CARPATHIAN J. MATH. 36 (2020), No. 3, 433 - 442
Online version at https://www.carpathian.cunbm.utcluj.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Selected paper presented at 13th International Conference of Fixed Point Theory and Applications (ICFPTA 2019), July 9-13, 2019, HeNan Normal University, XinXiang, HeNan, China

On admissible hybrid Geraghty contractions

ERDAL KARAPINAR[1,2], ADRIAN PETRUŞEL [3,4] and GABRIELA PETRUŞEL[5]

ABSTRACT. In this manuscript, we introduce the notion of admissible hybrid Geraghty contraction and we investigate the existence of fixed points of such mappings in the setting of complete metric spaces. Our results not only extend and generalize several results in the fixed point theory literature, but also unify most of them. We give some corollaries to illustrate the novelty of the main result.

1. INTRODUCTION AND PRELIMINARIES

In 1973, Geraghty [7] introduced an interesting class of auxiliary function to refine the Banach contraction mapping principle. Let $\mathcal{G}$ denote all functions $\beta : [0,\infty) \to [0,1)$ which satisfies the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

Using the above class of functions, Geraghty [7] proved the following remarkable theorem.

Theorem 1.1. (Geraghty [7].) *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. Suppose that there exists $\beta \in \mathcal{G}$ such that f satisfies the following inequality*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y), \text{ for any } x, y \in X. \tag{1.1}$$

Then f has a unique fixed point in X.

The concept of α-orbital admissible was proposed in [16] and it is a refinement of α-admissibility, defined in [21].

Let $\alpha : X \times X \to [0,\infty)$ be a function. We say that a mapping $f : X \to X$ is α-orbital admissible (see [16]) if

$$\alpha(x, fx) \geq 1 \Rightarrow \alpha(fx, f^2x) \geq 1. \tag{1.2}$$

An α-orbital admissible mapping f is called triangular α-orbital admissible (see [16]) if

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, fy) \geq 1 \Rightarrow \alpha(x, fy) \geq 1, \tag{1.3}$$

for every $x, y \in X$.

Lemma 1.1. Let X be a non-empty set. Suppose that $\alpha : X \times X \to [0,\infty)$ is a given function and $f : X \to X$ is a triangular α-orbital admissible mapping. If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, and $x_n = fx_{n-1}$ for $n = 0, 1, \ldots$, then, we have:

$$(a)\ \alpha(x_n, x_{n+1}) \geq 1, \text{ for each } n = 0, 1, 2, \ldots. \tag{1.4}$$

Received: 05.01.2020. In revised form: 12.05.2020. Accepted: 19.05.2020
2010 *Mathematics Subject Classification.* 47H10, 54H25, 46J10.
Key words and phrases. *Hybrid contraction, Geraghty type contraction, Fixed point theory, metric space.*
Corresponding author: Erdal Karapınar; karapinar@mail.cmuh.org.tw

$$(b)\ \alpha(x_n, x_{n+k}) \geq 1, \text{ for each } n = 0, 1, 2, \dots \text{ and } k = 1, 2, \dots. \tag{1.5}$$

Proof. On account of the assumptions of the theorem, there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Owing to the fact that f is α-orbital admissible, we find

$$\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1 \Rightarrow \alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geq 1.$$

By iterating the above inequality, we derive that

$$\alpha(x_n, x_{n+1}) = \alpha(fx_{n-1}, fx_n) \geq 1, \text{ for each } n = 0, 1, \dots.$$

On the other hand, since it is triangular α-orbital admissible mapping, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ and } \alpha(x_{n+1}, fx_{n+1}) = \alpha(x_{n+1}, x_{n+2}) \geq 1 \Rightarrow \alpha(x_n, x_{n+2}) \geq 1.$$

Notice that, recursively we can prove that

$$\alpha(x_n, x_{n+k}) \geq 1,$$

for each $n \in \{0, 1, 2, \dots\}$ and $k \in \{1, 2, \dots\}$. □

Definition 1.1. Let $\alpha : X \times X \to [0, \infty)$ be a mapping. The set X is called regular with respect to α if for a sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $x_n \to x \in X$ as $n \to \infty$ we have $\alpha(x_n, x) \geq 1$ for all n.

Lemma 1.2. cf.[17] Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n)$ is non-increasing and

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences

$$d(x_{m_k}, x_{n_k}),\ d(x_{m_k-1}, x_{n_k-1}),\ d(x_{m_k}, x_{n_k-1}),\ d(x_{m_k-1}, x_{n_k})$$

tend to ε when $k \to \infty$.

