CARPATHIAN J. MATH. **36** (2020), No. 3, 453 - 462
Online version at https://www.carpathian.cunbm.utcluj.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Existence of solutions for a fractional nonlocal boundary value problem

RODICA LUCA

ABSTRACT. We investigate the existence of solutions for a Riemann-Liouville fractional differential equation with a nonlinearity dependent of fractional integrals, subject to nonlocal boundary conditions which contain various fractional derivatives and Riemann-Stieltjes integrals. In the proof of our main results we use different fixed point theorems.

1. INTRODUCTION

We consider the nonlinear fractional differential equation

$$(E) \qquad D_{0+}^{\alpha} u(t) + f(t, u(t), I_{0+}^{p} u(t))) = 0, \ \ t \in (0,1),$$

with the nonlocal boundary conditions

$$(BC) \qquad u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ D_{0+}^{\beta_0} u(1) = \sum_{i=1}^{m} \int_0^1 D_{0+}^{\beta_i} u(t)\, dH_i(t),$$

where $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, n, $m \in \mathbb{N}$, $n \geq 2$, $\beta_i \in \mathbb{R}$ for all $i = 0, \ldots, m$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_m < \alpha - 1$, $\beta_0 \in [0, \alpha - 1)$, D_{0+}^{k} denotes the Riemann-Liouville derivative of order k (for $k = \alpha, \beta_0, \beta_1, \ldots, \beta_m$), $p > 0$, I_{0+}^{p} is the Riemann-Liouville integral of order p, f is a nonlinear function, and the integrals from the boundary condition (BC) are Riemann-Stieltjes integrals with H_i, $i = 1, \ldots, m$ functions of bounded variation.

We present conditions for the nonlinearity f such that problem $(E) - (BC)$ has at least one solution. The existence of multiple positive solutions for problem $(E) - (BC)$ with $n \geq 3$, $\beta_0 \geq \beta_m$ and the function f does not depend on the fractional integral $I_{0+}^{\alpha} u$, but it may change sign and be singular in the points $t = 0,\ 1$ and/or in the space variable u, was investigated in the paper [1]. In the proof of the main results of [1], the authors used various height functions of the nonlinearity of equation defined on special bounded sets, some properties of the corresponding Green functions, and two theorems from the fixed point index theory. The equation (E) with $f = f(t, u)$, and with a positive parameter λ supplemented with the boundary conditions

$$(BC_1) \qquad u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ D_{0+}^{p} u(1) = \sum_{i=1}^{m} a_i D_{0+}^{q} u(\xi_i),$$

where $\xi_i \in \mathbb{R}$, $i = 1, \ldots, m$, $0 < \xi_1 < \cdots < \xi_m < 1$, p, $q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0, p]$, has been studied in [9]. In paper [9], the nonlinearity f changes sign and it is singular only in $t = 0, 1$, and there the authors used the Guo-Krasnosel'skii fixed point theorem to prove the existence of positive solutions when the parameter belongs to various intervals. We also mention the monograph [8] and the papers [2], [3], [4], [5], [6], [7], [10], [11], [12], [14],

Received: 19.04.2019. In revised form: 31.01.2020. Accepted: 07.02.2020
2010 *Mathematics Subject Classification.* 34A08, 45G15.
Key words and phrases. *Riemann-Liouville fractional differential equation, fractional integrals, nonlocal boundary conditions, existence of solutions.*

[15], [16], [17], [18], [19], [20], [21] for some recent results on the existence, nonexistence and multiplicity of solutions for fractional differential equations and systems of fractional differential equations subject to various boundary conditions.

2. PRELIMINARY RESULTS

First we consider the fractional differential equation

$$D_{0+}^{\alpha}u(t)+y(t)=0,\quad t\in(0,1), \tag{2.1}$$

with the boundary conditions (BC), where $y\in C(0,1)\cap L^1(0,1)$. We denote by

$$\Delta=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_0)}-\sum_{i=1}^{m}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)}\int_0^1 s^{\alpha-\beta_i-1}\,dH_i(s).$$

Lemma 2.1. ([1]) *If $\Delta\neq 0$, then the unique solution $u\in C[0,1]$ of problem (2.1)-(BC) is given by*

$$\begin{aligned}u(t)=&-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-\beta_0)}\int_0^1(1-s)^{\alpha-\beta_0-1}y(s)\,ds\\&-\frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\beta_i)}\int_0^1\left(\int_0^s(s-\tau)^{\alpha-\beta_i-1}y(\tau)\,d\tau\right)dH_i(s),\quad t\in[0,1].\end{aligned} \tag{2.2}$$

Lemma 2.2. ([3]) *If $x\in C[0,1]$ then for $\theta>0$ we have*

$$|I_{0+}^{\theta}x(t)|\le\frac{\|x\|}{\Gamma(\theta+1)},\quad\forall\,t\in[0,1],$$

where $\|x\|=\sup_{t\in[0,1]}|x(t)|$.

