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Effect of voids in a heat-flux dependent theory for thermoelastic bodies with dipolar structure

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ABSTRACT. In our paper we formulate a theory for thermoelastic porous dipolar bodies in which we consider a new independent variable, namely the heat-flux vector. Furthermore, we add, to the differential equations that describe the behavior of the body, a new differential equation which is an equation of evolution which is satisfied by the components of the heat-flux vector. The basic system of the mixed initial-boundary value problem in this context consists of equations of the hyperbolic type. In order to ensure the consistency of the constructed theory, we formulate and prove an uniqueness result, with regards to the solution of the mixed problem.

1. INTRODUCTION

At the beginning we will say a few words about each of the three effects we consider in our study: heat-flux dependent theory, voids and dipolar structure. First, regarding the heat-flux dependent theory, it is known that in the classical theory of thermoelasticity there is a phenomenon that contradicts with real materials: there is no elastic term in the equation of heat conduction. Also, in the previous theories the partial differential equations are only of the parabolic type. In the present theory there is a hyperbolic equation for the balance of energy. The general purpose of all these new theories is the creation of some models which allow waves to propagate at finite speed. In some studies on these theories, the term "the second-sound" is used. Among the known studies in which the authors intended to avoid the above mentioned paradoxes, we must refer to papers [9]-[11] in which the authors considered three different aspects of thermoelasticiy. In the first of them the considerations are equivalent to those of the theory of classical thermoelasticity. In the context of Green-Naghdi's second theory, waves propagate at finite speed because the authors do not consider the energy dissipation. The theory of type III of Green and Naghdi is in fact a combination of the first two. Specific to this theory is the propagation for the heat waves at finite speed, even if energy dissipation is allowed. In the paper [4] Choudhuri replaced the Fourier law by a complicated energy balance in which is introduced a function for the thermal displacement and is included the thermal conductivity tensor and a tensor of rate of conductivity. In the books [15], [16] and [25] other considerations can be encountered regarding the use of a vector for heat-flux as an internal variable in the context of different generalizations of thermoelasticity.

As for the second effect, the one of the voids, we have to say that the theory of porous bodies is dedicated to the study of behavior of solids with voids in which the interstices are voids of the material and the skeletal is elastic. In first study for this kind of body, [26], Nunziato and Cowin considered the mass density as the product of the density of the matrix material and the volume fraction. This, in fact, means introducing an extra degree of

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freedom to characterize the behavior of bodies with small pores. The applications of this theory can be found in geological materials, like soil and rock or artificial manufactured materials with pores. In papers [26] and [5], it is considered only the isothermal case. Later Iesan in [14] extended the theory in order to cover the thermoelasticity of solids with pores.

Finally, the third effect considered in our study is that of the dipolar structure.

The theories of solids with microstructure, proposed by Eringen (see for instance [6], [7]), also aim to avoid the above paradoxes. After Eringen, these theories received a great consideration, [1]-[3], [13], [18], [19].

The theory of a dipolar medium occupies a special place among the theories dedicated to the microstructure. So, the studies [24] of Mindlin as well as of Green and Rivlin [12] are among the best known. Also, another known researcher is Gurtin that wrote many articles on bodies with dipolar structures. For instance, in the study [8] of Fried and Gurtin it is formulated a balance of energy for an interface regarding a solid and its outside world. Other results regarding the microstructure can be found in [20]-[23], [27], [28].

The observation that interest has increased in recent years for these theories can be explained by the fact that it is possible to use them in the investigation of deformation properties of solids for which the classical theory is inappropriate. As a concrete motivation of increasing interest for bodies with dipolar structure is that this theory is more realistic than the classical elasticity theory in approaching the problems of earth science.

Considering the large number of published works dedicated to the theory of porous dipolar bodies, it can be considered that it is useful for a large number of applications in continuum mechanics.

