

then X is called Choquet integrable on A.

If $X : \Omega \to \mathbb{R}$ is of arbitrary sign, then the Choquet integral is defined by (see [17], p. 233)

$$(C) \int_A X d\mu = \int_0^{+\infty} \mu(F_\alpha(X) \bigcap A) d\alpha + \int_{-\infty}^0 [\mu(F_\alpha(X) \bigcap A) - \mu(A)] d\alpha.$$

When μ is the Lebesgue measure, then the Choquet integral $(C) \int_A X d\mu$ reduces to the Lebesgue integral.

(iii) A possibility measure is a set function $P : \mathcal{P}(\Omega) \to [0, 1]$, satisfying the axioms $P(\emptyset) = 0$, $P(\Omega) = 1$ and $P(\bigcup_{i \in I} A_i) = \sup\{P(A_i); i \in I\}$ for all $A_i \subset \Omega$, and any I, an at most countable family of indices. Note that if $A, B \subset \Omega$, $A \subset B$, then the last property easily implies that $P(A) \leq P(B)$ and that

$$P(A \bigcup B) \leq P(A) + P(B).$$

A function $\lambda : \Omega \to [0, 1]$ is called possibility distribution if $\sup\{\lambda(\omega); \omega \in \Omega\} = 1$. Any possibility distribution λ on Ω, induces the possibility measure $P_\lambda : \mathcal{P}(\Omega) \to [0, 1]$, given by the formula $P_\lambda(A) = \sup\{\lambda(s); s \in A\}$, for all $A \subset \Omega$, $P_\lambda(\emptyset) = 0$ (see, e.g., [3], Ch. 1).

Some known properties of the Choquet integral are expressed by the following.

Remark 2.1. Let us suppose that μ is a monotone measure. Then, the following properties hold :

(i) $(C) \int_A$ is non-additive (i.e. $(C) \int_A (f + g) d\mu \neq (C) \int_A f d\mu + (C) \int_A g d\mu$) but it is positive homogeneous, i.e. for all $a \geq 0$ we have $(C) \int_A a f d\mu = a \cdot (C) \int_A f d\mu$ (for $f \geq 0$ see, e.g., [17], Theorem 11.2, (5), p. 228 and for f of arbitrary sign, see, e.g., [2], p. 64, Proposition 5.1, (ii)).

If $f \leq g$ on A then the Choquet integral is monotone, that is $(C)\int_A f d\mu \leq (C)\int_A g d\mu$ (see, e.g., [17], p. 228, Theorem 11.2, (3) for $f, g \geq 0$ and p. 232 for f, g of arbitrary sign).

If μ is submodular too (i.e. $\mu(A\bigcup B) + \mu(A\bigcap B) \leq \mu(A) + \mu(B)$ for all A, B) then the Choquet integral is subadditive, that is $(C)\int_A (f+g) d\mu \leq (C)\int_A f d\mu + (C)\int_A g d\mu$, for all f, g of arbitrary sign (see, e.g., [2], p. 75, Theorem 6.3).

If $\overline{\mu}$ denotes the dual measure of μ (that is $\overline{\mu}(A) = \mu(\Omega) - \mu(\Omega \setminus A)$, for all $A \in \mathcal{C}$), then for all f of arbitrary sign we have $(C)\int_A (-f) d\mu = -(C)\int_A f d\overline{\mu}$ (see, e.g., [17], Theorem 11.7, p. 233).

If $c \in \mathbb{R}$ and f is of arbitrary sign, then $(C)\int_A (f+c) d\mu = (C)\int_A f d\mu + c \cdot \mu(A)$ (see, e.g., [17], pp. 232-233, or [2], p. 65).

By the definition of the Choquet integral, if $F \geq 0$ and μ is subadditive, then it is immediate that

$$(C)\int_{A\bigcup B} F d\mu \leq (C)\int_A F d\mu + (C)\int_B F d\mu.$$

Note that if μ is submodular then it is clear that it is subadditive too.

(ii) Simple concrete examples of monotone and submodular set functions μ, can be obtained from a probability measure M on $\mathcal{P}(\mathbb{X})$ (i.e. $M(\emptyset) = 0$, $M(\mathbb{X}) = 1$ and M is countable additive), by the formula $\mu(A) = \gamma(M(A))$, where $\gamma : [0,1] \to [0,1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ (see, e.g., [2], pp. 16-17, Example 2.1). Concrete examples for $\gamma(x)$ are $\gamma(x) = \sqrt{x}$, $\gamma(x) = \frac{2x}{1+x}$, so on. If, in addition, M is the Lebesgue measure, then μ are called distorted Lebesgue measures.