We consider the Banach space $X=C([0,1])$ with the supremum norm $\|\cdot\|$, and define the operator $A:X\to X$ by

$$\begin{aligned}(Au)(t)=&-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s),I_{0+}^{p}u(s))\,ds\\&+\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-\beta_0)}\int_0^1(1-s)^{\alpha-\beta_0-1}f(s,u(s),I_{0+}^{p}u(s))\,ds\\&-\frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\beta_i)}\int_0^1\left(\int_0^s(s-\tau)^{\alpha-\beta_i-1}f(\tau,u(\tau),I_{0+}^{p}u(\tau))\,d\tau\right)dH_i(s),\quad t\in[0,1].\end{aligned} \tag{2.3}$$

By using Lemma 2.1, we see that u is a solution of problem $(E)-(BC)$ if and only if u is a fixed point of operator A. Therefore, next we will investigate the existence of fixed points of operator A.

3. EXISTENCE OF SOLUTIONS FOR PROBLEM $(E)-(BC)$

We introduce firstly the assumptions that we will use in our main existence theorems for problem $(E)-(BC)$.

- $(H1)$ $\alpha\in\mathbb{R}$, $\alpha\in(n-1,n]$, n, $m\in\mathbb{N}$, $n\ge 2$, $\beta_i\in\mathbb{R}$ for all $i=0,\dots,m$, $0\le\beta_1<\beta_2<\dots<\beta_m<\alpha-1$, $\beta_0\in[0,\alpha-1)$, $p>0$, $H_i:[0,1]\to\mathbb{R}$, $i=1,\dots,m$ are functions of bounded variation, and $\Delta=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_0)}-\sum_{i=1}^{m}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)}\int_0^1 s^{\alpha-\beta_i-1}\,dH_i(s)\neq 0$.
- $(H2)$ The function $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous and there exists $L_1>0$ such that $|f(t,x,y)-f(t,x_1,y_1)|\le L_1(|x-x_1|+|y-y_1|)$, for all $t\in[0,1]$, $x,y,x_1,y_1\in\mathbb{R}$.
- $(H3)$ There exists a function $g\in C([0,1],[0,\infty))$ such that $|f(t,x,y)|\le g(t)$, for all $(t,x,y)\in[0,1]\times\mathbb{R}^2$.

$(H4)$ The function $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous and there exist real constants $a_0>0$, $a_1\geq 0$, $a_2\geq 0$ such that
$|f(t,x,y)|\leq a_0+a_1|x|+a_2|y|$, for all $t\in[0,1]$, $x,y\in\mathbb{R}$.

$(H5)$ The function $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous and there exist the constants $b_0,\, b_1,\, b_2\geq 0$ with at least one nonzero, and $l_1,\, l_2\in(0,1)$ such that
$|f(t,x,y)|\leq b_0+b_1|x|^{l_1}+b_2|y|^{l_2}$, for all $t\in[0,1]$, $x,\, y\in\mathbb{R}$.

$(H6)$ The function $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous and there exist $c_0,\, c_1,\, c_2\geq 0$ with at least one nonzero, and nondecreasing functions $h_1,\, h_2\in C([0,\infty),[0,\infty))$ such that
$|f(t,x,y)|\leq c_0+c_1h_1(|x|)+c_2h_2(|y|)$, for all $t\in[0,1]$, $x,\, y\in\mathbb{R}$.

We denote by

$$\begin{aligned}
&M_1=\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|\Gamma(\alpha-\beta_0+1)}+\frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\beta_i+1)}\left|\int_0^1 s^{\alpha-\beta_i}dH_i(s)\right|,\\
&M_2=M_1-\frac{1}{\Gamma(\alpha+1)},\quad L_0=1+\frac{1}{\Gamma(p+1)}.
\end{aligned}\tag{3.4}$$

Theorem 3.1. *Assume that $(H1)$ and $(H2)$ hold. If $\Xi:=L_1L_0M_1<1$, then problem $(E)-(BC)$ has a unique solution on $[0,1]$, where L_0 and M_1 are given by (3.4).*

Proof. Let us fix $r>0$ such that $r\geq M_0M_1(1-L_1L_0M_1)^{-1}$, where $M_0=\sup_{t\in[0,1]}|f(t,0,0)|$. We consider the set $\overline{B}_r=\{u\in X,\ \|u\|_X\leq r\}$ and we show firstly that $A\overline{B}_r\subset\overline{B}_r$. Let $u\in\overline{B}_r$. By using $(H2)$ and Lemma 2.2, for $f(t,u(t),I_{0+}^pu(t))$ we obtain the following inequalities

$$\begin{aligned}
&|f(t,u(t),I_{0+}^pu(t))|\leq|f(t,u(t),I_{0+}^pu(t))-f(t,0,0)|+|f(t,0,0)|\\
&\leq L_1(|u(t)|+|I_{0+}^pu(t)|)+M_0\leq L_1\left(\|u\|+\frac{1}{\Gamma(p+1)}\|u\|\right)+M_0\\
&=L_1\left(1+\frac{1}{\Gamma(p+1)}\right)\|u\|+M_0\leq L_1L_0r+M_0,\ \ \forall\, t\in[0,1].
\end{aligned}$$

