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A generalized class of integral operators

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ABSTRACT. We introduce in the present note a unified approach to define integral operators, which include many well-known operators viz. Durrmeyer type operators, mixed hybrid operators as special cases. We also obtain the quantitative estimates between the difference of such integral operators with the discrete operators having same and different basis functions. Our operators proposed here give a very large class of integral operators, which have been discussed and proposed by several researchers in past seven decades.

1. THE OPERATORS

Miheşan in [15, (4.3)] proposed a discrete sequence of linear positive operators, which reproduce linear functions as follows

$$M_{n,\gamma}(f,x) = \sum_{k=0}^{\infty} m_{n,k}^{\gamma}(x) F_{n,k}(f), \tag{1.1}$$

where

$$m_{n,k}^{\gamma}(x) = \frac{(\gamma)_k}{k!} \frac{\left(\frac{nx}{\gamma}\right)^k}{\left(1+\frac{nx}{\gamma}\right)^{\gamma+k}}, F_{n,k}(f) = f\left(\frac{k}{n}\right).$$

These operators provide some of the well known operators as particular cases for different values of γ. Some operators of integral type and their approximation properties have been discussed in the book [10]. Also, very recently author in [7] and [8] proposed large families of linear positive operators for functions discretely defined at $f(0)$ and the link operators respectively. This paper is a continuation of these two article, here we propose generalized sequence of integral operators of usual Durrmeyer type operators, which include many well-known operators, including actual Durrmeyer and several hybrid operators as special cases. For $x \in \mathbb{R}^+ \equiv [0,\infty)$ and α, β non-zero real numbers or a function of n (as indicated below), we introduce

$$V_{n,\alpha,\beta}(f,x) = \sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x) G_{n,k}^{\beta}(f) \tag{1.2}$$

where $G_{n,k}^{\beta}(f) = \frac{n(\beta-1)}{\beta}\langle m_{n,k}^{\beta}, f\rangle$ with $\langle f,g\rangle = \int_0^{\infty} f(t)g(t)dt$ and $m_{n,k}^{\alpha}(x)$ defined in (1.1).

(1) If $\alpha=\beta=n/c$, $c \in \mathbb{N}_0$, we get the well known operators due to Heilmann–Müller (see [14]),
(2) If $\alpha=\beta=n$, we get Baskakov-Durrmeyer operators considered in [18],
(3) If $\alpha=\beta\to\infty$, we get the Szász-Durrmeyer operators (see [16]),
(4) If $\alpha=\beta=-n$, we get the Bernstein-Durrmeyer polynomials introduced in [4], in this case $x\in[0,1]$ and summation is for $0\le k\le n$,

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(5) If $\alpha \neq \beta$ and $\alpha = n, \beta \to \infty$, we get Baskakov-Szász operators [12],
(6) If $\alpha \neq \beta$ and $\alpha \to \infty, \beta = n$, we get the Szász-Baskakov operators [17],
(7) If $\alpha \neq \beta$ and $\alpha = nx, \beta = n$, we get the Lupaş -Baskakov operators [5] for the case $d = 1, c = 1$,
(8) If $\alpha \neq \beta$ and $\alpha = nx, \beta \to \infty$, we get the Lupaş -Szász operators [5] for the case $d = 0, c = 1$,
(9) $\alpha \neq \beta$ and $\alpha = nx, \beta = nt$, we get the Lupaş -Durrmeyer operators considered by Agratini [2], but in this case the weight i.e. integral $\int_0^\infty m_{n,k}^{\beta}(t)dt$ is not independent of k as considered in all above cases, here this integral depends on k, which cause problem in finding moments. Agratini in [2] calculated this integral in terms of Stirling number of first kind. Using Agratinis result, the operator can be written as follows:

$$\begin{aligned} V_{n,\alpha,\beta}(f,x) &= \sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x) \frac{\langle m_{n,k}^{\beta}, f\rangle}{\langle m_{n,k}^{\beta}, 1\rangle} \\ &= \sum_{k=0}^{\infty} \frac{m_{n,k}^{\alpha}(x)}{\left((n2^k.k!)^{-1}\sum_{i=0}^{k}(-1)^{k-i}s_{k,i}.i!(\log 2)^{-i-1}\right)} \langle m_{n,k}^{\beta}, f\rangle, \end{aligned}$$

due to such complicated form of the operators, such operators are not appropriate as far as convergence is concerned. In a similar way if $\alpha \neq \beta$ and $\alpha = n, \beta = nt$, one can define the hybrid Baskakov–Lupaş operators, and the similar problem arise for the weights under integral sign. So in all above cases, if $\beta = nt$ such problem arise. This can be considered as open problem for researchers, as far as the moment estimations of the operators are concerned.