Also, any possibility measure μ is monotone and submodular. While the monotonicity is immediate from the axiom $\mu(A\bigcup B) = \max\{\mu(A), \mu(B)\}$, the submodularity is immediate from the property $\mu(A\bigcap B) \leq \min\{\mu(A), \mu(B)\}$.

(iii) Many other properties of the Choquet integral can be found in, e.g., Chapter 11 in [17], or in [2].

Now, we present the following general approximation result which will be used in the next sections.

Theorem 2.1. ([5], Theorem 3.3 and Remark 3.5) *Denoting by $\mathcal{P}(\mathbb{R})$ the class of all subsets of $\mathbb{R}$, let $(\mathbb{R}, \mathcal{C})$ be a measurable space with $\mathcal{C} \subset \mathcal{P}(\mathbb{R})$ and $\mu_{n,x} : \mathcal{C} \to [0, +\infty)$, be a monotone and submodular family of set functions.*

For $\lambda_{n,x} : \mathbb{R} \to \mathbb{R}_+$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, Choquet densities with respect to $\mu_{n,x}$, (that is, $(C)\int_{\mathbb{R}} \lambda_{n,x}(t) d\mu_{n,x}(t) = 1$), let us define by $UC(\mathbb{R})$, the class of all functions $f : \mathbb{R} \to \mathbb{R}_+$, uniformly continuous on $\mathbb{R}$, such that $f \cdot \lambda_{n,x}$ are $\mathcal{C}$-measurable and $T_n(f)(x) < +\infty$, for all $n \in \mathbb{N}$, $x \in \mathbb{R}$, where

$$T_n(f)(x) = (C)\int_{\mathbb{R}} f(t) \cdot \lambda_{n,x}(t) d\mu_{n,x}(t).$$

Then, denoting $\varphi_x(t) = |t - x|$, for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $\delta > 0$ we have

$$|T_n(f)(x) - f(x)| \leq \left[1 + \frac{T_n(\varphi_x)(x)}{\delta}\right] \cdot \omega_1(f; \delta)_{\mathbb{R}}.$$

Also, choosing above $\delta = T_n(\varphi_x)(x)$, it follows

$$|T_n(f)(x) - f(x)| \leq 2\omega_1(f; T_n(\varphi_x)(x))_{\mathbb{R}}.$$

Remark 2.2. The above Theorem 2.1 remains valid for functions and operators defined on compact intervals too. Indeed, analysing the proof of Theorem 3.3 in [5], it is easily seen that it remains valid for $\lambda_{n,x} : I \to \mathbb{R}_+$, $f : I \to \mathbb{R}_+$ and $T_n(f)(x) = (C)\int_I f(t) \cdot \lambda_{n,x}(t) d\mu_{n,x}(t)$, where $I \subset \mathbb{R}$ is a compact subinterval. In fact, for $I = [0,1]$, Theorem 3.3 in [5] was implicitly used in the case of Bernstein-Durrmeyer-Choquet operators (see the proof of Theorem 3.1, (i) in [11]) and in the case of Bernstein-Kantorovich-Choquet operators (see the proof of Theorem 3.3 in [6]).

3. LANDAU-CHOQUET OPERATORS

The classical Landau linear operators $L_n(f)(x)$ defined in Introduction can be generalized to the nonlinear Landau-Choquet operators by the formula

$$L_{n,\mu_{n,x}}(f)(x) = \frac{1}{c(n,x,\mu_{n,x})}(C)\int_0^1 f(s)[1-(s-x)^2]^n d\mu_{n,x}(s),$$

where $f : [0,1] \to \mathbb{R}_+$, $\{\mu_{n,x}\}$, $n \in \mathbb{N}, x \in \mathbb{R}$, is a family of monotone and submodular set functions depending on n and x too, $F_\alpha([1-(s-x)^2]^n) = \{s \in [0,1]; [1-(s-x)^2]^n \geq \alpha\}$ and

$$c(n,x,\mu_{n,x}) = \int_0^\infty \mu_{n,x}[F_\alpha([1-(\cdot-x)^2]^n)]d\alpha.$$

In this section we study the approximation properties of the Landau-Choquet operators in the case when the family $\{\mu_{n,x}\}$ is defined as the possibility measures induced by the possibility distributions $\lambda_{n,x}(t) = [1-(t-x)^2]^n$, that is

$$\mu_{n,x}(A) = \sup\{\lambda_{n,x}(s); s \in A\} = \sup\{[1-(s-x)^2]^n; s \in A\}, \text{ for all } A \subset [0,1],$$

(see Definition 2.1, (iii)).