Then by the definition of operator A from (2.3), we deduce

$$\begin{aligned}
&|(Au)(t)|\leq\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(L_1L_0r+M_0)\,ds\\
&\ +\frac{t^{\alpha-1}}{|\Delta|\Gamma(\alpha-\beta_0)}\int_0^1(1-s)^{\alpha-\beta_0-1}(L_1L_0r+M_0)\,ds\\
&\ +\frac{t^{\alpha-1}}{|\Delta|}\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\beta_i)}\left|\int_0^1\left(\int_0^s(s-\tau)^{\alpha-\beta_i-1}(L_1L_0r+M_0)\,d\tau\right)dH_i(s)\right|\\
&=(L_1L_0r+M_0)\left\{\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha-1}}{|\Delta|\Gamma(\alpha-\beta_0+1)}\right.\\
&\ \left.+\frac{t^{\alpha-1}}{|\Delta|}\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\beta_i+1)}\left|\int_0^1 s^{\alpha-\beta_i}\,dH_i(s)\right|\right\},\ \ \forall\, t\in[0,1].
\end{aligned}$$

Therefore we conclude

$$\begin{aligned}
&\|Au\|\leq(L_1L_0r+M_0)\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|\Gamma(\alpha-\beta_0+1)}\right.\\
&\ \left.+\frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\beta_i+1)}\left|\int_0^1 s^{\alpha-\beta_i}\,dH_i(s)\right|\right]=(L_1L_0r+M_0)M_1\leq r.
\end{aligned}$$

So, we deduce that A maps $\overline{B}_r$ into itself.

Now for $u,\, v \in \overline{B}_r$ we have

$$
\begin{aligned}
|(Au)(t)-(Av)(t)| &\le \left| -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),I_{0+}^p u(s))\,ds \right.\\
&\quad \left. +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,v(s),I_{0+}^p v(s))\,ds \right|\\
&\quad +\frac{t^{\alpha-1}}{|\Delta|\Gamma(\alpha-\beta_0)}\int_0^1 (1-s)^{\alpha-\beta_0-1}|f(s,u(s),I_{0+}^p u(s))-f(s,v(s),I_{0+}^p v(s))|\,ds\\
&\quad +\frac{t^{\alpha-1}}{|\Delta|}\sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)}\left|\int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1}|f(\tau,u(\tau),I_{0+}^p u(\tau)) \right.\right.\\
&\quad \left.\left. -f(\tau,v(\tau),I_{0+}^p v(\tau))|\,d\tau\right) dH_i(s)\right|\\
&\le \frac{L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[|u(s)-v(s)|+|I_{0+}^p u(s)-I_{0+}^p v(s)|]\,ds\\
&\quad +\frac{t^{\alpha-1}L_1}{|\Delta|\Gamma(\alpha-\beta_0)}\int_0^1 (1-s)^{\alpha-\beta_0-1}[|u(s)-v(s)|+|I_{0+}^p u(s)-I_{0+}^p v(s)|]\,ds\\
&\quad +\frac{t^{\alpha-1}L_1}{|\Delta|}\sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)}\left|\int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1}\,[|u(\tau)-v(\tau)| \right.\right.\\
&\quad \left.\left. +|I_{0+}^p u(\tau)-I_{0+}^p v(\tau)|]\ d\tau\right) dH_i(s)\right|\\
&\le \frac{L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\|u-v\|+\frac{1}{\Gamma(p+1)}\|u-v\|\right]\,ds\\
&\quad +\frac{t^{\alpha-1}L_1}{|\Delta|\Gamma(\alpha-\beta_0)}\int_0^1 (1-s)^{\alpha-\beta_0-1}\left[\|u-v\|+\frac{1}{\Gamma(p+1)}\|u-v\|\right]\,ds\\
&\quad +\frac{t^{\alpha-1}L_1}{|\Delta|}\sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)}\left|\int_0^1\left(\int_0^s (s-\tau)^{\alpha-\beta_i-1}\left[\|u-v\|+\frac{1}{\Gamma(p+1)}\|u-v\|\right]d\tau\right)dH_i(s)\right|\\
&= L_1L_0\|u-v\|\left[\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha-1}}{|\Delta|\Gamma(\alpha-\beta_0+1)}\right.\\
&\quad \left. +\frac{t^{\alpha-1}}{|\Delta|}\sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i+1)}\left|\int_0^1 s^{\alpha-\beta_i}dH_i(s)\right|\right],\ \ \forall\, t\in[0,1].
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
\|Au-Av\| \le L_1L_0\|u-v\|\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|\Gamma(\alpha-\beta_0+1)}\right.\\
\left. +\frac{1}{|\Delta|}\sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i+1)}\left|\int_0^1 s^{\alpha-\beta_i}dH_i(s)\right|\right) = \Xi\|u-v\|.
\end{aligned}
$$

By using the condition $\Xi<1$, we deduce that operator A is a contraction. Then by the Banach contraction mapping principle, we conclude that operator A has a unique fixed point $u\in\overline{B}_r$, which is the unique solution of problem $(E)-(BC)$ on $[0,1]$. □

Theorem 3.2. *Assume that* $(H1)$, $(H2)$ *and* $(H3)$ *hold. If* $\Xi_1 := L_1L_0\frac{1}{\Gamma(\alpha+1)} < 1$, *then problem* $(E)-(BC)$ *has at least one solution on* $[0,1]$.

