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On optimality conditions for robust weak sharp solution in uncertain optimizations

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ABSTRACT. In this paper, we investigate the robust optimization problem involving nonsmooth and nonconvex real-valued functions. We firstly establish a necessary condition for the local robust weak sharp solution of considered problem under a constraint qualification. These optimality conditions are presented in terms of multipliers and Mordukhovich subdifferentials of the related functions. Then, by employing the robust version of the (KKT) condition, and some appropriate generalized convexity conditions, we also obtain some sufficient conditions for the global robust weak sharp solutions of the problem. In addition, some examples are presented for illustrating or supporting the results.

1. INTRODUCTION

In reality, it is common that the input data associated with the objective function and the constraints of programs are uncertain or incomplete due to prediction or measurement errors, that is, they are not known precisely when the problem is solved (see [1]). Robust optimization has come out as a noticeable determinism framework for investigating mathematical programming problems with uncertain data. Nowadays, theoretical and applied aspects in the area of robust optimization such as issues related to optimality conditionsfor solutions and characterization of solution sets; see, e.g. [1, 2, 3, 14, 15, 16, 11, 22, 23, 24] and other references therein.

On the other hand, the notion of a weak sharp minimizer in general mathematical programming problems was first introduced in [13]. It is an extension of a sharp minimizer in [20] to include the possibility of non-unique solution set. It has been acknowledged that the weak sharp minimizer plays important roles in stability/sensitivity analysis and convergence analysis of a wide range of numerical algorithms in mathematical programming (see, e.g., [5, 6, 18, 7] and references therein). Much attention has been paid to concerning sufficient and/or necessary conditions for weak sharp minimizers/solutions and characterizing weak sharp solution sets in various types of problems (see, [8, 9, 10, 27, 25, 26]). It might be seen, the study of optimality conditions for the weak sharp solution has been popular in many optimization problems. "How about the issue of this study, particularly, in a robust optimization?" According to this question, very recently, Kerdkaew and Wangkeeree [17] introduce robust weak sharp and robust sharp solution to a convex programming with the objective and constraint functions involved uncertainty. Then some optimality conditions for the robust weak sharp solution and the characterizations of the sets of all the robust weak sharp solutions were concerned.

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Motivated by above mentioned works, especially [17], we aim to to establish necessary and sufficient optimality conditions for the robust weak sharp solutions of the robust optimization problem formulated by nondifferentiable/nonconvex functions. Our obtained optimality conditions are presented in terms of multipliers and limiting/Mordukhovich subdifferential of the related functions. In addition, some examples are also provided for analyzing and illustrating the obtained results.

The rest of the paper is organized as follows. Section 2 contains some basic definitions from variational analysis and several auxiliary results. In Section 3, some necessary optimality conditions for local robust weak sharp solutions of the considered problem are presented. Some sufficient optimality conditions for robust weak sharp solutions of such problem are contained in Section 4.

2. Preliminaries

We begin this section by fixing notation and definitions including the notations generally used in variational analysis, the Mordukhovich generalized differentiation notions (see more details in [21, 19]), which are the main tools for our study. Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean space with dimension n. The inner product and norm in $\mathbb{R}^n$ are denoted by symbols $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The symbols $\mathbb{R}^n_+$, $\mathbf{B}$ and $B(x_0, r)$ stand for the nonnegative orthant of $\mathbb{R}^n$, closed unit ball in $\mathbb{R}^n$, and the open ball with center at x_0 and radius $r > 0$ for any $x_0 \in \mathbb{R}^n$, respectively. For a nonempty subset $S \subseteq \mathbb{R}^n$, the closure, boundary and convex hull of S are denoted by cl$S$, bd$S$ and co$S$, respectively, while the notation $x \xrightarrow{S} x_0$ means that $x \to x_0$ and $x \in S$.

Let a point $x_0 \in S$ be given. The set S is said to be closed around x_0 if there is a neighborhood U of x_0 such that $S \cap U$ is closed. Moreover, the set S is said to be locally closed if it is closed around every $x_0 \in S$.

Let S be closed around x_0. Recall that the contingent cone of S at x_0 is denoted by $T(S, x_0)$ and defined by

$$T(S, x_0) := \{v \in \mathbb{R}^n : \exists v_n \to v, \exists t_n \downarrow 0 \text{ s.t. } x_0 + t_n v_n \in S, \forall n \in \mathbb{N}\},$$

while the Fréchet (or regular) normal cone of S at x_0, which is a set of all the Fréchet normals, $\widehat{N}(S, x_0)$ is defined by

$$\widehat{N}(S, x_0) := \Big\{x^* \in \mathbb{R}^n : \limsup_{x \xrightarrow{S} x_0} \frac{\langle x^*, x - x_0 \rangle}{\|x - x_0\|} \leq 0\Big\}.$$

Note that $\widehat{N}(S, x_0)$ is a closed convex subset of $\mathbb{R}^n$ and we set $\widehat{N}(S, x_0) = \emptyset$ if $x_0 \notin S$. The notation $N(S, x_0)$ stands for the Mordukhovich (or basic, limiting) normal cone of S at x_0. It is defined by

$$N(S, x_0) := \Big\{x^* \in \mathbb{R}^n : \exists x_n \xrightarrow{S} x_0, \exists x_n^* \to x^* \text{ with } x_n^* \in \widehat{N}(S, x_n), \forall n \in \mathbb{N}\Big\}.$$

In the case that S is a convex set, then we obtain $\widehat{N}(S, x_0) = N(S, x_0) = T(S, x_0)^\circ = \{x^* \in \mathbb{R}^n : \langle x^*, x - x_0 \rangle \leq 0, \forall x \in S\}$.