We can write kernel of our operators as

$$\begin{aligned} K_{n,\alpha,\beta}(x,t) &= \sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x) m_{n,k}^{\beta}(t) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} . \frac{\left(\frac{nx}{\alpha}\right)^k}{\left(1+\frac{nx}{\alpha}\right)^{\alpha+k}} \frac{(\beta)_k}{k!} . \frac{\left(\frac{nt}{\beta}\right)^k}{\left(1+\frac{nt}{\beta}\right)^{\beta+k}} \\ &= \left(1+\frac{nx}{\alpha}\right)^{-\alpha}\left(1+\frac{nt}{\beta}\right)^{-\beta} \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(1)_k.k!} . \frac{\left(n^2xt\right)^k}{[(\alpha+nx)(\beta+nt)]^k} \\ &= \left(1+\frac{nx}{\alpha}\right)^{-\alpha}\left(1+\frac{nt}{\beta}\right)^{-\beta} {}_2F_1\left(\alpha,\beta;1;\frac{n^2xt}{(\alpha+nx)(\beta+nt)}\right), \end{aligned}$$

where ${}_2F_1(a,b;c;x) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r}\frac{x^r}{r!}, |x| < 1$ and $(p)_r$ is the Pochhammer symbol (rising factorial). This kernel can be linked with some special functions by assigning different values to α and β although such link can be found in different places separately, but for readers ready reference, we mention here for general case as follows:

(1) If $\alpha = \beta \to \infty$, we get the Szász-Durrmeyer operators

$$\begin{aligned} \lim_{\alpha,\beta\to\infty} K_{n,\alpha,\beta}(x,t) &= \lim_{\alpha,\beta\to\infty} \left(1+\frac{nx}{\alpha}\right)^{-\alpha}\left(1+\frac{nt}{\beta}\right)^{-\beta} {}_2F_1\left(\alpha,\beta;1;\frac{n^2xt}{(\alpha+nx)(\beta+nt)}\right) \\ &= e^{-n(x+t)} \sum_{v=0}^{\infty} \frac{(n^2xt)^v}{v!\Gamma(v+1)} = e^{-n(x+t)} I_0(2n\sqrt{xt}), \end{aligned}$$

where I_0 is the modified Bessel's function of first kind of zero order (see also [3]).

(2) If $\alpha = n, \beta \to \infty$, we get the Baskakov-Szász operators

$$\lim_{\alpha\to n,\beta\to\infty} K_{n,\alpha,\beta}(x,t) = \lim_{\alpha\to n,\beta\to\infty} \left(1+\frac{nx}{\alpha}\right)^{-\alpha}\left(1+\frac{nt}{\beta}\right)^{-\beta} {}_2F_1\left(\alpha,\beta;1;\frac{n^2xt}{(\alpha+nx)(\beta+nt)}\right)$$
$$= (1+x)^{-n}e^{-nt}\sum_{v=0}^{\infty}\frac{(n)_v}{v!}\frac{\left(\frac{nxt}{1+x}\right)^v}{v!} = (1+x)^{-n}e^{-nt}\,{}_1F_1\left(n;1;\frac{nxt}{1+x}\right)$$
$$= (1+x)^{-n}e^{-nt}e^{\frac{nxt}{1+x}}\,{}_1F_1\left(1-n;1;\frac{-nxt}{1+x}\right),$$

where ${}_1F_1(a;b;x) = \sum_{r=0}^{\infty}\frac{(a)_r}{(b)_r}\frac{x^r}{r!}$ is the confluent hypergeometric function of the first kind and we have applied Kummer's transformation

$${}_1F_1(b-a;b;z) = e^z\,{}_1F_1(a,b;-z).$$

Thus in case of Baskakov-Szász operators, kernel takes the following form

$$(1+x)^{-n}e^{\frac{-nt}{1+x}}L^0_{n-1}\left(\frac{-nxt}{1+x}\right),$$

where $L^0_n(x)$ is the generalized Laguerre function.