Proof. Let us fix $r_1 > 0$ such that $r_1 \geq M_1\|g\|$. We consider the set $\overline{B}_{r_1} = \{u \in X,\ \|u\| \leq r_1\}$, and we introduce the operators $A_1,\ A_2 : \overline{B}_{r_1} \to X$ defined by

$$(3.5)\qquad \begin{aligned} (A_1u)(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),I_{0+}^p u(s))\,ds,\ \ t\in[0,1],\\ (A_2u)(t) &= \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-\beta_0)}\int_0^1 (1-s)^{\alpha-\beta_0-1} f(s,u(s),I_{0+}^p u(s))\,ds\\ &\quad -\frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)}\int_0^1\left(\int_0^s (s-\tau)^{\alpha-\beta_i-1} f(\tau,u(\tau),I_{0+}^p u(\tau))\,d\tau\right) dH_i(s), \end{aligned}$$

for all $t \in [0,1]$ and $u \in \overline{B}_{r_1}$.

By using $(H3)$, we obtain for all $u,\ v \in \overline{B}_{r_1}$

$$\begin{aligned} &\|A_1u + A_2v\| \leq \|A_1u\| + \|A_2v\| \leq \frac{1}{\Gamma(\alpha+1)}\|g\|\\ &+\left(\frac{1}{|\Delta|\Gamma(\alpha-\beta_0+1)} + \frac{1}{|\Delta|}\sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i+1)}\left|\int_0^1 s^{\alpha-\beta_i}dH_i(s)\right|\right)\|g\| = M_1\|g\| \leq r_1. \end{aligned}$$

Hence $A_1u + A_2v \in \overline{B}_{r_1}$ for all $u,\ v \in \overline{B}_{r_1}$.

The operator A_1 is a contraction, because

$$\|A_1u - A_1v\| \leq L_1L_0\frac{1}{\Gamma(\alpha+1)}\|u-v\| = \Xi_1\|u-v\|,\ \ \forall\, u,\ v \in \overline{B}_{r_1},$$

and $\Xi_1 < 1$.

The continuity of f implies that the operator A_2 is continuous on $\overline{B}_{r_1}$. We prove next that A_2 is compact. The operator A_2 is uniformly bounded on $\overline{B}_{r_1}$, because

$$\|A_2u\| \leq \left(\frac{1}{|\Delta|\Gamma(\alpha-\beta_0+1)} + \frac{1}{|\Delta|}\sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i+1)}\left|\int_0^1 s^{\alpha-\beta_i}dH_i(s)\right|\right)\|g\| = M_2\|g\|,$$

for all $u \in \overline{B}_{r_1}$. Now we prove that A_2 is equicontinuous on $\overline{B}_{r_1}$. We denote by

$$(3.6)\qquad \Lambda_{r_1} = \sup\left\{|f(t,x,y)|,\ \ t\in[0,1],\ |x|\leq r_1,\ |y| \leq \frac{1}{\Gamma(p+1)}r_1\right\}.$$

Then for $u \in \overline{B}_{r_1}$ and $t_1,\ t_2 \in [0,1]$ with $t_1 < t_2$, we obtain

$$\begin{aligned} &|(A_2u)(t_2) - (A_2u)(t_1)| \leq \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})}{|\Delta|\Gamma(\alpha-\beta_0)}\int_0^1 (1-s)^{\alpha-\beta_0-1}\Lambda_{r_1}\,ds\\ &\quad+\frac{(t_2^{\alpha-1} - t_1^{\alpha-1})}{|\Delta|}\sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)}\left|\int_0^1\left(\int_0^s (s-\tau)^{\alpha-\beta_i-1}\Lambda_{r_1}\,d\tau\right)dH_i(s)\right|\\ &\leq \Lambda_{r_1}(t_2^{\alpha-1} - t_1^{\alpha-1})\left[\frac{1}{|\Delta|\Gamma(\alpha-\beta_0+1)} + \frac{1}{|\Delta|}\sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i+1)}\left|\int_0^1 s^{\alpha-\beta_i}dH_i(s)\right|\right]\\ &= \Lambda_{r_1}M_2(t_2^{\alpha-1} - t_1^{\alpha-1}). \end{aligned}$$

Therefore we conclude

$$|(A_2u)(t_2) - (A_2u)(t_1)| \to 0,\ \text{ as }\ t_2 \to t_1,\ \text{ uniformly with respect to }\ u \in \overline{B}_{r_1}.$$

We deduce that A_2 is equicontinuous on $\overline{B}_{r_1}$, and so, by using Arzela-Ascoli theorem, the set $A_2(\overline{B}_{r_1})$ is relatively compact. We conclude that operator A_2 is compact on $\overline{B}_{r_1}$. Then by the Krasnosel'skii fixed point theorem for the sum of two operators (see [13]) we deduce that there exists a fixed point of operator $A_1 + A_2$, which is a solution of the boundary value problem $(E) - (BC)$ on $[0,1]$. □

Theorem 3.3. *Assume that* $(H1)$, $(H2)$ *and* $(H3)$ *hold. If* $\Xi_2 := L_1L_0M_2 < 1$, *then problem* $(E)-(BC)$ *has at least one solution on* $[0,1]$.

Proof. We consider again a positive number $r_1 \geq M_1\|g\|$ and the operators $A_1,\ A_2$ defined on $\overline{B}_{r_1}$ given by (3.5). In a similar manner as in the proof of Theorem 3.2, we obtain that $A_1u + A_2v \in \overline{B}_{r_1}$ for all $u,\ v \in \overline{B}_{r_1}$.