Let $h : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be an extended real-valued function. The domain and the epigraph of the fuction h, are defined respectively by

$$\text{dom} h := \{x \in \mathbb{R}^n : h(x) < +\infty\} \text{ and } \text{epi} h := \{(x, \alpha) \in X \times \mathbb{R} \,|\, \alpha \geq h(x)\}.$$

For $x_0 \in \text{dom}h$ and $\varepsilon \geq 0$ be given, the analytic ε-subdifferential of function h at x_0, which has the form $\widehat{\partial}_\varepsilon h(x_0)$ is defined by

$$\widehat{\partial}_\varepsilon h(x_0) := \left\{ x^* \in \mathbb{R}^n : \liminf_{\substack{x \to x_0, \\ x \neq x_0}} \frac{h(x) - h(x_0) - \langle x^*, x - x_0 \rangle}{\|x - x_0\|} \geq -\varepsilon \right\}.$$

If $\varepsilon = 0$, then the analytic ε-subdifferential of h at x_0 reduces to the general Fréchet subdifferential of h at x_0, which is denoted by $\widehat{\partial} h(x_0)$. Besides, $\partial h(x_0)$ denotes the Mordukhovich subdifferential of h at x_0. It is defined by

$$\partial h(x_0) := \left\{ x^* \in \mathbb{R}^n : \exists x_n \xrightarrow{h} x_0, \exists x_n^* \to x^* \text{ with } x_n^* \in \widehat{\partial} h(x_n), \forall n \in \mathbb{N} \right\}$$

where $x_n \xrightarrow{h} x_0$ means $x_n \to x_0$ and $h(x_n) \to h(x)$. If $x \notin \text{dom}h$, then we set $\widehat{\partial} h(x_0) = \partial h(x_0) = \emptyset$. It is obvious that for any $x \in \mathbb{R}^n, \widehat{\partial} h(x_0) \subseteq \partial h(x_0)$. Specially, if h is a convex function, then $\widehat{\partial} h(x_0) = \partial h(x_0) = \{x^* \in \mathbb{R}^n : \langle x^*, x - x_0 \rangle \leq h(x) - h(x_0), \forall x \in \mathbb{R}^n\}$.

The distance function $d(\cdot, S) : \mathbb{R}^n \to \mathbb{R}$ and the indicator function $\delta(\cdot, S) : \mathbb{R}^n \to \overline{\mathbb{R}}$ of S are respectively defined by $d(x, S) := \inf_{y \in S} \|x - y\|, \forall x \in \mathbb{R}^n$, and

$$\delta(x, S) = \begin{cases} 0; & x \in S, \\ +\infty; & x \notin S. \end{cases}$$

By above notations and definitions, we get $\widehat{\partial}\delta(x_0, S) = \widehat{N}(S, x_0)$ and $\partial\delta(x_0, S) = N(S, x_0)$. Simultaneously, $\widehat{\partial} d(x_0, S) = \mathbf{B} \cap \widehat{N}(S, x_0)$ and $\partial d(x_0, S) \subseteq \mathbf{B} \cap N(S, x_0)$.

Next, we recall some useful and important propositions and definitions for this paper.

Lemma 2.1. [19, Corollary 1.81] If $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is locally Lipschitz at x_0, with modulus $l > 0$, then we always have $\|x^*\| \leq l, \ \forall x^* \in \partial h(x_0)$.

Theorem 2.1 (The generalized Fermat rule)**.** [21, 19] Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. If h attains a local minimum at $x_0 \in \mathbb{R}^n$, then $0_{\mathbb{R}^n} \in \widehat{\partial} h(x_0)$, which implies $0_{\mathbb{R}^n} \in \partial h(x_0)$.

Theorem 2.2 (fuzzy sum rule for the Fréchet subdifferential and the sum rule for the Mordukhovich subdifferential)**.** [21, 19] Let $f, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous around $x_0 \in \text{dom}f \cap \text{dom}h$. If f is Lipschitz continuous around x_0, then

(1) for every $x^* \in \widehat{\partial}(f + h)(x_0)$ and every $\varepsilon > 0$, there exist $x_1, x_2 \in B(x_0, \varepsilon)$ such that

$$|f(x_1) - f(x_0)| < \varepsilon, |h(x_2) - h(x_0)| < \varepsilon \text{ and } x^* \in \widehat{\partial} f(x_1) + \widehat{\partial} h(x_2) + \varepsilon\mathbf{B}.$$

(2) $\partial(f + h)(x_0) \subseteq \partial f(x_0) + \partial h(x_0)$.

We conclude this section by the following problems and solution concepts.