First, we will state the basic equations and conditions of the mixed initial-boundary value problem for our bodies. The independent variables of these equations and conditions are v_i, ϕ_{ij} and ϑ, that is, the components of displacement, the components of couple displacement and the absolute temperature, respectively. All these are regular functions depending on the variable time t and on the position variable x, of the form $f = f(t, x)$. This thermoelasticity theory is extended by attaching the variable heat-flux among the independent variables. It can be considered that our work is inspired by the study [17] of Lebon. Also, our considerations are a counterpart of the theory of thermoelasticity, proposed by Lord and Shulman in [18]. In the following, we first obtain the basic equations specific to the heat-flux theory of thermoelasticity for a dipolar porous medium, namely the constitutive equations. For this we will use only some thermodynamical considerations. As a second step, we deduce the equations that govern the evolution of our body, and certain restrictions on the constants of the material. To prove the consistency and applicability of our theory we obtain an uniqueness result of solution for our mixed problem, even in the general case of a medium with anisotropy. It is important to notice that the uniqueness theorem, is proven by using some auxiliary inequalities which involve the constitutive variable heat-flux.

2. Basic notions, equations and conditions

We will consider a thermoelastic dipolar body with pores which initially occupies the regular and bounded domain D of the three-dimensional Euclidean space R^3. We will use the notation ∂D for the boundary of D and suppose that it is a piecewise smooth surface so that we can apply the divergence theorem. Points of the domain D are identifiable by the variables x_1, x_2, x_3 and we will denote by x the triplet (x_1, x_2, x_3). The functions used in our study are defined for all points of the cylinder $\bar{D} \times (0, \infty)$, with $\bar{D} = D \cup \partial D$. We will use Einstein convention of summation with regards to the repeated indices. The

Latin indices are taken over the values $1, 2, 3$. To designate the derivative of the function f regarding the variable t we will write $\dot{f} = \partial f/\partial t$, so, a dot over the function. Also, the partial differentiation of a function f regarding a spatial variable x_j is denoted by $f_{,j} = \partial f/\partial x_j$, that is, a comma followed by the respective subscript.
We will define a mathematical model which is based on some linear differential equations for the theory of thermoelasticity for dipolar bodies with pores.
In order to characterize the evolution of this medium will use the variables $(v_i, \phi_{ij}, \vartheta)$, above described. If we denote by ρ the mass density, by γ the matrix density and by ν the matrix volume fraction, then, according to [5], these quantities are related by

$$\rho = \nu\gamma. \tag{2.1}$$

By using the technique of Green and Rivlin we will obtain the local form of the energy laws. According to this technique, over the usual motion, it is superposed another deformation of our solid which has, in addition to the initial motion, a rotation having a rigid body angular velocity. Also, we must suppose that the other characteristic properties are not affected by this overlap.
So, first we can obtain the kinematic (or geometric) relations in which the strain tensors ε_{ij}, γ_{ij} and χ_{ijk} are related to the variables of motion (see Eringen [7]):

$$e_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right), \ \mu_{ij} = v_{j,i} - \phi_{ij}, \ \chi_{ijk} = \phi_{jk,i}. \tag{2.2}$$

The novelty of our model is the introduction of the heat flux among the constitutive variables. The components of this heat flux vector are denoted by q_i relative to the area unit of the initial state of the solid.
As a consequence, since other unknown functions appeared, we need some extra evolutionary equations. According to Lebon [17], we must assume that $\dot{q}_m$ is a function which is depending on the variables e_{ij}, μ_{ij}, χ_{ijk}, ϑ, $\vartheta_{,i}$ and q_i:

$$\dot{q}_m = Q_m\left(e_{ij}, \mu_{ij}, \chi_{ijk}, \vartheta, \vartheta_{,i}, q_i\right), \tag{2.3}$$

which satisfies the following supplementary condition (suggested by Lebon in [17]):

$$\frac{\partial \dot{Q}_m}{\partial q_j} = 0, \ \ \forall m \neq j. \tag{2.4}$$

Let us denote by r the internal strength of the heat source, by φ the variation of the volume fraction with regards to the initial volume distribution, that is, $\varphi = \nu - \nu_0$, by τ_{ij}, σ_{ij} and γ_{ijk} the stress tensors and by h_i the components of the equilibrated stress vector. Then, with the help of the procedure of Green and Rivlin we can obtain the first local balance law of energy, according to which the internal energy E satisfies the following equation:

$$\rho\dot{E} = \rho r + \tau_{ij}\dot{e}_{ij} + \sigma_{ij}\dot{\mu}_{ij} + \gamma_{ijk}\dot{\chi}_{ijk} + h_i\dot{\varphi}_{,i} - g\dot{\varphi} + q_{i,i}. \tag{2.5}$$