The operator A_2 is a contraction because

$$\|A_2u - A_2v\| \leq L_1L_0\left(\frac{1}{|\Delta|\Gamma(\alpha-\beta_0+1)} + \frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\beta_i+1)}\left|\int_0^1 s^{\alpha-\beta_i}\,dH_i(s)\right|\right)$$
$$\times\|u-v\| = L_1L_0M_2\|u-v\| = \Xi_2\|u-v\|,\ \ \forall\, u,\ v \in \overline{B}_{r_1},$$

with $\Xi_2 < 1$.

Then the continuity of f implies that the operator A_1 is continuous on $\overline{B}_{r_1}$. We prove now that A_1 is compact. The operator A_1 is uniformly bounded on $\overline{B}_{r_1}$ because

$$\|A_1u\| \leq \frac{1}{\Gamma(\alpha+1)}\|g\|,\ \ \forall\, u \in \overline{B}_{r_1}.$$

Now we show that A_1 is equicontinuous on $\overline{B}_{r_1}$. By using Λ_{r_1} (defined in the proof of Theorem 3.2), we obtain for $u \in \overline{B}_{r_1}$ and $t_1,\ t_2 \in [0,1]$ with $t_1 < t_2$

$$\begin{aligned}
&|(A_1u)(t_2) - (A_1u)(t_1)| = \left|-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,u(s),I_{0+}^p u(s))\,ds\right.\\
&\left.+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,u(s),I_{0+}^p u(s))\,ds\right|\\
&=\left|-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]f(s,u(s),I_{0+}^p u(s))\,ds\right.\\
&\left.-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,u(s),I_{0+}^p u(s))\,ds\right|\\
&\leq \frac{\Lambda_{r_1}}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\,ds + \frac{\Lambda_{r_1}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds\\
&=\frac{\Lambda_{r_1}}{\Gamma(\alpha+1)}[-(t_2-t_1)^{\alpha}+t_2^{\alpha}-t_1^{\alpha}] + \frac{\Lambda_{r_1}}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}\\
&\leq \frac{\Lambda_{r_1}}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha}).
\end{aligned}$$

Then we deduce

$$|(A_1u)(t_2) - (A_1u)(t_1)| \to 0,\ \text{ as }\ t_2 \to t_1,\ \text{ uniformly with respect to }\ u \in \overline{B}_{r_1}.$$

We conclude that A_1 is equicontinuous on $\overline{B}_{r_1}$, and by using Arzela-Ascoli theorem, the set $A_1(\overline{B}_{r_1})$ is relatively compact. We deduce that operator A_1 is compact on $\overline{B}_{r_1}$. By the Krasnosel'skii fixed point theorem for the sum of two operators, we obtain that there exists a fixed point of operator $A_1 + A_2$, which is a solution of the boundary value problem $(E)-(BC)$ on $[0,1]$. □

Theorem 3.4. *Assume that* $(H1)$ *and* $(H4)$ *hold. If* $\Xi_3 := M_1\left(a_1 + \frac{a_2}{\Gamma(p+1)}\right) < 1$, *then the boundary value problem* $(E)-(BC)$ *has at least one solution on* $[0,1]$.

Proof. We consider the operator $A : X \to X$ defined in (2.3). We firstly prove that A is completely continuous. By the continuity of f we deduce that A is a continuous operator.

We show next that A is a compact operator. Let $\Omega \subset X$ be a bounded set. Then there exist a positive constant L_2 such that

$$|f(t,u(t),I_{0+}^p u(t))| \leq L_2,\ \ \forall\, t \in [0,1],\ \ u \in \Omega.$$

Therefore we obtain as in the proof of Theorem 3.1 that $|(Au)(t)| \le L_2 M_1$, for all $t \in [0,1]$ and $u \in \Omega$. So, $A(\Omega)$ is uniformly bounded.

We will show next that $A(\Omega)$ is equicontinuous. Let $u \in \Omega$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then by using the operators A_1 and A_2 defined on Ω (given by (3.5)), and based on a similar approach as that used in the proof of Theorem 3.2, we obtain

$$\begin{aligned}
&|(Au)(t_2) - (Au)(t_1)| \le |(A_1u)(t_2) - (A_1u)(t_1)| + |(A_2u)(t_2) - (A_1u)(t_1)| \\
&\le \frac{L_2}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha) + L_2 M_2 (t_2^{\alpha-1} - t_1^{\alpha-1}).
\end{aligned}$$

Then $|(Au)(t_2) - (Au)(t_1)| \to 0$ as $t_2 \to t_1$ uniformly with respect to $u \in \Omega$. Thus $A(\Omega)$ is equicontinuous. By Arzela-Ascoli theorem, we deduce that $A(\Omega)$ is relatively compact, and so A is compact. Therefore A is completely continuous.