Let Ω be a nonempty locally closed subset of $\mathbb{R}^n$. For $q_0, q_i \in \mathbb{N}, i = 1, \ldots, m$, let $\mathcal{U}$ and $\mathcal{V}_i, i = 1, \ldots, m$ be nonempty compact subsets of $\mathbb{R}^{q_0}$ and $\mathbb{R}^{q_i}$, respectively. We consider the following *uncertain optimization problem*:

(UP) Minimize $f(x, u)$ subject to $g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m, x \in \Omega,$

where $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathcal{V}_i \to \mathbb{R}, \ i = 1, \ldots, m$ are given real-valued functions, x is the vector of decision variable, u and $v_i, i = 1, \ldots, m$ are uncertain parameters belonging to the specified compact uncertainty sets $\mathcal{U}$ and $\mathcal{V}_i$, respectively. In fact, the uncertainty sets can be apprehended in the sense that the parameter vectors u and all v_i are not known exactly at the time of the decision. For examining the uncertain optimization problem (UP), one usually associates with it, namely *robust counterpart*, is the following problem:

(RUP) Minimize $\max_{u\in\mathcal{U}} f(x,u)$ subject to $g_i(x,v_i) \le 0,\ \forall v_i \in \mathcal{V}_i,\ i=1,\dots,m, x\in\Omega.$

The *robust feasible set* K is denoted by $K := \{x\in\Omega : g_i(x,v_i)\le 0, \forall v_i\in\mathcal{V}_i,\ i=1,\dots,m\}$.

The following concept of robust solutions can be found in the literature; see e.g.,[16].

Definition 2.1. A point $x_0 \in K$ is said to be a *local robust solution* for (UP) if it is a local solution for (RUP) i.e., if there exists a neighborhood U of x_0 such that $\max_{u\in\mathcal{U}} f(x,u) - \max_{u\in\mathcal{U}} f(x_0,u) \ge 0,\ \forall x\in K\cap U$. In addition, if $U=\mathbb{R}^n$, then $x_0\in K$ is said to be a global *robust solution* for (UP).

In [17], a new concept of a solution, which is related to the weak sharpness, namely the *(local/global) robust weak sharp solution* was introduced.

Definition 2.2. A point $x_0 \in K$ is said to be a *local robust weak sharp solution* for (UP) if it is a local weak sharp solution for (RUP) i.e., there exist a neighborhood U of x_0 and a real number $\eta > 0$ such that

$$\max_{u\in\mathcal{U}} f(x,u) - \max_{u\in\mathcal{U}} f(x_0,u) \ge \eta d(x,\widetilde{K}),\ \forall x\in K\cap U, \tag{2.1}$$

where $\widetilde{K} := \left\{x\in K : \max_{u\in\mathcal{U}} f(x,u) = \max_{u\in\mathcal{U}} f(x_0,u)\right\}$. Specially, if $U=\mathbb{R}^n$, then $x_0\in K$ is said to be a *global robust weak sharp solution* for (UP).

It is simple to see that every (local) robust weak sharp solution must be also a (local) robust solution. In contrast, the converse implication need not to be true.

3. Necessary optimality conditions for robust weak sharp solutions

In this section, we focus our attention on establishing some optimality conditions for local (global) robust sharp solutions in uncertain optimization problems in terms of the advanced tools of variational analysis and generalized differentiation. Given arbitrary $x_0\in\Omega$, we set $\mathcal{U}(x_0) := \{u^*\in\mathcal{U} : f(x_0,u^*) = \max_{u\in\mathcal{U}} f(x_0,u)\}$, $\mathcal{V}_i(x_0) := \{v_i^*\in\mathcal{V}_i : g_i(x_0,v_i^*) = \max_{v_i\in\mathcal{V}_i} g_i(x_0,v_i)\}$, and $\mathcal{I}(x_0) := \{i=1,\dots,m : g_i(x_0,v_i)=0,\ \forall v_i\in\mathcal{V}_i\}$.

In what follows, throughout this section, we assume $g_i : \mathbb{R}^n\times\mathcal{V}_i\to\mathbb{R}$ is a function such that for each fixed $v_i\in\mathcal{V}_i$, $i=1,\dots,m$, $g_i(\cdot,v_i)$ is locally Lipschitz continuous and assume function $f:\mathbb{R}^n\times\mathcal{U}\to\mathbb{R}$ satisfies the following conditions:

(C1) For a fixed $x_0\in\Omega$, there exists $r_{x_0} > 0$ such that the function $f(x,\cdot):\mathcal{U}\to\mathbb{R}$ is upper semicontinuous for all $x\in B(x_0,r_{x_0})$ and $f(\cdot,u)$ is Lipschitz continuous in x, uniformly for $u\in\mathcal{U}$; i.e., for some real number $l>0$, for all $x,y\in\Omega$ and $u\in\mathcal{U}$, one has $\|f(x,u)-f(y,u)\| \le l\|x-y\|$.

(C2) The multifunction $\partial_x f(\cdot,\cdot) : \mathbb{R}^n\times\mathcal{U}\to 2^{\mathbb{R}^n}$ is closed at (x_0,u) for each $u\in\mathcal{U}(x_0)$, where the symbol ∂_x stands for the Mordukhovich subdifferential operation with respect to x.

In order to obtain the necessary and sufficient optimality condition for local robust sharp solutions of (UP), the following constraint qualification is essential.

Definition 3.3. Given arbitrary $x_0\in\Omega$, the constraint qualification (CQ) is said to be satisfied at x_0 if there do not exist $\mu_i\ge 0$ and $v_i\in\mathcal{V}_i, i\in\mathcal{I}(x_0)$ such that $\sum_{i\in\mathcal{I}(x_0)}\mu_i \ne 0$ and $0_{\mathbb{R}^n} \in \sum_{i\in\mathcal{I}(x_0)} \mu_i\partial_x g_i(x_0,v_i) + N(\Omega,x_0)$.

Remark 3.1. We can see that the (CQ) defined in Definition 3.3 reduces to the constraint qualification defined in [12, Definition 3.2] when $\Omega = \mathbb{R}^n$. Also, it is reduces to the extended Mangasarian-Fromovitz constraint qualification (see [4]) in the smooth setting when $\Omega = \mathbb{R}^n$.