(3) If $\alpha \to \infty, \beta = n$, we get the Szász-Baskakov operators and it can be represented in terms of generalized Laguerre function as follows

$$\lim_{\alpha\to\infty,\beta\to n} K_{n,\alpha,\beta}(x,t) = (1+t)^{-n}e^{\frac{-nx}{1+t}}L^0_{n-1}\left(\frac{-nxt}{1+t}\right).$$

(4) If $\alpha = nx, \beta \to \infty$, we get the Lupaş-Szász operators, which can be written in terms of generalized Laguerre function as

$$\lim_{\alpha\to nx,\beta\to\infty} K_{n,\alpha,\beta}(x,t) = 2^{-nx}e^{-nt/2}L^0_{nx-1}\left(\frac{-nt}{2}\right).$$

Lemma 1.1. *The r-th ($r \in \mathbb{N}$) order moment with $e_r(t) = t^r$ (except the above case (9)) can be represented as*

$$V_{n,\alpha,\beta}(e_r,x) = \frac{\Gamma(\beta-r-1)\Gamma(r+1)}{\Gamma(\beta-1)}\left(\frac{\beta}{n}\right)^r {}_2F_1\left(\alpha,-r;1;\frac{-nx}{\alpha}\right).$$

In particular

$$V_{n,\alpha,\beta}(e_0,x) = 1,\quad V_{n,\alpha,\beta}(e_1,x) = \frac{\beta(1+nx)}{n(\beta-2)},$$
$$V_{n,\alpha,\beta}(e_2,x) = \frac{\beta^2[2\alpha+4\alpha nx+(\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)}.$$

Proof. We have

$$\begin{aligned}
V_{n,\alpha,\beta}(e_r,x) &= \frac{n(\beta-1)}{\beta}\sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)\int_0^{\infty} m_{n,k}^{\beta}(t)t^r\,dt \\
&= \frac{n(\beta-1)}{\beta}\sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)\int_0^{\infty} \frac{(\beta)_k}{k!}.\frac{\left(\frac{nt}{\beta}\right)^k}{\left(1+\frac{nt}{\beta}\right)^{\beta+k}}t^r\,dt \\
&= \frac{n(\beta-1)}{\beta}\sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)\frac{(\beta)_k}{k!}.B(k+r+1,\beta-r-1)\left(\frac{\beta}{n}\right)^{r+1} \\
&= (\beta-1)\left(\frac{\beta}{n}\right)^r\sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)\frac{\Gamma(k+\beta)}{\Gamma(\beta).k!}.\frac{\Gamma(k+r+1)\Gamma(\beta-r-1)}{\Gamma(\beta+k)} \\
&= (\beta-1)\Gamma(\beta-r-1).\left(\frac{\beta}{n}\right)^r\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}\frac{\left(\frac{nx}{\alpha}\right)^k}{\left(1+\frac{nx}{\alpha}\right)^{\alpha+k}}\frac{\Gamma(k+r+1)}{\Gamma(\beta).k!} \\
&= \frac{\Gamma(\beta-r-1)\Gamma(r+1)}{\Gamma(\beta-1)}\left(\frac{\beta}{n}\right)^r\sum_{k=0}^{\infty}\frac{(\alpha)_k(r+1)_k}{k!(1)_k}\frac{\left(\frac{nx}{\alpha}\right)^k}{\left(1+\frac{nx}{\alpha}\right)^{\alpha+k}} \\
&= \frac{\Gamma(\beta-r-1)\Gamma(r+1)}{\Gamma(\beta-1)}\left(\frac{\beta}{n}\right)^r\left(1+\frac{nx}{\alpha}\right)^{-\alpha}{}_2F_1\left(\alpha,r+1;1;\frac{nx}{nx+\alpha}\right).
\end{aligned}$$

Applying the well-known Kummer's transformation

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\;{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right),$$

we immediately get

$$V_{n,\alpha,\beta}(e_r,x) = \frac{\Gamma(\beta-r-1)\Gamma(r+1)}{\Gamma(\beta-1)}\left(\frac{\beta}{n}\right)^r\;{}_2F_1\left(\alpha,-r;1;\frac{-nx}{\alpha}\right).$$

The other consequences follow from the above hypergeometric representation. □

The main motivation to study and define these operators based on two parameters α and β is that, one may have an idea of many Durrmeyer type and hybrid operators at one place, rather than studying several papers independently. Although approximation properties hold good for cases (1)-(8), but for case (9) still its not possible. Actually moments play an important role in order to determine the convergence estimates. One can easily obtain the moments of different operators, using Lemma 1.1, by assigning different values to α and β.