It is easy to see that any possibility measure $\mu_{n,x}$ is bounded, monotone and submodular, therefore we are under the hypothesis of Theorem 2.1.

We have the following.

Theorem 3.2. *Let $\mu_{n,x}$ for all $n \in \mathbb{N}$, $x \in [0,1]$ as above. If $f : [0,1] \to \mathbb{R}_+$ is continuous on $[0,1]$, then*

$$|L_{n,\mu_{n,x}}(f)(x) - f(x)| \leq 2\omega_1\left(f; 1/\sqrt{2n+1}\right)_{[0,1]},$$

for all $n \in \mathbb{N}$ and $x \in [0,1]$, where

$$\omega_1(f;\delta)_{[0,1]} = \sup\{|f(x)-f(y)|; |x-y| \leq \delta, x,y \in [0,1]\}$$

represents the modulus of continuity of f.

Proof. By Theorem 2.1 and Remark 2.2, for all $n \in \mathbb{N}$ and $x \in [0,1]$ we get the estimate

$$|L_{n,\mu_{n,x}}(f)(x) - f(x)| \leq 2\omega_1(f; L_{n,\mu_{n,x}}(\varphi_x)(x))_{[0,1]}, \tag{3.1}$$

Therefore, the convergence of $L_{n,\mu_{n,x}}(f)$ to f one relies on the convergence to zero, as $n \to \infty$, of the quantity

$$L_{n,\mu_{n,x}}(\varphi_x)(x) = \frac{1}{c(n,x,\mu_{n,x})} \cdot (C)\int_0^1 |t-x| \cdot [1-(t-x)^2]^n d\mu_{n,x}(t)$$

$$= \frac{1}{c(n,x,\mu_{n,x})} \cdot \int_0^\infty \mu_{n,x}[\{t \in [0,1]; |t-x| \cdot [1-(t-x)^2]^n \geq \alpha\}]d\alpha.$$

Firstly, we calculate $c(n,x,\mu_{n,x})$. We get

$$c(n,x,\mu_{n,x}) = (C)\int_0^1 [1-(t-x)^2]^n d\mu_{n,x}(t) = \int_0^\infty \mu_{n,x}(\{t \in [0,1]; [1-(t-x)^2]^n \geq \alpha\})d\alpha$$

$$\int_0^1 \mu_{n,x}(\{t \in [0,1]; [1-(t-x)^2]^n \geq \alpha\})d\alpha + \int_1^\infty \mu_{n,x}(\{t \in [0,1]; [1-(t-x)^2]^n \geq \alpha\})d\alpha$$

$$= \int_0^1 \mu_{n,x}(\{t \in [0,1]; [1-(t-x)^2]^n \geq \alpha\})d\alpha$$

$$= \int_0^1 \sup\{[1-(t-x)^2]^n; t \in [0,1], [1-(t-x)^2]^n \geq \alpha\} d\alpha = \int_0^1 1 d\alpha = 1.$$

In what follows, we calculate

$$(C)\int_0^1 |t-x| \cdot [1-(t-x)^2]^n d\mu_{n,x}(t).$$

For that purpose, taking into account that

$$|t-x| \cdot [1-(t-x)^2]^n \leq \sup\{t \in [0,1]; |t-x| \cdot [1-(t-x)^2]^n\},$$

we will calculate

$$A_{n,x} := \sup\{t \in [0,1]; |t-x| \cdot [1-(t-x)^2]^n\}.$$

We have

$$A_{n,x} = \max\{\sup\{(t-x)[1-(t-x)^2]^n; t \in [x,1]\}, \quad \sup\{(x-t)[1-(t-x)^2]^n; t \in [0,x]\}\}.$$