Here we denoted by ρ the mass density of the initial state of the solid, and $\rho > 0$.
Now we denote by S the specific entropy, so that by using the second balance law of energy we deduce the Clausius-Duhem inequality (which is also called the entropy production inequality):

$$\rho\dot{S} \geq \rho\frac{r}{\vartheta} + \frac{\vartheta_{,i}}{\vartheta}q_i. \tag{2.6}$$

The internal strength of the heat source r, the specific entropy S and the internal energy E are considered in an arbitrary material element and are referred to the unit mass of the

respective element.
If we substitute r from Eq. (2.5) in inequality (2.6), we are led to the following inequality

$$\rho\dot{E} \leq \rho\vartheta\dot{S} + \tau_{ij}\dot{e}_{ij} + \sigma_{ij}\dot{\mu}_{ij} + \gamma_{ijk}\dot{\chi}_{ijk} + h_i\dot{\varphi}_{,i} - g\dot{\varphi} + \frac{\vartheta_{,i}}{\vartheta}q_i,$$

which can be reformulated as follows:

$$\rho\left(\dot{E} - \vartheta\dot{S}\right) \leq \tau_{ij}\dot{e}_{ij} + \sigma_{ij}\dot{\mu}_{ij} + \gamma_{ijk}\dot{\chi}_{ijk} + h_i\dot{\varphi}_{,i} - g\dot{\varphi} + \frac{\vartheta_{,i}}{\vartheta}q_i. \tag{2.7}$$

From this inequality we are inspired to introduce the free energy function

$$\mathcal{E} = E - \vartheta S, \tag{2.8}$$

which is also called the Helmholtz's function.
Taking into account Eq. (2.8), the entropy production inequality (2.7) receives the following form:

$$-\rho\left(\dot{\mathcal{E}} + S\dot{\vartheta}\right) + \tau_{ij}\dot{e}_{ij} + \sigma_{ij}\dot{\mu}_{ij} + \gamma_{ijk}\dot{\chi}_{ijk} + h_i\dot{\varphi}_{,i} - g\dot{\varphi} + \frac{\vartheta_{,i}}{\vartheta}q_i \geq 0. \tag{2.9}$$

The procedure used by Green and Rivlin in the case of classical elasticity, can be used to obtain the balance laws in the context of thermoelasticty of porous dipolar bodies, namely:
- the motion equations:

$$\left(\tau_{ij} + \sigma_{ij}\right)_{,j} + \rho F_i = \rho\ddot{v}_i, \ \gamma_{ijk,i} + \sigma_{jk} + \rho G_{jk} = I_{js}\ddot{\phi}_{ks}; \tag{2.10}$$

- the balance of equilibrated charge:

$$h_{i,i} + g + \rho L = \rho\kappa\ddot{\varphi}. \tag{2.11}$$

In these equations $F = (F_i)$ is the body force vector per unit mass, $G = (G_{jk})$ is the dipolar body force tensor per unit mass, $I = (I_{jk})$ is of the micro-inertia tensor per unit mass, κ is the equilibrated inertia, g is the intrinsic equilibrated body force, and L is the extrinsic equilibrated body force.

3. Main results

In this section we will deduce the main equations and basic conditions. All our further considerations will be made in the context of a linear theory. As a consequence, we will develop in MacLaurin series the free energy function $\mathcal{E}$ and retain only the linear and quadratic terms, that is,