From Lemma 1.1, we observe that the convergence takes place (except the case (9)) for the general operators, by the well-known theorem due to Korovkin, when $\alpha=\alpha_{n,x},\beta=\beta_n$ and n is large enough. The values of α,β are indicated in cases (1)-(8) above. Also, the quantitative difference estimates between these integral operators and the discrete operators are estimated in next section.

2. Difference estimates

Let $U_n, V_n, W_n : D(\mathbb{R}^+) \to C(\mathbb{R}^+)$ be three positive linear operators

$$\begin{aligned} U_n(f;x): &= \sum_{k=0}^{\infty} u_{n,k}(x)F_{n,k}(f), \\ W_n(f;x): &= \sum_{k=0}^{\infty} u_{n,k}(x)G_{n,k}(f), \\ V_n(f;x): &= \sum_{k=0}^{\infty} v_{n,k}(x)G_{n,k}(f), \end{aligned}$$

where $D(\mathbb{R}^+)$ be the set of all $f \in C(\mathbb{R}^+)$ for which the above operators preserve the constant functions only. Here the operators U_n, W_n have same basis function namely $u_{n,k}(x)$, while the operator V_n has different basis function $v_{n,k}(x)$. We use the notation $(J_{n,k} = \{F_{n,k}, G_{n,k}\})$

$$b^{J_{n,k}} := J_{n,k}(e_1), \mu_m^{J_{n,k}} = J_{n,k}(e_1 - b^{J_{n,k}}e_0)^m, m \in \mathbb{N}.$$

Also, for $f \in C_B(\mathbb{R}^+)$, the class of bounded and continuous functions on the interval $\mathbb{R}^+$, the norm is defined as $||f|| = \sup\{|f(x)| : x \in \mathbb{R}^+\} < \infty$. Recently Acu-Rasa [1], Gupta [6], Gupta-Tachev [13] and Gupta et al [11] etc. established/presented some results for the difference of operators.

For the operators having same basis functions, the following theorem was provided by the author:

Theorem 2.1. [6] *Let $f^{(s)} \in C_B(\mathbb{R}^+), s \in \{0,1,2\}$ and $x \in \mathbb{R}^+$, then for $n \in \mathbb{N}$, we have*

$$|(U_n - W_n)(f,x)| \leq \frac{A(x)}{2}||f''|| + \frac{\omega(f'',\eta_1)}{2}(1 + A(x)) + 2\omega(f,\eta_2(x)),$$

where

$$A(x) = \sum_{k=0}^{\infty} u_{n,k}(x)(\mu_2^{F_{n,k}} + \mu_2^{G_{n,k}})$$

and

$$\eta_1^2 = \sum_{k=0}^{\infty} u_{n,k}(x)(\mu_4^{F_{n,k}} + \mu_4^{G_{n,k}}), \eta_2^2 = \sum_{k=0}^{\infty} u_{n,k}(x)(b^{F_{n,k}} - b^{G_{n,k}})^2.$$

For the operators having different basis functions, we have the following theorem:

Theorem 2.2. [9] *If $f \in D(\mathbb{R}^+)$ with $f'' \in C_B(\mathbb{R}^+)$, then*

$$|(U_n - V_n)(f;x)| \leq B(x)\|f''\| + 2\omega(f;\delta_1(x)) + 2\omega(f;\delta_2(x)),$$

where $\omega(f,.)$ denotes the usual modulus of continuity and

$$\begin{aligned} B(x) &= \frac{1}{2}\sum_{k=0}^{\infty}\left(u_{n,k}(x)\mu_2^{F_{n,k}} + v_{n,k}(x)\mu_2^{G_{n,k}}\right), \\ \delta_1^2(x) &= \sum_{k=0}^{\infty} u_{n,k}(x)\left(b^{F_{n,k}} - x\right)^2, \ \ \delta_2^2(x) = \sum_{k=0}^{\infty} v_{n,k}(x)\left(b^{G_{n,k}} - x\right)^2. \end{aligned}$$

The above two theorems can be applied for the operators (1.2) and (1.1), except for the cases $\alpha = \beta = -n$ and $\beta = nt$. We provide below the exact quantitative estimates, which hold for difference of many operators.