Denote

$$H_{n,x}(t) = (t-x)[1-(t-x)^2]^n, t \in [x,1].$$

We get

$$H'_{n,x}(t) = [1-(t-x)^2]^{n-1}[1-(t-x)^2(2n+1)],$$

which by $H'_{n,x}(t) = 0$ implies $(t-x)^2 = \frac{1}{2n+1}$ and therefore

$$t = x + \frac{1}{\sqrt{2n+1}}$$

is a maximum point of $H_{n,x}$. Therefore, the maximum value of $H_{n,x}(t)$ for $t \in [x,1]$ is

$$H_{n,x}(x + 1/\sqrt{2n+1}) = \frac{1}{\sqrt{2n+1}}\left[1 - \frac{1}{2n+1}\right]^n \leq \frac{1}{\sqrt{2n+1}}.$$

Also, denoting

$$G_{n,x}(t) = (x-t)[1-(t-x)^2]^n, t \in [0,x],$$

by

$$G'_{n,x}(t) = [1-(t-x)^2]^{n-1}(-1+(2n+1)(x-t)^2) = 0,$$

it follows that $t = x - \frac{1}{\sqrt{2n+1}}$ is a maximum pount for $G_{n,x}(t)$ and using similar reasonings as above, has the maximum value

$$G_{n,x}(x - 1/\sqrt{2n+1}) \leq \frac{1}{\sqrt{2n+1}}.$$

Therefore,

$$(C)\int_0^1 |t-x| \cdot [1-(t-x)^2]^n d\mu_{n,x}(t) \leq (C)\int_0^1 \frac{1}{\sqrt{2n+1}} d\mu_{n,x}(t) = \frac{1}{\sqrt{2n+1}} \cdot (C)\int_0^1 1 d\mu_{n,x}$$

$$= \frac{1}{\sqrt{2n+1}} \cdot \mu_{n,x}([0,1]) = \frac{1}{\sqrt{2n+1}} \cdot \sup\{[1-(s-x)^2]^n; s \in [0,1]\} = \frac{1}{\sqrt{2n+1}}.$$

Concluding, by (3.1) it immediately follows the estimate in the statement. □

Remark 3.3. Note that the positivity of function f in Theorem 3.2 is necessary because of the positive homogeneity of the Choquet integral used in its proof. However, if f is of arbitrary sign on $[0,1]$ and lower bounded, then the statement of Theorem 3.2 can be restated for the slightly modified operator defined by

$$\overline{L}_{n,\mu_{n,x}}(f)(x) = L_{n,\mu_{n,x}}(f-c)(x) + c,$$

where $f(x) \geq c$, for all $x \in [0,1]$.

Indeed, this is immediate from the fact that $\omega_1(f-c;\delta)_{[0,1]} = \omega_1(f;\delta)_{[0,1]}$ and from the equality

$$\overline{L}_{n,\mu_{n,x}}(f)(x) - f(x) = L_{n,\mu_{n,x}}(f-c)(x) - (f(x)-c).$$

Remark 3.4. Following the lines in [14], p. 312, it easily follows that the order of approximation of f by the classical Landau operators defined in Introduction is $\omega_1(f;1/\sqrt{n})_{[0,1]}$. Therefore, by Theorem 3.2 it follows that the Landau-Choquet operators give the same order of approximation. However, for large classes of functions, the Landau-Choquet operators can give the essential better approximation order $\omega_1(f;1/n)_{[0,1]}$.

In this sense, let us consider $f:[0,1]\to\mathbb{R}_+$, with the properties $0 < f(x)$, for all $x\in[0,1]$, f is nondecreasing on $[0,1]$ and $\ln(f(x))$ is a Lipschitz function on $[0,1]$ with the Lipschitz constant 2. Keeping the same notation, we can extend f on $[1,2]$ by continuity, taking $f(x)=f(1)$, for all $x\in[1,2]$, such that f remains nondecreasing and that $\ln[f(x)]$ remains Lipschitz function with the Lipschitz constant 2, on the larger interval $[0,2]$.

We will show that we have

$$f(x) \le L_{n,\mu_{n,x}}(f)(x) \le f\left(x+\frac{1}{n}\right),$$

for all $x\in[0,1]$. Then, for the order of approximation we will get

$$0 \le L_{n,\mu_{n,x}}(f)(x) - f(x) \le f\left(x+\frac{1}{n}\right) - f(x) \le \omega_1\left(f;\frac{1}{n}\right)_{[0,1]}, \text{ for all } x\in[0,1], \tag{3.2}$$

which is essentially better than the order $\mathcal{O}(\omega_1(f;1/\sqrt{n})_{[0,1]})$ given by Theorem 3.2 and by the classical Landau operator.