Now we will prove that the set $F = \{u \in X,\ \ u = \nu A(u),\ \ 0 < \nu < 1\}$ is bounded. Let $u \in F$, that is $u = \nu A(u)$ for some $\nu \in (0,1)$. Then we have

$$|u(t)| = |\nu (Au)(t)| \le |(Au)(t)|, \ \ \forall\, t \in [0,1].$$

By (H4) we obtain

$$\begin{aligned}
|u(t)| &\le |(Au)(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [a_0 + a_1|u(s)| + a_2 |I_{0+}^p u(s)|]\, ds \\
&+ \frac{t^{\alpha-1}}{|\Delta|\Gamma(\alpha-\beta_0)} \int_0^1 (t-s)^{\alpha-\beta_0-1} [a_0 + a_1|u(s)| + a_2|I_{0+}^p u(s)|]\, ds \\
&+ \frac{t^{\alpha-1}}{|\Delta|} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1} [a_0 + a_1|u(\tau)| + a_2|I_{0+}^p u(\tau)|]\, d\tau \right) dH_i(s) \right| \\
&\le \left(a_0 + a_1\|u\| + \frac{a_2}{\Gamma(p+1)}\|u\| \right) \left[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha-1}}{|\Delta|\Gamma(\alpha-\beta_0+1)} \right. \\
&\left. + \frac{t^{\alpha-1}}{|\Delta|} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i+1)} \left| \int_0^1 s^{\alpha-\beta_i} dH_i(s) \right| \right].
\end{aligned}$$

Therefore we deduce

$$\|u\| \le M_1 \left(a_0 + a_1\|u\| + \frac{a_2}{\Gamma(p+1)}\|u\| \right).$$

Because $\Xi_3 < 1$, we obtain

$$\|u\| \le M_1 a_0 \left(1 - M_1 a_1 - \frac{M_1 a_2}{\Gamma(p+1)} \right)^{-1}.$$

Hence we deduce that the set F is bounded.

By using the Leray-Schauder alternative, we conclude that the operator A las at least one fixed point, which is a solution for our problem $(E) - (BC)$. □

Theorem 3.5. *Assume that $(H1)$ and $(H5)$ hold. Then problem $(E) - (BC)$ has at least one solution.*

Proof. Let $\overline{B}_R = \{u \in X,\ \ \|u\| \le R\}$, where

$$R \ge \max \left\{ 3b_0 M_1, (3b_1 M_1)^{\frac{1}{1-l_1}}, \left(\frac{3b_2 M_1}{(\Gamma(p+1))^{l_2}} \right)^{\frac{1}{1-l_2}} \right\}.$$

We prove now that $A : \overline{B}_R \to \overline{B}_R$. For $u \in \overline{B}_R$ we deduce

$$|(Au)(t)| \le \left(b_0 + b_1 R^{l_1} + \frac{b_2}{(\Gamma(p+1))^{l_2}} R^{l_2} \right) M_1 \le \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R, \ \ \forall\, t \in [0,1],$$

and then $\|Au\| \leq R$, which implies that $A(\overline{B}_R) \subset \overline{B}_R$.

From the continuity of the function f we can easily show that the operator A is continuous. The functions from $A(\overline{B}_R)$ are uniformly bounded and equicontinuous. Indeed, by using the notation (3.6), with r_1 replaced by R, we obtain for any $u \in \overline{B}_R$ and $t_1,\, t_2 \in [0,1],\ t_1 < t_2$ that

$$|(Au)(t_2) - (Au)(t_1)| \leq \frac{\Lambda_R}{\Gamma(\alpha+1)}(t_2^{\alpha} - t_1^{\alpha}) + \Lambda_R M_2 (t_2^{\alpha-1} - t_1^{\alpha-1}).$$

Therefore $|(Au)(t_2) - (Au)(t_1)| \to 0$ as $t_2 \to t_1$ uniformly with respect to $u \in \overline{B}_R$. By Arzela-Ascoli theorem, we conclude that $A(\overline{B}_R)$ is relatively compact, and then A is a completely continuous operator. By the Schauder fixed point theorem, we deduce that operator A has at least one fixed point u in $\overline{B}_R$ which is a solution of our problem $(E)-(BC)$. □

Theorem 3.6. *Assume that* $(H1)$ *and* $(H6)$ *hold. If there exists* $\Xi_0 > 0$ *such that*

$$\left(c_0 + c_1 h_1(\Xi_0) + c_2 h_2\left(\frac{\Xi_0}{\Gamma(p+1)}\right)\right) M_1 < \Xi_0, \tag{3.7}$$

where $c_0,\, c_1,\, c_2,\, h_1,\, h_2$ *are given in* $(H6)$*, then problem* $(E)-(BC)$ *has at least one solution on* $[0,1]$.

Proof. We consider the set $\overline{B}_{\Xi_0} = \{u \in X,\ \|u\| \leq \Xi_0\}$, where Ξ_0 is given in the assumptions of the theorem. We will show that $A : \overline{B}_{\Xi_0} \to \overline{B}_{\Xi_0}$. For $u \in \overline{B}_{\Xi_0}$ and $t \in [0,1]$, we obtain

$$|(Au)(t)| \leq \left(c_0 + c_1 h_1(\Xi_0) + c_2 h_2\left(\frac{\Xi_0}{\Gamma(p+1)}\right)\right) M_1 < \Xi_0.$$

Then $A(\overline{B}_{\Xi_0}) \subset \overline{B}_{\Xi_0}$. In a similar manner as in the proof of Theorem 3.5 we can show that operator A is completely continuous.