Remark 2.1. Obviously for the operators (1.1), we have

$$b^{F_{n,k}} = F_{n,k}(e_1) = \frac{k}{n},$$

$$\mu_m^{F_{n,k}} := F_{n,k}(e_1 - b^{F_{n,k}} e_0)^m = 0, m \in \mathbb{N}.$$

For $x \geq 0, n \in \mathbb{N}$, we have

$$M_{n,\gamma}(e_0, x) = 1, M_{n,\gamma}(e_1, x) = x, M_{n,\gamma}(e_2, x) = \frac{x^2(\gamma+1)}{\gamma} + \frac{x}{n},$$

$$M_{n,\gamma}(e_3, x) = \frac{x^3(\gamma+1)(\gamma+2)}{\gamma^2} + \frac{3x^2(\gamma+1)}{n\gamma} + \frac{x}{n^2},$$

$$M_{n,\gamma}(e_4, x) = \frac{x^4(\gamma+1)(\gamma+2)(\gamma+3)}{\gamma^3} + \frac{6x^3(\gamma+1)(\gamma+2)}{n\gamma^2} + \frac{7x^2(\gamma+1)}{n^2\gamma} + \frac{x}{n^3}.$$

Remark 2.2. For the weights of our generalized operators (1.2), we have

$$\begin{aligned} G_{n,k}^{\beta}(e_r) &= \frac{n(\beta-1)}{\beta} \int_0^\infty m_{n,k}^{\beta}(t) t^r dt \\ &= \left(\frac{\beta}{n}\right)^r \frac{\Gamma(\beta - r - 1)(k+r)!}{\Gamma(\beta-1).k!}. \end{aligned}$$

Thus, we have

$$b^{G_{n,k}^{\beta}} = G_{n,k}(e_1) = \frac{(k+1)\beta}{n(\beta-2)},$$

$$\begin{aligned} \mu_2^{G_{n,k}^{\beta}} &= G_{n,k}^{\beta}(e_1 - b^{G_{n,k}^{\beta}} e_0)^2 \\ &= \frac{\beta^2[k^2 + k\beta + \beta - 1)]}{n^2(\beta-2)^2(\beta-3)}. \end{aligned}$$

and similarly

$$\begin{aligned} \mu_4^{G_{n,k}^{\beta}} &= G_{n,k}^{\beta}(e_1 - b^{G_{n,k}^{\beta}} e_0)^4 \\ &= \frac{\beta^4(k+4)(k+3)(k+2)(k+1)}{n^4(\beta-2)(\beta-3)(\beta-4)(\beta-5)} - 4\frac{\beta^4(k+3)(k+2)(k+1)^2}{n^4(\beta-2)^2(\beta-3)(\beta-4)} \\ &\quad +6\frac{\beta^4(k+2)(k+1)^3}{n^4(\beta-2)^3(\beta-3)} - 3\frac{\beta^4(k+1)^4}{n^4(\beta-2)^4}. \end{aligned}$$

Corresponding to Theorem 2.1, we have the following quantitative estimate for differences having same basis

Proposition 2.1. *Let $f^{(s)} \in C_B(\mathbb{R}^+), s \in \{0, 1, 2\}$ and $x \in [0, \infty)$, then for $n \in \mathbb{N}$, we have*

$$|(M_{n,\alpha} - V_{n,\alpha,\beta})(f, x)| \leq \frac{A(x)}{2}||f''|| + \frac{\omega(f'', \eta_1)}{2}(1 + A(x)) + 2\omega(f, \eta_2(x)),$$

where

$$A(x) = \frac{\beta^2}{(\beta-2)^2(\beta-3)}\left[x^2 + \frac{x(nx+\alpha)}{n\alpha}\right] + \frac{\beta^3 x}{n(\beta-2)^2(\beta-3)} + \frac{\beta^2(\beta-1)}{n^2(\beta-2)^2(\beta-3)},$$