The following necessary optimality condition for local robust weak sharp solutions of (UP) is obtained under the (CQ).

Theorem 3.3. Let $x_0 \in K$ and the constraint qualification (CQ), defined in Definition 3.3, be satisfied at x_0. If x_0 is a local robust weak sharp solution for (UP), then there exists a real number $\eta > 0$ such that
(3.2)

$$\eta \mathbf{B} \cap \widehat{N}(K, x_0) \subseteq \text{ co } \left(\bigcup_{u \in \mathcal{U}(x_0)} \partial_x f(x_0, u) \right) + \bigcup_{\mu_i \in M_i(x_0)} \left(\sum_{i=1}^{m} \mu_i \partial_x g_i(x_0, v_i) \right) + N(\Omega, x_0),$$

where $M_i(x_0) = \{\mu_i \geq 0 \,:\, \mu_i g_i(x_0, v_i) = 0, v_i \in \mathcal{V}_i\}$ for all $i = 1, \ldots, m$.

Proof. Suppose that x_0 is a local robust sharp solution for (UP). Then, there exist real numbers $\eta, r_1 > 0$ such that

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(x_0, u) \geq \eta d(x, \widetilde{K}), \forall x \in K \cap B(x_0, r_1).$$

Let $x^* \in \mathbf{B} \cap \widehat{N}(K, x_0)$ be given. It follows from $\widehat{\partial} d(x_0, K) = \mathbf{B} \cap \widehat{N}(K, x_0)$ that $x^* \in \widehat{\partial} d(x_0, K)$. By the definition of $\widehat{\partial} d(\cdot, K)$, for any $\varepsilon > 0$, there exists $r_2 \in (0, \frac{1}{2} r_1)$ such that

$$\langle x^*, x - x_0 \rangle \leq d(x, \widetilde{K}) + \varepsilon \|x - x_0\|, \forall x \in B(x_0, r_2).$$

It is obvious that $B(x_0, r_2) \subseteq B(x_0, r_1)$, so we obtain $\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(x_0, u) + \eta \varepsilon \|x - x_0\| \geq \eta \langle x^*, x - x_0 \rangle$ for all $x \in K \cap B(x_0, r_2)$. Consider the following function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\varphi(x) := -\eta \langle x^*, x - x_0 \rangle + \phi(x) + \eta \varepsilon \|x - x_0\| + \delta(x, K), \forall x \in \mathbb{R}^n,$$

where $\phi(x) := \max\limits_{u \in \mathcal{U}} f(x, u) - \max\limits_{u \in \mathcal{U}} f(x_0, u), \forall x \in \mathbb{R}^n$. Observe that for each $x \in K \cap B(x_0, r_2), \varphi(x) \geq 0$, while $\varphi(x_0) = 0$. This means the function φ attains its local minimum point at x_0. Further, we can get by the properties of $f(\cdot, u)$, $\| \cdot - x_0\|$ and $\delta(\cdot, K)$ that the function φ is lower semicontinuous around x_0. Therefore, it follows from Theorem 2.1 that $0_{\mathbb{R}^n} \in \widehat{\partial} \varphi(x_0)$. Moreover, from Theorem 2.2 (i), for each $y \in \widehat{\partial} \varphi(x_0)$ and each $\varepsilon > 0$, there exist $x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon \in B(x_0, \varepsilon)$, such that

$$|\phi(x_1^\varepsilon)| < \varepsilon, \quad \eta\varepsilon \|x_2^\varepsilon - x_0\| < \varepsilon, \quad \delta(x_3^\varepsilon, K) < \varepsilon \text{ and } y \in \widehat{\partial} \phi(x_1^\varepsilon) + \eta \varepsilon \widehat{\partial} \|x_2^\varepsilon - x_0\| + \widehat{\partial} \delta(x_3^\varepsilon, K) + \varepsilon \mathbf{B}.$$

Since $0_{\mathbb{R}^n} \in \widehat{\partial} \varphi(x_0), \eta x^* \in \widehat{\partial} \phi(x_1^\varepsilon) + \eta \varepsilon \widehat{\partial} \|x_2^\varepsilon - x_0\| + \widehat{\partial} \delta(x_3^\varepsilon, K) + \varepsilon \mathbf{B}$. Observe that $x_3^\varepsilon \in K$ and $\widehat{\partial} \delta(x_3^\varepsilon, F) = \widehat{N}(K, x_3^\varepsilon)$. It follows from the definition of ϕ that it is Lipschitz continuous around x_0 with a constant l. So, due to [19, Proposition 1.85], for all sufficiently small $\varepsilon > 0$, one has $\widehat{\partial} \phi(x_1^\varepsilon) \subseteq l\mathbf{B}$. Similarly, we also get $\widehat{\partial} \|x_2^\varepsilon - x_0\| \subseteq \mathbf{B}$. According to these inclusions, the compactness of $\mathbf{B}$, and $x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon \in B(x_0, \varepsilon)$ yields $x_1 \xrightarrow{\phi} x_0, \quad x_2^\varepsilon \xrightarrow{\|\cdot - x_0\|} x_0, \quad x_3^\varepsilon \xrightarrow{K} x_0$, as $\varepsilon \downarrow 0$. It follows that

$$\eta x^* \in \partial \phi(x_0) + N(K, x_0). \tag{3.3}$$

As f satisfies (C1) and (C2), by the same fashion of proof in Theorem 3.3 of [11], we obtain

$$\partial \phi(x_0) \subseteq \text{ co } \left(\bigcup_{u \in \mathcal{U}(x_0)} \partial_x f(x_0, u) \right). \tag{3.4}$$