In this paper, we will introduce the notion of admissible hybrid Geraghty contraction and we investigate the existence of fixed points of such mappings in the setting of complete metric spaces. Our theorems not only generalize several results in the fixed point theory literature, but also unify some of them, see [4], [5], [6], [8], [9], [13], [15], [16], [17], [18], [22]. Our results are also related to the interpolative approach in fixed point theory, see [1], [2], [3], [10], [11], [12]. Finally, we will deduce some corollaries to illustrate the novelty of the main results.

2. Main results

We start with a definition of a new notion, namely that of admissible hybrid contraction, as follows.

Definition 2.2. Let (X, d) be a metric space. A self-mapping f is called an admissible hybrid Geraghty contraction if there exist $\beta \in \mathcal{G}$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x, y)d(fx, fy) \leq \beta\left(\mathcal{H}_f^q(x, y)\right)\mathcal{H}_f^q(x, y), \tag{2.6}$$

where $q \geq 0$ and $\lambda_i \geq 0, i \in \{1, 2, 3, 4, 5\}$ such that $\sum_{i=1}^{5} \lambda_i = 1$ and, for every $x, y \in X$, we have

(2.7)
$$\mathcal{H}_f^q(x,y) := \begin{cases} \left[\lambda_1 d^q(x,y) + \lambda_2 d^q(x,fx) + \lambda_3 d^q(y,fy) + \lambda_4 \left(\frac{d(y,fy)(1+d(x,fx))}{1+d(x,y)}\right)^q \right. \\ \qquad \left. +\lambda_5 \left(\frac{d(y,fx)(1+d(x,fy))}{1+d(x,y)}\right)^q\right]^{\frac{1}{q}}, \text{ for } q>0, \\ [d(x,y)]^{\lambda_1}\cdot[d(x,fx)]^{\lambda_2}\cdot[d(y,fy)]^{\lambda_3}\cdot\left[\frac{d(y,fy)(1+d(x,fx))}{1+d(x,y)}\right]^{\lambda_4}\cdot\left[\frac{d(x,fy)+d(y,fx)}{2}\right]^{\lambda_5}, \\ \qquad \text{for } q=0. \end{cases}$$

The main result of this manuscript is the following theorem.

Theorem 2.2. *Let (X,d) be a complete metric space and let $f: X \to X$ be an admissible hybrid Geraghty contraction. Suppose also that:*

(i) *f is triangular $\alpha-$orbital admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(iii_a) *f is continuous*
or
(iii_b) *f^2 is continuous and $\alpha(fx,x) \geq 1$ for any $x \in X$ with $d(x,fx) > 0$.*

Then, f has at least one fixed point in X.

Proof. On account of the assumption (ii), there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Hence, starting this point, we shall construct an iterative sequence $\{x_n\}$ as follows:

$$x_0 := x \text{ and } x_n = fx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{2.8}$$

We suppose that

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{2.9}$$

Indeed, if for some $n \in \mathbb{N}$ we have the inequality $x_n = fx_{n-1} = x_{n-1}$, then, the proof is completed.

By Lemma 1.1, we have

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ and } \alpha(x_n, x_m) \geq 1, \text{ for each } n, m \in \mathbb{N}_0 \text{ with } n > m. \tag{2.10}$$

By substituting $x = x_{n-1}$ and $y = x_n$ in the inequality (2.6), we derive that

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) \leq \beta\left(\mathcal{H}_f^q(x_{n-1}, x_n\right)\mathcal{H}_f^q(x_{n-1}, x_n). \tag{2.11}$$

Case 1. For the case $q > 0$ we have

$$\begin{aligned} \mathcal{H}_f^q(x_{n-1},x_n) &= [\lambda_1 d^q(x_{n-1},x_n) + \lambda_2 d^q(x_{n-1},fx_{n-1}) + \lambda_3 d^q(x_n,fx_n) + \\ &\qquad +\lambda_4\left(\tfrac{d(x_n,fx_n)(1+d(x_{n-1},fx_{n-1}))}{1+d(x_{n-1},x_n)}\right)^q + \lambda_5\left(\tfrac{d(x_n,fx_{n-1})(1+d(x_{n-1},fx_n))}{1+d(x_{n-1},x_n)}\right)^q\Bigg]^{\frac{1}{q}} \\ &= [\lambda_1 d^q(x_{n-1},x_n) + \lambda_2 d^q(x_{n-1},x_n) + \lambda_3 d^q(x_n,x_{n+1}) + \\ &\qquad +\lambda_4\left(\tfrac{d(x_n,x_{n+1})(1+d(x_{n-1},x_n))}{1+d(x_{n-1},x_n)}\right)^q + \lambda_5\left(\tfrac{d(x_n,x_n)(1+d(x_{n-1},x_{n+1}))}{1+d(x_{n-1},x_n)}\right)^q\Bigg]^{\frac{1}{q}} \\ &= \left[\lambda_1 d^q(x_{n-1},x_n) + \lambda_2 d^q(x_{n-1},x_n) + \lambda_3 d^q(x_n,x_{n+1}) + \lambda_4\left(d(x_n,x_{n+1})\right)^q\right]^{\frac{1}{q}} \\ &= [(\lambda_1+\lambda_2)d^q(x_{n-1},x_n) + (\lambda_3+\lambda_4)d^q(x_n,x_{n+1})]^{1/q}, \end{aligned}$$