We suppose now that there exists $u \in \partial B_{\Xi_0}$ such that $u = \nu A(u)$ for some $\nu \in (0,1)$. We obtain as above that $\|u\| \leq \|Au\| < \Xi_0$, which is a contradiction, because $u \in \partial B_{\Xi_0}$. Then by the nonlinear alternative of Leray-Schauder type, we conclude that operator A has a fixed point $u \in \overline{B}_{\Xi_0}$, and so problem $(E)-(BC)$ has at least one solution. □

4. Examples

Let $\alpha = \frac{5}{2}$ $(n=3)$, $p = \frac{10}{3}$, $m=2$, $\beta_0 = \frac{6}{5}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{3}{4}$, $H_1(t) = t^2$ for all $t \in [0,1]$, $H_2(t) = \{0,\ \text{if } t \in [0,1/2);\ 3,\ \text{if } t \in [1/2,1]\}$.

We consider the fractional differential equation

$$D_{0+}^{5/2} u(t) + f(t, u(t), I_{0+}^{10/3} u(t)) = 0,\ \ 0 < t < 1, \tag{E_0}$$

with the boundary conditions

$$u(0) = u'(0) = 0,\ \ D_{0+}^{6/5} u(1) = 2\int_0^1 t D_{0+}^{1/3} u(t)\, dt + 3 D_{0+}^{3/4} u\left(\frac{1}{2}\right). \tag{BC_0}$$

We obtain here $\Delta \approx -1.87462428 \neq 0$, $L_0 \approx 1.1079852$, $M_1 \approx 1.16312084$ and $M_2 \approx 0.86221973$. So assumption $(H1)$ is satisfied.

Example 4.1. We consider the function

$$f(t,x,y) = \frac{|x|}{2(t+1)^2(1+|x|)} - \frac{t}{4}\arctan y - \frac{3t}{t^2+4},\ \ t \in [0,1],\ \ x,\, y \in \mathbb{R}.$$

Here we have $L_1 = \frac{1}{2}$, and then $\Xi \approx 0.64436034 < 1$. Therefore assumption $(H2)$ is satisfied, and by Theorem 3.1 we deduce that problem $(E_0)-(BC_0)$ has a unique solution $u(t),\ t \in [0,1]$.

Example 4.2. We consider the function

$$f(t,x,y) = \frac{1}{\sqrt{4+t^2}}\sin t + \frac{|x|}{3(2+|x|)} - \frac{1}{2(1+t)}\sin^2 y, \ \ t \in [0,1], \ \ x,\, y \in \mathbb{R}.$$

In this case we have $L_1 = 1$ and $|f(t,x,y)| \le g(t)$ for all $t \in [0,1]$ and $x,\, y \in \mathbb{R}$, where $g(t) = \frac{|\sin t|}{\sqrt{4+t^2}} + \frac{1}{3} + \frac{1}{2(1+t)}$, for all $t \in [0,1]$. So assumptions $(H2)$ and $(H3)$ are satisfied, and in addition we obtain $\Xi_1 \approx 0.33339398 < 1$. Then by Theorem 3.2 we conclude that problem $(E_0)-(BC_0)$ has at least one solution on $[0,1]$.

Example 4.3. We consider the function

$$f(t,x,y) = \frac{t}{t^2+1}\left(4\cos t + \frac{1}{2}\sin x\right) - \frac{1}{(t+1)^3}y, \ \ \forall\, t \in [0,1], \ \ x,\, y \in \mathbb{R}.$$

Because we have $|f(t,x,y)| \le 2+\frac{1}{4}|x|+|y|$, for all $t \in [0,1], \ \ x,\, y \in \mathbb{R}$, the assumption $(H4)$ is satisfied with $a_0 = 2$, $a_1 = \frac{1}{4}$ and $a_2 = 1$. In addition we obtain $\Xi_3 \approx 0.41638005 < 1$, and then by Theorem 3.4 we deduce that problem $(E_0)-(BC_0)$ has at least one solution on $[0,1]$.

Example 4.4. We consider the function

$$f(t,x,y) = \frac{e^{-t}}{1+t^3} - \frac{1}{3}x^{2/3} + \frac{1}{4(3+t)}\arctan y^{1/5}, \ \ t \in [0,1], \ \ x,\, y \in \mathbb{R}.$$

Because we obtain $|f(t,x,y)| \le 1 + \frac{1}{3}|x|^{2/3} + \frac{1}{12}|y|^{1/5}$, for all $t \in [0,1], \ \ x,\, y \in \mathbb{R}$, then assumption $(H5)$ is satisfied with $b_0 = 1$, $b_1 = \frac{1}{3}$, $b_2 = \frac{1}{12}$, $l_1 = \frac{2}{3}$, $l_2 = \frac{1}{5}$. Then by Theorem 3.5 we deduce that problem $(E_0)-(BC_0)$ has at least one solution on $[0,1]$.