On the other hand, $\Pi := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, v_i \in \mathcal{V}_i, i \in \mathcal{I}\}$. Hence, $K = \Omega \cap \Pi$. As $0_{\mathbb{R}^n} \in N(\Omega, x_0)$, the following inclusion always holds:

$$\bigcup_{\mu_i \in M_i(x_0)} \left(\sum_{i \in \mathcal{I}(x_0)} \mu_i \partial_x g_i(x_0, v_i) \right) \subseteq \bigcup_{\mu_i \in M_i(x_0)} \left(\sum_{i \in \mathcal{I}(x_0)} \mu_i \partial_x g_i(x_0, v_i) \right) + N(\Omega, x_0).$$

Since the (CQ) is satisfied at x_0, there do not exist $\mu_i \geq 0$ and $v_i \in \mathcal{V}_i, i \in \mathcal{I}(x_0)$ such that $\sum_{i \in \mathcal{I}(x_0)} \mu_i \neq 0$ and $0_{\mathbb{R}^n} \in \sum_{i \in \mathcal{I}(x_0)} \mu_i \partial_x g_i(x_0, v_i) + N(\Omega, x_0)$. Applying [19, Corollary 4.36], we have

$$N(\Pi, x_0) \subseteq \bigcup_{\mu_i \in M_i(x_0)} \left(\sum_{i \in \mathcal{I}(x_0)} \mu_i \partial_x g_i(x_0, v_i) \right).$$

It follows from [19, Corollary 3.37] that $N(K, x_0) = N(\Omega \cap \Pi, x_0) \subseteq N(\Omega, x_0) + N(\Pi, x_0)$. Setting $\mu_i = 0$ for every $i \in \mathcal{I} \setminus \mathcal{I}(x_0)$, we arrive the following inclusion:

$$N(K, x_0) \subseteq \bigcup_{\mu_i \in M_i(x_0)} \left(\sum_{i \in \mathcal{I}} \mu_i \partial_x g_i(x_0, v_i) \right) + N(\Omega, x_0). \tag{3.5}$$

As $x^* \in \mathbf{B} \cap \widehat{N}(K, x_0)$ was arbitrary, we verify (3.2) by combining (3.3), (3.4) and (3.5). □

The following example shows that the (CQ) being satisfied around $x_0 \in K$ is essential for Theorem 3.3.

Example 3.1. Let $f : \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V} \to \mathbb{R}$ be defined by

$$f(x, u) = \begin{cases} -u; & x = 0, \\ x^{-2} - u; & \text{otherwise}, \end{cases}$$

and $g(x, v) := v - x^3$ where $x \in \mathbb{R}, u \in \mathcal{U} := [0, 1]$ and $v \in \mathcal{V} := [-1, 0]$. Take $\Omega := [-1, 1]$ and consider the problem (UP). It is not hard to see that f satisfies (C1) and (C2), and the robust feasible set is $K = [0, 1]$. Consider $x_0 := 0 \in K$ with its neighborhood $U = (-\frac{1}{2}, \frac{1}{2})$. Choosing a positive real number $\eta = 1 > 0$, we can verify that x_0 is a local robust weak sharp solution of the problem (UP). Simultaneously, we get from direct calculating that

$$\partial_x f(x_0, u) = \{0\},\ u \in \mathcal{U},\ \partial_x g(x_0, v) = \{0\},\ v \in \mathcal{V}, N(\Omega, x_0) = \{0\}$$

and $N(K, x_0) = -\mathbb{R}_+$. It follows that the (CQ) is not satisfied at x_0. Furthermore, we get $\eta \mathbf{B}_{\mathbb{R}^2} \cap \widehat{N}(K, x_0) = [-\eta, 0]$ while

$$\text{co} \left(\bigcup_{u \in \mathcal{U}(x_0)} \partial_x f(x_0, u) \right) + \bigcup_{\mu \in M(x_0)} \mu \partial_x g(x_0, v) + N(\Omega, x_0) = \{0\},$$

which shows that (3.2) does not hold for every $\eta, \delta > 0$. Hence, the assumption that (CQ) being satisfied is essential. Observe that the functions $f(\cdot, u)$ and $g(\cdot, v)$ is not convex with $x_1 = \frac{1}{2}, x_2 = 0$, and $\lambda = \frac{1}{2}$. Therefore, [17, Theorem 3.2] is not applicable for this example.

Now, we state a type of the robust version of Karush-Kuhn-Tucker (KKT) conditions as the following definition.

Definition 3.4. the robust version of the (KKT) condition is satisfied at $x_0 \in K$ if there exist $\lambda > 0$ and $\mu \in \mathbb{R}^m_+$ such that $\lambda + \sum_{i=1}^m \mu_i = 1$,

$$0 \in \lambda \,\text{co} \left(\bigcup_{u \in \mathcal{U}(x_0)} \partial_x f(x_0, u) \right) + \sum_{i=1}^m \mu_i \,\text{co} \left(\bigcup_{v_i \in \mathcal{V}_i(x_0)} \partial_x g_i(x_0, v_i) \right),$$

and $\mu_i \sup_{v_i \in \mathcal{V}_i} g_i(x_0, v_i) = 0,\ i \in \mathcal{I}$.

The following example illustrates that only satisfying the robust version of (KKT) condition is not sufficient for a point to be a (local) robust weak sharp solution of problem (UP).