Indeed, let $x\in[0,1]$ be fixed. By the definition of the Choquet integral in the formula for the Landau-Choquet operators, we easily obtain

$$L_{n,\mu_{n,x}}(f)(x) = \int_0^\infty \sup\{[1-(s-x)^2]^n; s\in[0,1], f(s)\cdot[1-(s-x)^2]^n \ge \alpha\}d\alpha.$$

We firstly show that for all $s\in[0,1]$, we have

$$f(s)\cdot[1-(s-x)^2]^n \le f\left(x+\frac{1}{n}\right).$$

If $s\le x$ then

$$f(s)\le f(x)\le f\left(x+\frac{1}{n}\right),$$

which implies

$$f(s)\cdot[1-(s-x)^2]^n \le f\left(x+\frac{1}{n}\right).$$

Also, when $s>x$, let us denote $s=x+h$, with $h>0$. We have two cases : (i) $h\le\frac{1}{n}$; (ii) $h>\frac{1}{n}$.

In the case (i), since

$$f(x+h)\le f\left(x+\frac{1}{n}\right),$$

we immediately get

$$f(s)\cdot[1-(s-x)^2]^n \le f\left(x+\frac{1}{n}\right).$$

Let us consider now the case (ii). The inequality required to be proved is evidently equivalent to

$$0\le \ln[f(x+h)] - \ln[f(x+1/n)] \le -n\ln(1-h^2), \text{ for all } x\in[0,1], 1>h>\frac{1}{n}.$$

By hypothesis, we also have

$$0 \leq \ln(f(x+h)) - \ln(f\,(x+1/n)) \leq 2(h - \frac{1}{n}).$$

Therefore, if we would prove that

$$2(h-1/n) \leq -n\ln(1-h^2),$$

then that would lead again to the above required inequality.

For this goal, denoting

$$G(h) = 2(h-1/n) + n\ln(1-h^2), 1 > h \geq \frac{1}{n},$$

we get

$$G(1/n) = n\ln(1-1/n^2) < 0$$

and

$$G'(h) = \frac{2}{1-h^2}(1-h^2-nh) \leq \frac{2}{1-h^2}[1-\frac{1}{n^2}-1] = \frac{2}{1-h^2}[-\frac{1}{n^2}] < 0,$$

for all $\frac{1}{n} \leq h < 1$. This means that G is nonincreasing and therefore

$$G(h) \leq 0, \text{ for all } \frac{1}{n} \leq h < 1,$$

which implies the required inequality.

In continuation, we easily get

$$L_{n,\mu_{n,x}}(f)(x) = \int_0^{+\infty} \sup\{[1-(s-x)^2]^n; s \in [0,1], f(s)\cdot[1-(s-x)^2]^n \geq \alpha\}d\alpha$$

$$= \int_0^{f(x+1/n)} \sup\{[1-(s-x)^2]^n; s \in [0,1], f(s)[1-(s-x)^2]^n \geq \alpha\}d\alpha$$

$$+ \int_{f(x+1/n)}^{+\infty} \sup\{[1-(s-x)^2]^n; s \in [0,1], f(s)\cdot[1-(s-x)^2]^n \geq \alpha\}d\alpha$$

$$= \int_0^{f(x+1/n)} \sup\{[1-(s-x)^2]^n; s \in [0,1], f(s)\cdot[1-(s-x)^2]^n \geq \alpha\}d\alpha := E_{n,x}(f).$$

Now, it is easy to see that

$$E_{n,x}(f) \leq f(x+1/n)$$

and that

$$E_{n,x}(f)(x) \geq \int_0^{f(x)} \sup\{[1-(s-x)^2]^n; s \in [0,1], f(s)\cdot[1-(s-x)^2]^n \geq \alpha\}d\alpha$$

$$= \int_0^{f(x)} 1 d\alpha = f(x).$$

It is clear that all the strictly positive, differentiable and non-decreasing functions, with

$$0 \leq \frac{f'(x)}{f(x)} \leq 2, \text{ for all } x \in [0,1],$$

belong to the above class of functions.

Remark 3.5. It is of interest to study quantitative estimates in approximation by Landau-Choquet integra operators with respect to other submodular set functions, like for example the distorted Lebesgue measures defined by Remark 2.1, (ii). A generic example would be

$$\mu(A) = \sqrt{M(A)},$$

where M is the Lebesgue measure. Also, the method used in this paper suggests to introduce and study the approximation properties for Choquet variants of other integral operators, different from those studied by the papers mentioned in References.

Acknowledgement. We thank the referees for their useful comments.

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