Example 4.5. We consider the function

$$f(t,x,y) = \frac{(1-t)^2}{10} + \frac{(1-t)x^2}{15(1+x^2)} - \frac{t^4y^3}{5}, \ \ \forall\, t \in [0,1], \ x,\, y \in \mathbb{R}.$$

Because we have $|f(t,x,y)| \le \frac{1}{10} + \frac{1}{15}|x|^2 + \frac{1}{5}|y|^3$, for all $t \in [0,1], \ x,\, y \in \mathbb{R}$, then assumption $(H6)$ is satisfied with $h_1(x) = x^2$ and $h_2(x) = x^3$ for $x \in [0,\infty)$, $c_0 = \frac{1}{10}$, $c_1 = \frac{1}{15}$ and $c_2 = \frac{1}{5}$. For $\Xi_0 = 2$ the condition (3.7) is satisfied, because $\left(c_0 + c_1h_1(2) + c_2h_2\left(\frac{2}{\Gamma(p+1)}\right)\right)M_1$ $\approx 0.428821 < 2$. Therefore by Theorem 3.6 we conclude that problem $(E_0)-(BC_0)$ has at least one solution on $[0,1]$.

Acknowledgement. The author thanks the referee for his/her valuable comments and suggestions.

References

[1] Agarwal, R. P. and Luca, R., *Positive solutions for a semipositone singular Riemann-Liouville fractional differential problem*, Inter. J. Nonlinear Sci. Num. Simul., **20** (2019), (7-8), 823–832

[2] Ahmad, B. and Luca, R., *Existence of solutions for a sequential fractional integro-differential system with coupled integral boundary conditions*, Chaos Solitons Fractals, **104** (2017), 378–388

[3] Ahmad, B. and Luca, R., *Existence of solutions for a system of fractional differential equations with coupled nonlocal boundary conditions*, Fract. Calc. Appl. Anal., **21** (2018), No. 2, 423–441

[4] Ahmad, B. and Luca, R., *Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions*, Appl. Math. Comput., **339** (2018), 516–534

[5] Ahmad B. and Ntouyas, S. K., *Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions*, Appl. Math. Comput., **266** (2015), 615–622

[6] Aljoudi, S., Ahmad, B, Nieto, J. J. and Alsaedi, A., *A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions*, Chaos Solitons Fractals, **91** (2016), 39–46

[7] Guo, L., Liu, L. and Wu, Y., *Iterative unique positive solutions for singular p-Laplacian fractional differential equation system with several parameters*, Nonlinear Anal. Model. Control, **23** (2018), No. 2, 182–203

[8] Henderson, J. and Luca, R., *Boundary Value Problems for Systems of Differential, Difference and Fractional Equations. Positive solutions*, Elsevier, Amsterdam, 2016

[9] Henderson, J. and Luca, R., *Existence of positive solutions for a singular fractional boundary value problem*, Nonlinear Anal. Model. Control, **22** (2017), No. 1, 99–114

[10] Henderson, J. and Luca, R., *Systems of Riemann-Liouville fractional equations with multi-point boundary conditions*, Appl. Math. Comput., **309** (2017), 303–323

[11] Henderson, J., Luca, R. and Tudorache, A., *On a system of fractional differential equations with coupled integral boundary conditions*, Fract. Calc. Appl. Anal., **18** (2015), No. 2, 361–386

[12] Henderson, J., Luca, R. and Tudorache, A., *Existence and nonexistence of positive solutions for coupled Riemann-Liouville fractional boundary value problems*, Discrete Dyn. Nature Soc., **2016** Article ID 2823971 (2016), 1–12

[13] Krasnosel'skii, M. A., *Two remarks on the method of successive approximations*, Uspekhi Mat. Nauk. **10** (1955), 123–127

[14] Liu, L., Li, H., Liu, C. and Wu, Y., *Existence and uniqueness of positive solutions for singular fractional differential systems with coupled integral boundary value problems*, J. Nonlinear Sci. Appl., **10** (2017), 243–262

[15] Liu, S., Liu, J., Dai, Q. and Li, H., *Uniqueness results for nonlinear fractional differential equations with infinite-point integral boundary conditions*, J. Nonlinear Sci. Appl., **10** (2017), 1281–1288

[16] Luca, R., *Positive solutions for a system of Riemann-Liouville fractional differential equations with multi-point fractional boundary conditions*, Bound. Value Prob., **2017** (2017), No. 102, 1–35

[17] Pu, R., Zhang, X., Cui, Y., Li, P. and Wang, W., *Positive solutions for singular semipositone fractional differential equation subject to multipoint boundary conditions*, J. Funct. Spaces, **2017**, Article ID 5892616 (2017), 1–7

[18] Shen, C., Zhou, H. and Yang, L., *Positive solution of a system of integral equations with applications to boundary value problems of differential equations*, Adv. Difference Equ., **2016** (2016), No. 260, 1–26

[19] Xu, J. and Wei, Z., *Positive solutions for a class of fractional boundary value problems*, Nonlinear Anal. Model. Control, **21** (2016), 1–17

[20] Zhang, X., *Positive solutions for a class of singular fractional differential equation with infinite-point boundary conditions*, Appl. Math. Lett., **39** (2015), 22–27

[21] Zhang, X. and Zhong, Q., *Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables*, Appl. Math. Lett., **80** (2018), 12–19

DEPARTMENT OF MATHEMATICS
GH. ASACHI TECHNICAL UNIVERSITY
11 BLVD. CAROL I, 700506, IAŞI, ROMANIA
Email address: rluca@math.tuiasi.ro