and from (2.11) we get

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) \leq \beta\left(\mathcal{H}_f^q(x_{n-1}, x_n)\right) \\
&< \mathcal{H}_f^q(x_{n-1}, x_n) \\
&< [(\lambda_1 + \lambda_2) d^q(x_{n-1}, x_n) + (\lambda_3 + \lambda_4) d^q(x_n, x_{n+1})]^{1/q}.
\end{aligned} \tag{2.12}
$$

If we suppose that $d(x_{n-1}, x_n) \leq d(x_n, x_{n-1})$, then we get

$$
\begin{aligned}
d(x_n, x_{n+1}) &< (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^{1/q} d(x_n, x_{n+1}) \\
&< d(x_n, x_{n+1}),
\end{aligned} \tag{2.13}
$$

a contradiction. Therefore, for every $n \in \mathbb{N}$ we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

Consequently, we deduce that $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$, for each n. Since the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing.

As a next step, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Indeed, since $\{d(x_n, x_{n+1})\}$ is non-increasing and bounded below, we conclude that it converges to some non-negative real numbers, say r.

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

It is evident that

$$\lim_{n\to\infty} \mathcal{H}_f^q(x_{n-1}, x_n) = r.$$

We assert that $r = 0$. Suppose, on the contrary that $r \neq 0$.

Letting $n \to \infty$ in the equation (2.12), we find

$$\lim_{n\to\infty} \beta(\mathcal{H}_f^q(x_{n-1}, x_n)) = 1 \Rightarrow \lim_{n\to\infty} \mathcal{H}_f^q(x_{n-1}, x_n) = 0.$$

As a consequence, $r = 0$ and so

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.14}$$

In what follows, we claim sequence $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence then there exists $\varepsilon > 0$ and sequences $\{x_{n_k}\}$, $\{x_{m_k}\}$;
$n_k > m_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{2.15}$$

$$d(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{2.16}$$

Now take $x = x_{m_k-1}$ and $y = x_{n_k-1}$ in (2.6), we have

$$\alpha(x_{m_k-1}, x_{n_k-1}) \geq 1 \; for \; all \; k$$

implies

$$
\begin{aligned}
d(fx_{m_k-1}, fx_{n_k-1}) &\leq \alpha(x_{m_k-1}, x_{n_k-1}) d(fx_{m_k-1}, fx_{n_k-1}) \\
&\leq \beta\left(\mathcal{H}_f^q(x_{m_k-1}, x_{n_k-1})\right) \mathcal{H}_f^q(x_{m_k-1}, x_{n_k-1}) \\
&< \mathcal{H}_f^q(x_{m_k-1}, x_{n_k-1}),
\end{aligned} \tag{2.17}
$$

where
(2.18)

$$\mathcal{H}_f^q(x_{m_k-1},x_{n_k-1})=\left[\begin{array}{c}\lambda_1 d^q(x_{m_k-1},x_{n_k-1})+\lambda_2 d^q(x_{m_k-1},fx_{m_k-1})+\lambda_3 d^q(x_{n_k-1},fx_{n_k-1})\\ +\lambda_4\left(\frac{d(x_{n_k-1},fx_{n_k-1})(1+d(x_{m_k-1},fx_{m_k-1}))}{1+d(x_{m_k-1},x_{n_k-1})}\right)^q\\ +\lambda_5\left(\frac{d(x_{n_k-1},fx_{m_k-1})(1+d(x_{m_k-1},fx_{n_k-1}))}{1+d(x_{m_k-1},x_{n_k-1})}\right)^q\end{array}\right]^{\frac{1}{q}}.$$

Due to Lemma 1.2, we have
(2.19)

$$\lim_{k\to\infty} d(x_{m_k},x_{n_k}) = \lim_{k\to\infty} d(x_{m_k-1},x_{n_k-1}) = \lim_{k\to\infty} d(x_{m_k},x_{n_k-1}) = \lim_{k\to\infty} d(x_{m_k-1},x_{n_k}) = \varepsilon.$$

If we letting $k\to\infty$ in (2.17) and keeping (2.18),(2.19) in mind, we get

$$\varepsilon = \lim_{k\to\infty} d(fx_{m_k-1},fx_{n_k-1}) \leq \lim_{k\to\infty} \mathcal{H}_f^q(x_{m_k-1},x_{n_k-1}) = (\lambda_1+\lambda_4+\lambda_5)^{\frac{1}{q}}\varepsilon,$$

which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence on a complete metric space, so that, there exists z such that

$$\lim_{n\to\infty} d(x_n,z)=0. \tag{2.20}$$

We shall indicate that z is a fixed point of f. If f is continuous, (due to assumption (iii))

$$\lim_{n\to\infty} d(x_{n+1},fz) = \lim_{n\to\infty} d(x_n,fx_n)=0,$$

so, we get that $fz=z$, that is, z is a fixed point of f.