$$\begin{aligned} \eta_1^2(x) = \frac{3\beta^4}{\alpha^3 n^4(\beta-2)^4(\beta-3)(\beta-4)(\beta-5)} \Big[& n^4(\alpha+1)(\alpha+2)(\alpha+3)(\beta+4)x^4 \\ & + 2n^3\alpha(\alpha+1)(\alpha+2)(\beta+3)(\beta+4)x^3 + n^2\alpha^2(\alpha+1)(\beta^3 + 14\beta^2 + 29\beta + 28)x^2 \\ & + n\alpha^3(\beta+1)(5\beta^2 + 3\beta + 4)x + \alpha^3(\beta-1)(3\beta^2 - 5\beta + 4)\Big], \end{aligned}$$

and

$$\eta_2^2 = \left(x^2 + \frac{x}{\alpha} + \frac{x}{n}\right)\frac{4}{(\beta-2)^2} + \frac{\beta^2}{n^2(\beta-2)^2} - \frac{4\beta x}{n(\beta-2)^2}.$$

Proof. Applying Remark 2.1 and Remark 2.2, we have

$$\begin{aligned}
A(x) &= \sum_{k=0}^{\infty}\left(m_{n,k}^{\alpha}(x)\mu_2^{F_{n,k}} + m_{n,k}^{\alpha}(x)\mu_2^{G_{n,k}^{\beta}}\right)\\
&= \sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)\frac{\beta^2[k^2+k\beta+\beta-1)]}{n^2(\beta-2)^2(\beta-3)}\\
&= \frac{\beta^2}{(\beta-2)^2(\beta-3)}\left[x^2 + \frac{x(nx+\alpha)}{n\alpha}\right] + \frac{\beta^3 x}{n(\beta-2)^2(\beta-3)} + \frac{\beta^2(\beta-1)}{n^2(\beta-2)^2(\beta-3)}.
\end{aligned}$$

Next, by Remark 2.1, we have

$$\begin{aligned}
\eta_1^2(x) &= \sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)(\mu_4^{F_{n,k}} + \mu_4^{G_{n,k}^{\beta}})\\
&= \sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)\left[\frac{\beta^4(k+4)(k+3)(k+2)(k+1)}{n^4(\beta-2)(\beta-3)(\beta-4)(\beta-5)} - 4\frac{\beta^4(k+3)(k+2)(k+1)^2}{n^4(\beta-2)^2(\beta-3)(\beta-4)}\right.\\
&\left. + 6\frac{\beta^4(k+2)(k+1)^3}{n^4(\beta-2)^3(\beta-3)} - 3\frac{\beta^4(k+1)^4}{n^4(\beta-2)^4}\right]\\
&= \frac{3\beta^4}{\alpha^3 n^4(\beta-2)^4(\beta-3)(\beta-4)(\beta-5)}\Big[n^4(\alpha+1)(\alpha+2)(\alpha+3)(\beta+4)x^4\\
&+ 2n^3\alpha(\alpha+1)(\alpha+2)(\beta+3)(\beta+4)x^3 + n^2\alpha^2(\alpha+1)(\beta^3+14\beta^2+29\beta+28)x^2\\
&+ n\alpha^3(\beta+1)(5\beta^2+3\beta+4)x + \alpha^3(\beta-1)(3\beta^2-5\beta+4)\Big].
\end{aligned}$$

Finally by Remark 2.2 and Remark 2.1, we get

$$\begin{aligned}
\eta_2^2(x) &= \sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)(b^{F_{n,k}} - b^{G_{n,k}^{\beta}})^2\\
&= \sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)\left(\frac{k}{n} - \frac{(k+1)\beta}{n(\beta-2)}\right)^2\\
&= \left(x^2 + \frac{x}{\alpha} + \frac{x}{n}\right)\frac{4}{(\beta-2)^2} + \frac{\beta^2}{n^2(\beta-2)^2} - \frac{4\beta x}{n(\beta-2)^2}.
\end{aligned}$$

□

Remark 2.3. We can immediately obtain the difference between different discrete operators and the Durrmeyer variants having same basis by assigning different values to α and β in above Proposition 2.1. For example if we take $\alpha \to \infty, \beta \to \infty$, we immediately get the difference estimate between Szász operators and Szász-Durrmeyer operators (see [6, Th. 5]). Similarly one can obtain many results using our operators for different values.