Example 3.2. Let $f : \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V} \to \mathbb{R}$ be defined by $f(x,u) = -x^2 - u$ and $g(x,v) = v \max\{x, 0\}$, where $x \in \mathbb{R}, u \in \mathcal{U} := [0,1]$ and $v \in \mathcal{V} := [-1, 0]$. By taking $\Omega = \mathbb{R}$, we have $K = \mathbb{R}$. Consider $x_0 := 0 \in K$, then we have

$$\partial_x f(x_0, u) = \{0\}, u \in \mathcal{U}, \partial_x g(x_0, v) = \{0\}, v \in \mathcal{V}.$$

The robust version of the (KKT) condition is satisfied at x_0 with $\lambda = \mu = \frac{1}{2} > 0$. However, this x_0 is not a (local) robust weak sharp solution of our considered problem since there is no $\eta > 0$ satisfy $\max_{u\in\mathcal{U}} f(x,u) - \max_{u\in\mathcal{U}} f(x_0,u) = -x^2 \geq \eta d(x, \widetilde{K})$ for all $x \in \mathbb{R}$.

4. Sufficient optimality conditions for robust weak sharp solutions

In this section, we focus on sufficient optimality conditions for robust weak sharp solutions of problem (UP). In order to formulate the conditions, we need to introduce concepts of generalized convexity at a given point for a family of real-valued functions. We set $g := (g_1, \ldots, g_m)$ for convenience in the sequel.

Definition 4.5. (f, g) is said to be *generalized convex* at $x_0 \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n, z_u^* \in \partial_x f(x_0, u), u \in \mathcal{U}(x_0)$, and $x_v^* \in \partial g_i(x_0, v), v \in \mathcal{V}_i(x_0), i = 1, \ldots, m$, there exists $w \in \mathbb{R}^n$ such that $f(x,u) - f(x_0,u) \geq \langle z_u^*, w\rangle, \quad g_i(x,v) - g_i(x_0,v) \geq \langle x_v^*, w\rangle$.

Remark 4.2. If $f(\cdot, u), u \in \mathcal{U}$ and $g_i(\cdot, v), v \in \mathcal{V}_i, i = 1, \ldots, m$ are convex, then (f, g) is generalized convex at any $x_0 \in \mathbb{R}^n$ with $w := x - x_0$ for each $x \in \mathbb{R}^n$.

The following example demonstrates that the class of generalized convex functions at a given point is properly wider than the one of convex functions.

Example 4.3. Let $f : \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V} \to \mathbb{R}$ be defined by $f(x,u) = 2x + u$ and

$$g(x,v) = \begin{cases} vx; & x \geq 0, \\ x - v; & \text{otherwise} \end{cases}$$

where $x \in \mathbb{R}, u \in \mathcal{U} := [0,1] \subseteq \mathbb{R}$, and $v \in \mathcal{V} := [-1,1] \subseteq \mathbb{R}$. Consider $x_0 := 0 \in \mathbb{R}$. Observe that $\partial_x f(x_0,u) = \{2\}$ for all $u \in \mathcal{U}, \partial_x g(x_0,v) = \{v, 1\}$ for all $v \in \mathcal{V}$. We see that (f,g) is generalized convex at $x_0 = 0 \in \mathbb{R}$ as follows:
<u>Case I:</u> If $x \geq 0$, then there exists $w := x \in \mathbb{R}$ such that $f(x,u) - f(x_0,u) = 2x = \langle 2, x\rangle$ and $g(x,v) - g(x_0,v) = vx = \langle v, x\rangle$.
<u>Case II:</u> If $x < 0$, then there exists $w = x + v \in \mathbb{R}$ such that $f(x,u) - f(x_0,u) = 2x \geq \langle 2, x - v\rangle$, and $g(x,v) - g(x_0,v) = x - v = \langle 1, x - v\rangle$.
However, $g(\cdot, 0)$ is not a convex function as follows: let $x_1 = 1, x_2 = -1 \in \mathbb{R}$, and choose $\lambda = \frac{1}{2} \in [0,1]$, we have $g(\lambda x_1 + (1-\lambda)x_2, 0) > \lambda g(x_1, 0) + (1-\lambda) g(x_2, 0)$.

By means of the robust version of the (KKT) condition and the generalized convexity, we established the following result.

Theorem 4.4. Let $x_0 \in K$ and the robust version of the (KKT) condition be satisfied at x_0. If (f,g) is generalized convex at x_0, then x_0 is a robust weak sharp solution for the problem (UP).

Proof. Since the robust version of the (KKT) condition is satisfied at x_0, there exist $\lambda_1 \geq 0, \lambda_{1_k} \geq 0, z_{1_k}^* \in \partial_x f(x_0, u_{1_k}), u_{1_k} \in \mathcal{U}(x_0), \sum_{k=1}^{k_1} \lambda_{1_k} = 1, k = 1, \ldots, k_1, k_1 \in \mathbb{N}$, and

$\mu \in \mathbb{R}^m_+, \mu_{i_j} \geq 0, x^*_{i_j} \in \partial_x g_i(x_0, v_{i_j}), v_{i_j} \in \mathcal{V}_i(x_0), \sum_{j=1}^{j_i} \mu_{i_j} = 1, j = 1, \ldots, j_i, j_i \in \mathbb{N}$, such that $\lambda_1 + \sum_{i=1}^m \mu_i = 1$ and

$$0 = \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} z^*_{1_k} \right) + \sum_{i=1}^m \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} x^*_{i_j} \right), \tag{4.6}$$

$$\mu_i \sup_{v_i \in \mathcal{V}_i} g_i(x_0) = 0, i = 1, \ldots, m. \tag{4.7}$$

Assume on the contrary that x_0 is not a robust weak sharp solution for the problem (UP). Then, there exists $\tilde{x} \in K$ such that for all $\eta > 0$

$$\max_{u \in \mathcal{U}} f(\tilde{x}, u) - \max_{u \in \mathcal{U}} f(x_0, u) < \eta d(\tilde{x}, \widetilde{K}). \tag{4.8}$$