In the alternative hypothesis, that f^2 is continuous we have $f^2z=\lim_{n\to\infty} f^2x_n = z$ and we want to show that $fz=z$. Supposing that, on the contrary, $fz\neq z$, we have from (2.6)

$$\begin{aligned}
d(z,fz) &= d(f^2z,fz) \leq \alpha(fz,z)d(fz,z) \leq \beta\left(\mathcal{H}_f^q(fz,z)\right)\mathcal{H}_f^q(fz,z) < \mathcal{H}_f^q(fz,z)\\
&= \Bigg[\lambda_1 d^q(fz,z)+\lambda_2 d^q(fz,f^2z)+\lambda_3 d^q(z,fz)+\lambda_4\left(\tfrac{d(z,fz)(1+d(fz,f^2z))}{1+d(fz,z)}\right)^q\\
&\qquad +\lambda_5\left(\tfrac{d(z,f^2z)(1+d(fz,fz))}{1+d(fz,z)}\right)^q\Bigg]^{\frac{1}{q}}\\
&= \Bigg[\lambda_1 d^q(fz,z)+\lambda_2 d^q(fz,z)+\lambda_3 d^q(z,fz)+\lambda_4\left(\tfrac{d(z,fz)(1+d(fz,z))}{1+d(fz,z)}\right)^q\\
&\qquad +\lambda_5\left(\tfrac{d(z,z)(1+d(fz,fz))}{1+d(fz,z)}\right)^q\Bigg]^{\frac{1}{q}}\\
&= \left[(\lambda_1+\lambda_2+\lambda_3+\lambda_4)d^q(fz,z)\right]^{\frac{1}{q}}\\
&= \left[(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\right]^{\frac{1}{q}} d(fz,z)\\
&< d(fz,z).
\end{aligned}$$

This is a contradiction, so that $fz=z$.

Case 2. For the case $q = 0$ taking $x = x_{n-1}$ and $y = x_n$ we have

$$\begin{aligned}\mathcal{H}_f^q(x_{n-1}, x_n) &= [d(x_{n-1}, x_n)]^{\lambda_1} \cdot [d(x_{n-1}, fx_{n-1})]^{\lambda_2} \cdot [d(x_n, fx_n)]^{\lambda_3} \\ &\quad \cdot \left[\tfrac{d(x_n, fx_n)(1+d(x_{n-1}, fx_{n-1}))}{1+d(x_{n-1}, x_n)}\right]^{\lambda_4} \cdot \left[\tfrac{d(x_{n-1}, fx_n)+d(x_n, fx_{n-1}))}{2}\right]^{\lambda_5} \\ &\leq [d(x_{n-1}, x_n)]^{\lambda_1} \cdot [d(x_{n-1}, x_n)]^{\lambda_2} \cdot [d(x_n, x_{n+1})]^{\lambda_3} \cdot [d(x_n, x_{n+1})]^{\lambda_4} \\ &\quad \cdot \left[\tfrac{d(x_{n-1}, x_n)+d(x_n, x_{n+1})+d(x_n, x_n))}{2}\right]^{\lambda_5} \\ &\leq [d(x_{n-1}, x_n)]^{\lambda_1+\lambda_2} \cdot [d(x_n, x_{n+1})]^{\lambda_3+\lambda_4} \cdot \left[\tfrac{[d(x_{n-1}, x_n)+d(x_n, x_{n+1})]}{2}\right]^{\lambda_5}.\end{aligned}$$

From (2.6)

(2.21)
$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n)d(fx_{n-1}, fx_n) \leq \beta(\mathcal{H}_f^q(x_{n-1}, x_n))\mathcal{H}_f^q(x_{n-1}, x_n) < \mathcal{H}_f^q(x_{n-1}, x_n).$$

As in the first case, we have that $d(x_{n-1}, x_n) > d(x_n, x_{n+1})$ since in the contrary case we have a contradiction. Indeed, if we suppose *ad absurdum* that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, we have

$$d(x_n, x_{n+1}) \leq \mathcal{H}_f^q(x_{n-1}, x_n) < [d(x_n, x_{n+1}))]^{\lambda_1+\lambda_2\lambda_3+\lambda_4+\lambda_5} = d(x_n, x_{n+1}),$$

which is a contradiction. Thus, we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{2.22}$$

By using the same arguments as the case $q > 0$ we shall easily obtain that $\{x_n\}$ is a Cauchy sequence in a complete metric space and so, there exists z such that $\lim_{n\to\infty} x_n = z$.