Based on Theorem 2.2, we have the following quantitative estimate for the difference between the operators (1.1) and (1.2).

Proposition 2.2. *Let $f \in C_B(\mathbb{R}^+)$ and $x \in [0,\infty)$. Then for $n \in \mathbb{N}$, we have*

$$|(M_{n,\gamma} - V_{n,\alpha,\beta})(f,x)| \leq B(x)\|f''\| + 2\omega(f;\delta_1(x)) + 2\omega(f;\delta_2(x)),$$

where

$$B(x) = \frac{\beta^2}{2(\beta-2)^2(\beta-3)}\left[x^2 + \frac{x(nx+\alpha)}{n\alpha}\right] + \frac{\beta^3 x}{2n(\beta-2)^2(\beta-3)} + \frac{\beta^2(\beta-1)}{2n^2(\beta-2)^2(\beta-3)},$$

$$\delta_1^2(x) = \frac{x(nx+\gamma)}{n\gamma},$$

$$\delta_2^2(x) = x^2\left[\frac{\beta^2}{(\beta-2)^2} + \frac{\beta^2}{\alpha(\beta-2)^2} + 1 - \frac{2\beta}{(\beta-2)}\right] + x\left[\frac{3\beta^2}{n(\beta-2)^2} - \frac{2\beta}{n(\beta-2)}\right] + \frac{\beta^2}{n^2(\beta-2)^2}.$$

Proof. Applying Remarks 2.1 and 2.2, we have

$$\begin{aligned} B(x) &= \frac{1}{2}\sum_{k=0}^{\infty}\left(m_{n,k}^{\gamma}(x)\mu_2^{F_{n,k}} + m_{n,k}^{\alpha}(x)\mu_2^{G_{n,k}^{\beta}}\right) \\ &= \frac{1}{2}\sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)\frac{\beta^2[k^2+k\beta+\beta-1)]}{n^2(\beta-2)^2(\beta-3)} \\ &= \frac{\beta^2}{2(\beta-2)^2(\beta-3)}\left[x^2 + \frac{x(nx+\alpha)}{n\alpha}\right] + \frac{\beta^3 x}{2n(\beta-2)^2(\beta-3)} + \frac{\beta^2(\beta-1)}{2n^2(\beta-2)^2(\beta-3)}. \end{aligned}$$

Next, by Remark 2.1, we have

$$\delta_1^2(x) = \sum_{k=0}^{\infty} m_{n,k}^{\gamma}(x)\left(b^{F_{n,k}} - x\right)^2 = \frac{x(nx+\gamma)}{n\gamma}.$$

Finally by Remark 2.2 and Remark 2.1, we get

$$\begin{aligned} \delta_2^2(x) &= \sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x)\left(b^{G_{n,k}^{\beta}} - x\right)^2 \\ &= x^2\left[\frac{\beta^2}{(\beta-2)^2} + \frac{\beta^2}{\alpha(\beta-2)^2} + 1 - \frac{2\beta}{(\beta-2)}\right] + x\left[\frac{3\beta^2}{n(\beta-2)^2} - \frac{2\beta}{n(\beta-2)}\right] + \frac{\beta^2}{n^2(\beta-2)^2}. \end{aligned}$$

□

Remark 2.4. One can obtain the difference between different operators having different basis in summation by assigning different values to α, β and γ in Proposition 2.2. For example if we take $\gamma = n, \alpha \to \infty, \beta = n$, we may get the difference estimate between Baskakov operators and Szász-Baskakov operators (see [9, Prop.3.9]). Similarly if we consider $\gamma = nx, \alpha \to \infty, \beta \to \infty$, we may get the difference estimate between Lupaş operators and Szász-Durrmeyer operators (see [9, Prop. 3.12]).

Remark 2.5. Miheşan in [15, (4.3)] claimed that the operators (1.1) are valid for $x \geq 0, \gamma \in \mathbb{R}, \gamma + nx > 0$. We point out here that this condition holds good for Baskakov, Szász and Lupaş operators, but in case of Bernstein polynomials if $\gamma = -n$, then one has $nx > n$, which is not true for $x \in [0,1]$, in which interval the Bernstein polynomials are defined. Therefore it is sufficient to assume that γ is non-zero real number or a function of n.

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