It follows from the generalized convexity of (f, g) and (4.6) that there exists $w \in \mathbb{R}^n$ such that

$$\begin{aligned} 0 &= \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} \langle z^*_{1_k}, w \rangle \right) + \sum_{i=1}^m \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} \langle x^*_{i_j}, w \rangle \right) \\ &\leq \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} [f(\tilde{x}, u_{1_k}) - f(x_0, u_{1_k})] \right) + \sum_{i=1}^m \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} [g_i(\tilde{x}, v_{i_j}) - g_i(x_0, v_{i_j})] \right). \end{aligned} \tag{4.9}$$

Therefore,

$$\begin{aligned} &\lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(x_0, u_{1_k}) \right) + \sum_{i=1}^m \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} g_i(x_0, v_{i_j}) \right) \\ \leq\ &\lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(\tilde{x}, u_{1_k}) \right) + \sum_{i=1}^m \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} g_i(\tilde{x}, v_{i_j}) \right) \end{aligned} \tag{4.10}$$

Since $v_{i_j} \in \mathcal{V}_i(x_0), g_i(x_0, v_{i_j}) = \sup_{v_i \in \mathcal{V}_i} g_i(x_0, v_i)$ for all $i = 1, \ldots, m, \forall j = 1, \ldots, j_i$. From (4.7), we have $\mu_i g_i(x_0, v_{i_j}) = 0$ for $i = 1, \ldots, m$ and $j = 1, \ldots, j_i$. Furthermore, for each $\tilde{x} \in K, \mu_i g_i(\tilde{x}, v_{i_j}) \leq 0$ for $i = 1, \ldots, m$ and $j = 1, \ldots, j_i$. Hence, by (4.10) we have

$$\begin{aligned} \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(x_0, u_{1_k}) \right) &= \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(x_0, u_{1_k}) \right) + \sum_{i=1}^m \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} g_i(x_0, v_{i_j}) \right) \\ &\leq \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(\tilde{x}, u_{1_k}) \right). \end{aligned}$$

This together with $u_{1_k} \in \mathcal{U}(x_0)$ imply $\sum_{k=1}^{k_1} \lambda_{1_k} \max_{u \in \mathcal{U}} f(x_0, u) \leq \sum_{k=1}^{k_1} \lambda_{1_k} f(\tilde{x}, u_{1_k}) \leq \sum_{k=1}^{k_1} \lambda_{1_k} \max_{u \in \mathcal{U}} f(\tilde{x}, u)$, which yields $\max_{u \in \mathcal{U}} f(x_0, u) - \max_{u \in \mathcal{U}} f(\tilde{x}, u) \leq \eta d(\tilde{x}, \widetilde{K})$, for all $\eta > 0$. Thus for any $\eta > 0, \max_{u \in \mathcal{U}} f(\tilde{x}, u) - \max\limits_{u \in \mathcal{U}} f(x_0, u) \geq \eta d(\tilde{x}, \widetilde{K})$. This contradicts (4.8) and hence x_0 is a robust weak sharp solution of (UP). □

Remark 4.3. To establish the result in Theorem 4.4, the assumptions of the convexities of objective function, constrains and parameter uncertain sets are dropped. However, these assumptions are employed to obtain several results on optmality conditions for robust optimal solutions of uncertain optimization problems obtained in recent literature (see, e.g., [16, 22, 23, 24, 17]).

The next example assert the importance of the generalized convexity of (f, g) imposed in Theorem 4.4.

Example 4.4. Let $f : \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V} \to \mathbb{R}$ be defined by

$$f(x, u) = x^3 + u \text{ and } g(x, v) = 1 - (v + x^4),$$

where $x \in \mathbb{R}, u \in \mathcal{U} := [-1, 0], v \in \mathcal{V} := [1, 2]$ and let $\Omega := [-2, 2]$. It can be seen that conditions (C1) and (C2) are satisfied and the robust feasible set is $K = \mathbb{R}$. By taking $x_0 := 0 \in K$, we see that $\partial_x f(x_0, u) = \{0\}$ for all $u \in \mathcal{U}$ and $\partial_x g(x_0, v) = \{0\}$ for all $v \in \mathcal{V}$. By the same way in Example 3.2, we have that the robust version of the (KKT) condition is satisfied at x_0. However, the generalized convexity of (f, g) is not satisfied at x_0. Indeed, there exists $z = -\frac{1}{2} \in K$ such that for each $w \in \mathbb{R}$, $f(z, u) - f(x_0, u) = \left(-\frac{1}{2}\right)^3 < 0 = \langle 0, w\rangle$ and $g(z, v) - g(x_0, v) = 1 - \left(v + (-\frac{1}{2})^4\right) < 0 = \langle 0, w\rangle$. Notice that x_0 is not a (local) robust weak sharp solution of (UP) as there is no $\eta > 0$ satisfy $\max_{u\in\mathcal{U}} f(x, u) - \max_{u\in\mathcal{U}} f(x_0, u) = x^3 \geq \eta d(x, \widetilde{K}),\ \forall x \in \mathbb{R}$. Therefore, the conclusion of the Theorem 4.4 may fail if the generalized convexity has been dropped.