We claim that z is a fixed point of f.

Under the assumption that f is continuous we have

$$\lim_{n\to\infty} d(x_{n+1}, fz) = \lim_{n\to\infty} d(fx_n, fz) = 0,$$

and together with the uniqueness of limit, $fz = z$. Also, if f^2 is continuous, as in case (1) we have that $fz = z$ and then

$$\begin{aligned}d(z, fz) = d(f^2z, fz) &\leq \alpha(fz, z)d(f^2z, fz) \leq \beta(\mathcal{H}_f^q(f^2z, fz))\mathcal{H}_f^q(f^2z, fz) < \mathcal{H}_f^q(f^2z, fz) \\ &\leq [d(z, fz)]^{\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5} \\ &< d(z, fz).\end{aligned}$$

This contradiction shows us that $z = fz$.

□

If, instead the continuity condition on f, we propose the regularity condition, then we get the following result.

Theorem 2.3. *Let (X, d) be a complete metric space and let $f : X \to X$ be an admissible hybrid Geraghty contraction. Suppose also that:*

(1) f is triangular $\alpha-$orbital admissible;
(2) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;
(3) (X, d) is regular with respect to α.

Then, f possesses at least one fixed point.

Proof. Following the lines in the proof of the Theorem 2.2, we already know that for any $q \geq 0$, the sequence $\{x_n\}$ is Cauchy. Due to the completeness of the metric space (X, d), there exists a point $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = 0$. Since the space X is regular with respect to α, inequality (2.6) together with the triangle inequality gives us

$$\begin{aligned} d(z, fz) &\le d(z, x_{n+1}) + d(x_{n+1}, fz) \\ &\le d(z, x_{n+1}) + \alpha(x_n, z)d(fx_n, fz) \\ &\le d(z, x_{n+1}) + \beta\left(\mathcal{H}_f^q(x_n, z)\right)\mathcal{H}_f^q(x_n, z) \\ &\le d(z, x_{n+1}) + \mathcal{H}_f^q(x_n, z). \end{aligned} \tag{2.23}$$

Again, we have to consider two separate cases. For the case $q > 0$,

$$\begin{aligned} \mathcal{H}_f^q(x_n, z) &= \Bigg[\lambda_1 d^q(x_n, z) + \lambda_2 d^q(x_n, fx_n) + \lambda_3 d^q(z, fz) + \lambda_4\left(\tfrac{d(z,fz)(1+d(x_n,fx_n))}{1+d(x_n,z)}\right)^q + \\ &\qquad +\lambda_5\left(\tfrac{d(z,fx_n)(1+d(x_n,fz))}{1+d(x_n,z)}\right)^q\Bigg]^{\frac{1}{q}} \\ &= \Bigg[\lambda_1 d^q(x_n, z) + \lambda_2 d^q(x_n, x_{n+1}) + \lambda_3 d^q(z, fz) + \lambda_4\left(\tfrac{d(z,fz)(1+d(x_n,x_{n+1}))}{1+d(x_n,z)}\right)^q + \\ &\qquad +\lambda_5\left(\tfrac{d(z,x_{n+1})(1+d(x_n,fz))}{1+d(x_n,z)}\right)^q\Bigg]^{\frac{1}{q}}. \end{aligned}$$

Since $\lim_{n\to\infty} \mathcal{H}_f^q(x_n, z) = (\lambda_3 + \lambda_4)d(z, fz)$, letting $n \to \infty$ in (2.23) we obtain $d(z, fz) \le (\lambda_3 + \lambda_4)d(z, fz)$ which implies that $d(z, fz) = 0$.
Similarly, for the case $q = 0$, we get $\lim_{n\to\infty} \mathcal{H}_f^q(x_n, z) = 0$ and then $d(z, fz) = 0$. □

Remark 2.1. In the case $q = 0$ of Definition 2.7, if $\tilde{x} \in X$ is a fixed point for f, then $\mathcal{H}_f^q(\tilde{x}, y) = 0$, for every $y \in X$. Thus, f is a constant function. Hence, trivially we have a fixed point. On the other hand, we implicitly exclude this trivial case in our proof by assuming $x_n \neq x_{n+1} = fx_n$.

Theorem 2.4. *Let $f : X \to X$ be a mapping on the complete metric space (X, d) endowed with a partial order $\preceq$ on X. Suppose that there exists $\beta \in G$, such that, for all $x, y \in X$ with $x \preceq y$, we have*

$$\alpha(x, y)d(fx, fy) \le \beta\left(\mathcal{H}_f^q(x, y)\right)\mathcal{H}_f^q(x, y),$$

where $\mathcal{H}_f^q(x, y)$ is defined as in Theorem 2.2. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;*
(ii) *f is continuous or $(X, \preceq, d)$ is regular.*

Then f has at least one fixed point.

Proof. It is sufficient to define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x \preceq y \text{ or } x \succeq y, \\ 0 \text{ otherwise.} \end{cases}$$

Clearly, f is an admissible hybrid Geraghty contraction. From condition (i), we have $\alpha(x_0, fx_0) \ge 1$. Moreover, for all $x, y \in X$, from the monotone property of f, we have

$$\alpha(x_0, fx_0) \ge 1 \Leftrightarrow x_0 \preceq fx_0 \Rightarrow fx_0 \preceq f^2x_0 \Leftrightarrow \alpha(fx_0, f^2x_0) \ge 1.$$

The rest is satisfied in a straightway. □

Theorem 2.5. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Suppose there exists $\beta \in \mathcal{G}$ such that*

$$d(fx, fy) \le \beta\left(\mathcal{H}_f^q(x, y)\right)\mathcal{H}_f^q(x, y), \tag{2.24}$$

where $q > 0$ *and* $\lambda_i \geq 0, i \in \{1, 2, 3, 4, 5\}$ *are such that* $\sum_{i=1}^{5} \lambda_i = 1$*. Then the fixed point of* f *is unique.*

Proof. Let $v \in X$ be another fixed point of f, different from z. By replacing in (2.6), and taking into account the additional hypotheses, we have

$$\begin{aligned} d(z,v) &= d(fz,fv) \leq \beta(\mathcal{H}_f^q(z,v))\mathcal{H}_f^q(z,v) < \mathcal{H}_f^q(z,v) \\ &= \Bigg[\lambda_1 d^q(z,v) + \lambda_2 d^q(z,fz) + \lambda_3 d^q(v,fv) + \lambda_4 \left(\tfrac{d(v,fv)(1+d(z,fz))}{1+d(z,v)}\right)^q \\ &\qquad + \lambda_5 \left(\tfrac{d(v,fz)(1+d(z,fv))}{1+d(z,v)}\right)^q\Bigg]^{\frac{1}{q}} \\ &= d(z,v)(\lambda_1+\lambda_5)^{1/q} \leq d(z,v), \end{aligned}$$

which is a contradiction. Thus, $z = v$, so that f possesses exactly one fixed point. □

Corollary 2.1. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be a mapping. If there exists* $\beta \in \mathcal{G}$ *such that*

$$d(fx,fy) \leq \beta\left(\mathcal{A}_f^q(x,y)\right)\mathcal{A}_f^q(x,y), \tag{2.25}$$

where $q > 0$, $\lambda_i \geq 0, i \in \{1, 2, 3\}$ *are such that* $\sum_{i=1}^{3} \lambda_i = 1$ *and, for* $x, y \in X$*, we denote*

$$\mathcal{A}_f^q(x,y) := \left[\lambda_1 d^q(x,y) + \lambda_2 d^q(x,fx) + \lambda_3 d^q(y,fy)\right]^{\frac{1}{q}}, \tag{2.26}$$

then the fixed point of f *is unique.*

Corollary 2.2. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be a mapping. Suppose there exists* $\beta \in \mathcal{G}$ *such that*

$$d(fx,fy) \leq \beta\left(\mathcal{B}_f^q(x,y)\right)\mathcal{B}_f^q(x,y), \tag{2.27}$$

where $q > 0$, $\lambda_i \geq 0, i \in \{1, 2, 3\}$ *are such that* $\sum_{i=1}^{3} \lambda_i = 1$ *and, for* $x, y \in X$*, we denote*

$$\mathcal{B}_f^q(x,y) := \left[\lambda_1 d^q(x,y) + \lambda_2 \left(\tfrac{d(y,fy)(1+d(x,fx))}{1+d(x,y)}\right)^q + \lambda_3 \left(\tfrac{d(y,fx)(1+d(x,fy))}{1+d(x,y)}\right)^q\right]^{\frac{1}{q}}. \tag{2.28}$$