It is not hard to see that the functions$f(\cdot, u)$ and $g(\cdot, v)$ are not convex. In fact, the convexities of them are not satisfied when $x_1 = -\frac{1}{2}, x_2 = 0$, and $\lambda = \frac{1}{2}$. Therefore, this problem cannot be solved by [17, Theorem 3.2].

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REFERENCES

[1] Ben-Tal, A., Ghaoui, L. E. and Nemirovski, A., *Robust optimization*, Princeton series in applied mathematics, Priceton, NJ: Priceton University Press, 2009

[2] Bertsimas, D., Brown, D. and Caramanis, C., *Theory and applications of robust optimization*, SIAM Rev., **53** (2011), 464–501

[3] Beck, A. and Ben-Tal, A., *Duality in robust optimization: primal worst equals dual best*, Oper. Res. Lett., **37** (2009), 1–6

[4] Bonnans, J. F. and Shapiro, A., *Perturbation Analysis of Optimization Problems*, Springer-Verlag, New York, 2000

[5] Burke, J. V., Lewis, A. and Overton, M., *Optimization matrix stability*, Proc. Am. Math. Soc., **129** (2000), 1635–1642

[6] Burke, J. V., Lewis, A. and Overton, M., *Optimization stability and eigenvalue multiplicity*, Found. Comput. Math., **1** (2001), 205–225

[7] Burke, J. V. and Ferris, M. C., *Weak sharp minima in mathematical programming*, SIAM J. Control Optim., **36** (1993), 1340–1359

[8] Burke, J. V. and Deng, S., *Weak sharp minima revisited, part I: Basic theory*, Control Cybern., **31** (2002), 439–469

[9] Burke, J. V. and Deng, S., *Weak sharp minima revisited, part II: Application to linear regularity and error bounds*, Math. Program., Ser. B, **104** (2005), 235–261

[10] Burke, J. V. and Deng, S., *Weak sharp minima revisited, part III: Error bounds for differentiable convex inclusions*, Math. Program., Ser. B, **116** (2009), 37–56

[11] Chuong, T. D., *Optimality and duality for robust multiobjective optimization problem*, Nonlinear Analysis., **134** (2016), 127–143

[12] Chuong, T. D. and Kim, D. S., *Optimality conditions and duality in nonsmooth multiobjective optimization problems*, Ann Oper Res., **217** (2014), 117–136

[13] Ferris, M. C., *Weak sharp minima and penalty functions in mathematical programming*, Ph.D. thesis, Universiy of Cambridge, Cambridge, UK, 1988

[14] Jeyakumar, V. and Li, G., *Characterizing robust set containments and solutions of uncertain linear programs without qualifications*, Oper. Res. Lett., **38** (2010), 188–194

[15] Jeyakumar, V., Li, G. Y. and Lee, G. M., *Robust duality for generalized convex programming problems under data uncertainty*, Nonlinear Anal., **75** (2012), No. 3, 1362–1373

[16] Jeyakumar, V., Lee, G. M. and Li, G., *Characterizing robust solution sets of convex programs under data uncertainty*, J. Optim Theory., **164** (2015), 407–435

[17] Kerdkaew, J. and Wangkeeree, R., *Characterizing robust weak sharp solution sets of uncertain convex optimizations*, Journal of Industrial and Management Optimization, doi: 10.3934/jimo.2019074 (2019)

[18] Lewis, A. S. and Pang, J. S., Error bounds for convex inequality systems, *In: Crouzeix, J. P. (ed.) Proceedings of the Fifth Symposium on Generalized Convexity*, Luminy-Marseille, 1996

[19] Mordukhovich, B. S., *Variational Analysis and Generalized Differentiation. I: Basic Theory*, Springer, Berlin, 2006

[20] Polyak, B .T., *Sharp Minima*, Institute of Control Sciences Lecture Notes, Moscow, USSR, 1979; Presented at the IIASA Workshop on Generalized Lagrangians and Their Applications, IIASA, Laxenburg, Austria, 1979

[21] Rockafellar, R. T. and Wets, R. J.-B., *Variational Analysis*, Springer, Berlin, 1998

[22] Sun, X.-K., Peng, Z.-Y. and Guo, X.-L., *Some characterizations of robust optimal solutionsfor uncertain convex optimization problems*, Optim. Lett., **10** (2016), No. 7, 1463–1478

[23] Sun,X.-K., Long, X.-K., Fu, H.-Y. and Li, X.-B., *Some characterizations of robust optimal solutions for uncertain fractional optimization and applications*, J. Ind. Manag. Optim., **13** (2017), No. 2, 803–824

[24] Sun, X.-K., Teo, K. L. and Tang, L., *Dual approaches to characterize robust optimal solution sets for a class of uncertain optimization problems*, J. Optim. Theory Appl., **182** (2019), No. 3, 984–1000

[25] Zheng, X. Y. and Ng, K. F., *Strong KKT conditions and weak sharp minima in convex-composite optimization*, Math. Program., **126** (2009), 259–279

[26] Zheng, X. Y. and Yang, X. Q., *Weak sharp minima for semi-infinite optimization problems with applications*, SIAM J. Optim., **18** (2004), 573–588

[27] Zhu, S. K., *Weak sharp efficiency in multiobjective optimization*, Optim. Lett., **10** (2016), 1287–1301

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