Then there exists a unique fixed point of f*.*

Corollary 2.3. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be a mapping. If there exists* $\beta \in \mathcal{G}$ *such that*

$$d(fx,fy) \leq \beta\left(\mathcal{S}_f^q(x,y)\right)\mathcal{S}_f^q(x,y), \tag{2.29}$$

where $q > 0$, $\lambda_i \geq 0, i \in \{1, 2\}$ *are such that* $\sum_{i=1}^{2} \lambda_i = 1$ *and, for* $x, y \in X$*, we denote*

$$\mathcal{S}_f^q(x,y) := \left[\lambda_1 d^q(x,fx) + \lambda_2 d^q(y,fy)\right]^{\frac{1}{q}}, \text{ for } q > 0, \tag{2.30}$$

then there exists a unique fixed point of f*.*

Remark 2.2. If we take $\lambda_i = 0$ for $i \in \{2, 3, 4, 5\}$ and $q = 1$, then we derive the original theorem of Geraghty, that is, Theorem 1.1. It is clear that for different choices of λ_i, $i \in \{1, 2, 3, 4, 5\}$ and for q, we can list more corollaries.

Corollary 2.4. *Let* (X, d) *be a complete metric space and the functions* $\beta \in \mathcal{G}$ *and* $\alpha : X \times X \to [0, \infty)$*. Let* f *be a self map on* X *such that:*

- (i) f *is triangular* $\alpha-$*orbital admissible;*
- (ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$*;*
  - $(iiia)$ *either,* f *is continuous,*
  - $(iiib)$ *or* f^2 *is continuous and* $\alpha(fx, x) \geq 1$ *for any* $x \in X$ *with* $d(x, fx) > 0$*,*

$(iiic)$ or (X,d) is regular with respect to α.

If one of the below conditions (c_1)-(c_3) is satisfied, then f has at least one fixed point $z \in X$.

(c_1) *$\alpha(x,y)d(x,y) \le \beta\left(\mathcal{C}_f^q(x,y)\right)\mathcal{C}_f^q(x,y)$, where $a_1,a_2,a_3,a_4 \in [0,\infty)$ are such that $a_1 + a_2 + a_3 + a_4 = 1$ and*

(2.31)
$$\mathcal{C}_f^q(x,y) := \begin{cases} \left[a_1 d^q(x,y)(x,y) + a_2 d^q(x,fx) + a_3 d^q(y,fy) + a_4\left(\frac{d(y,fy)(1+d(x,fx))}{1+d(x,y)}\right)^q\right]^{\frac{1}{q}}, & \text{for } q>0, \\ [d(x,y)]^{a_1}\cdot[d(x,fx)]^{a_2}\cdot[d(y,fy)]^{a_3}\cdot\left[\frac{d(y,fy)(1+d(x,fx))}{1+d(x,y)}\right]^{a_4}, & \text{for } q=0; \end{cases}$$

(c_2) *$\alpha(x,y)d(x,y) \le \beta\left(\mathcal{D}_f^q(x,y)\right)\mathcal{D}_f^q(x,y)$, where $b_1,b_2,b_3 \in [0,\infty)$ are such that $b_1 + b_2 + b_3 = 1$ and*

$$\mathcal{D}_f^q(x,y) = \begin{cases} [b_1 d^q(x,y)(x,y) + b_2 d^q(x,fx) + b_3 d^q(y,fy)]^{\frac{1}{q}}, & \text{for } q>0, \\ [d(x,y)]^{b_1}\cdot[d(x,fx)]^{b_2}\cdot[d(y,fy)]^{b_3}, & \text{for } q=0; \end{cases} \tag{2.32}$$

(c_3) *$\alpha(x,y)d(x,y) \le \beta\left(\mathcal{E}_f^q(x,y)\right)\mathcal{E}_f^q(x,y)$, where $c_1,c_2 \in [0,\infty)$ are such that $c_1 + c_2 = 1$ and*

$$\mathcal{E}_f^q(x,y) = \begin{cases} [c_1 d^q(x,fx) + c_2 d^q(y,fy)]^{\frac{1}{q}}, & \text{for } q>0, \\ [d(x,fx)]^{c_1}\cdot[d(y,fy)]^{c_2}, & \text{for } q=0. \end{cases} \tag{2.33}$$

Remark 2.3. As we get Theorem 2.5 by $\alpha(x,y) = 1$ in Theorem 2.2 and Theorem 2.3, we can conclude several corollaries by letting $\alpha(x,y) = 1$ in Corollary 2.4. Furthermore, for different combinations of $\lambda_i,\ i = 1,2,3,4,5$ and q, we can list more corollaries. For instance, by taking $\lambda_i = 0$ for $i = 2,3,4,5$ and $q = 1$ in Theorem 2.5, we derive original theorem of Geraghty, that is, Theorem 1.1.

Example 2.1. Let $X := \{1,3,5,7,9\}$ and $f : X \to X$ be defined by

$$f(x) := \begin{cases} 1, & x \in \{1,9\} \\ 3, & x \in \{5,7\} \\ 9, & x = 3, \end{cases}$$

Then all the conditions of Theorem 2.5 are satisfied with $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ and

$$\alpha(x,y) := \begin{cases} 1, & x \text{ and } y \in \{1,5,7,9\} \\ 0, & x \text{ or } y \in \{3\}, \end{cases}$$

and

$$\beta(t) := \begin{cases} \frac{1}{1+t}, & t \in [0,1[\\ \frac{1}{2}, & t \in [1,\infty[. \end{cases}$$

Moreover, the unique fixed point is $x^* = 1$.

References

[1] Agarwal, R. and Karapınar, E., *Interpolative Rus-Reich-Ciric type contractions via simulation functions*, An. St. Univ. Ovidius Constanta, Ser. Mat., **27** (2019), No. 3, 137–152
[2] Aydi, H., Chen, C. M. and Karapınar, E., *Interpolative Ćirić-Reich-Rus type contractions via the Branciari distance*, Mathematics, **7** (2019), No. 1, 84; https://doi.org/10.3390/math7010084
[3] Aydi, H., Karapınar, E. and Roldan Lopez de Hierro, A. F., *ω-Interpolative Ćirić-Reich-Rus-type contractions*, Mathematics **7** (2019), No. 1, 84; https://doi.org/10.3390/math7010084
[4] Bianchini, R. M. and Grandolfi, M. , *Transformazioni di tipo contracttivo generalizzato in uno spazio metrico*, Atti Acad. Naz. Lincei, VII. Ser. Rend. Cl. Sci. Fis. Mat. Natur., **45** (1968), 212–216
[5] Ćirić, Lj. , *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273
[6] Dhage, B. C., *Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations*, Diff. Equations and Appl., **5** (2013), 155–184
[7] Geraghty, M., *On contractive mappings*, Proc. Amer. Math. Soc., **40** (1973), 604–608
[8] Hardy, G. E. and Rogers, T. D., *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., **16** (1973), 201–206
[9] Kannan, R, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60** (1968), 71–76
[10] Karapınar, E., Alqahtani, O. and Aydi, H., *On interpolative Hardy-Rogers type contractions*, Symmetry, **11** (2019), No. 1, 8; https://doi.org/10.3390/sym11010008
[11] Karapınar, E., *Revisiting the Kannan type contractions via interpolation*, Adv. Theory Nonlinear Anal. Appl., **2** (2018), 85–87
[12] Karapınar, E., Agarwal, R. and Aydi, H., *Interpolative Reich-Rus-Ćirić type contractions on partial metric spaces*, Mathematics, **6** (2018), No. 11, 256; https://doi.org/10.3390/math6110256
[13] O'Regan D. and Petruşel, A., *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl., **341** (2008), 1241–1252
[14] Petruşel, A. and Rus, I. A., *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc., **134** (2006), 411–418
[15] Petruşel, A. and Rus, I. A., *Fixed point theory in terms of a metric and of an order relation*, Fixed Point Theory, **20** (2019), 601–622
[16] Popescu, O., *Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces*, Fixed Point Theory Appl., 2014, 2014:190
[17] Radenovic, S., Kadelburg, Z., Jandrlic, D. and Jandrlic, A., *Some results on weakly contractive maps*, Bull. Iranian Math. Soc., **38** (2012), 625–645
[18] Reich, S., *Some remarks concerning contraction mappings*, Canad. Math. Bull., **14** (1971), 121–124
[19] Rus, I. A., *Generalized Contractions and Applications*, Cluj University Press, Clui-Napoca, Romania, 2001
[20] Rus, I. A., Petruşel, A. and Petruşel, G., *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, 2008
[21] Samet, B., Vetro, C. and Vetro, P., *Fixed point theorems for $\alpha - \psi$-contractive type mappings*, Nonlinear Anal. TMA, **75** (2012), 2154–2165
[22] Watanabe, T., *Fixed point theorems in ordered metric spaces and applications to nonlinear boundary value problems*, Fixed Point Theory, **20** (2019), 349–364

[1]Department of Medical Research
China Medical University
China Medical University Hospital
40402, Taichung, Taiwan

[2]Çankaya University
Department of Mathematics
06790, Etimesgut, Ankara, Turkey
Email address: erdalkarapinar@yahoo.com

[3]Department of Mathematics
Babeş-Bolyai University
Cluj-Napoca, Romania

[4]Academy of Romanian Scientists, Bucharest, Romania
Email address: petrusel@math.ubbcluj.ro

[5]Department of Business Babeş-Bolyai University
Cluj-Napoca, Romania
Email address: gabi.petrusel@tbs.ubbcluj.ro