$$\begin{aligned}\rho\mathcal{E} = {} & \frac{1}{2}A_{ijmn}e_{ij}e_{mn} + G_{ijmn}e_{ij}\mu_{mn} + F_{ijmnr}e_{ij}\chi_{mnr} + \frac{1}{2}B_{ijmn}\mu_{ij}\mu_{mn} + \\ & + D_{ijmnr}\mu_{ij}\chi_{mnr} + \frac{1}{2}C_{ijkmnr}\chi_{ijk}\chi_{mnr} + B_{ij}e_{ij}\varphi + C_{ij}\mu_{ij}\varphi + D_{ijk}\chi_{ijk}\varphi + \\ & + \frac{1}{2}\xi\varphi^2 + d_{ijm}e_{ij}\varphi_{,m} + e_{ijm}\mu_{ij}\varphi_{,m} + f_{ijkm}\chi_{ijk}\varphi_{,m}! + \frac{1}{2}A_{ij}\varphi_{,i}\varphi_{,j} + d_i\varphi\varphi_{,i} + \\ & + m_{ij}q_i\varphi_{,j} + m_iq_i\varphi + P_{ijm}e_{ij}q_m + Q_{ijm}\mu_{ij}q_m + R_{ijkm}\chi_{ijk}q_m + \frac{1}{2}S_{ij}q_iq_j - \\ & - b_{ij}e_{ij}\vartheta - c_{ij}\mu_{ij}\vartheta - d_{ijk}\chi_{ijk}\vartheta - a_i\varphi_{,i}\vartheta - \xi_iq_i\vartheta - m\varphi\vartheta - \frac{1}{2}a\vartheta^2.\end{aligned} \tag{3.12}$$

On the other hand, $\dot{q}_m$ is depending on the variables e_{ij}, μ_{ij}, χ_{ijk}, ϑ, $\vartheta_{,i}$ and q_i, so that we will expand it in MacLaurin series and retain only up to quadratic terms. So, we obtain the equation:

$$\dot{q}_m = \frac{1}{\alpha}\left(\kappa_{mn}\vartheta_{,n} - q_m\right), \tag{3.13}$$

where α is the time of thermal relaxation and is a positive constant.
In Eqs. (3.12)-(3.13) we introduced the thermoelastic coefficients A_{ijmn}, G_{ijmn}, ...,a_i, ξ, m which characterize the mechanical and thermal properties of the medium, the thermal conductivity tensor κ_{ij} and the heat capacity a.
Given that the strain tensor e_{ij} is a symmetrical tensor, by its definition, we can deduce the following symmetry relations satisfied by the thermoelastic coefficients from (3.12)-(3.13):

$$A_{ijmn}=A_{mnij}=A_{jimn}, G_{ijmn}=G_{jimn}, F_{ijmnr}=F_{jimnr}, B_{ijmn}=B_{mnij}, d_{ijm}=d_{jim},$$
$$C_{ijkmnr}=C_{mnrijk}, b_{ij}=b_{ji}, P_{ijm}=P_{jim}, A_{ij}=A_{ji}, B_{ij}=B_{ji}, S_{ij}=S_{ji}, \kappa_{ij}=\kappa_{ji}. \tag{3.14}$$

Now we can tackle the problem of determining the constitutive equations as well as establishing the energy balance.

Theorem 3.1. *Consider that the free energy function $\mathcal{E}$ has the heat-flux included in its internal variables. Then the constitutive equations for our dipolar porous medium can be written in the following form:*

$$\tau_{ij} = \rho\frac{\partial\mathcal{E}}{\partial e_{ij}},\ \sigma_{ij} = \rho\frac{\partial\mathcal{E}}{\partial \mu_{ij}},\ \gamma_{ijk} = \rho\frac{\partial\mathcal{E}}{\partial \chi_{ijk}}, S = -\frac{\partial\mathcal{E}}{\partial \vartheta},\ g = -\rho\frac{\partial\mathcal{E}}{\partial \varphi},\ h_i = \rho\frac{\partial\mathcal{E}}{\partial \varphi_{,i}}. \tag{3.15}$$

Furthermore, for the equation of energy we deduce the expression:

$$\frac{\partial\mathcal{E}}{\partial q_k}\dot{q}_k + \dot{S}\vartheta - r = \frac{1}{\rho}q_{k,k}. \tag{3.16}$$

Proof. First, we take into account that the function $\mathcal{E}$ is depending on the independent variables e_{ij}, μ_{ij}, χ_{ijk}, φ, $\varphi_{,i}$, ϑ, $\vartheta_{,i}$, q_i so that, for its derivative with respect to time variable, we must carry out the differentiation with respect to these variables. As such, we are led to the inequality

$$\left(\rho\frac{\partial\mathcal{E}}{\partial e_{ij}} - \tau_{ij}\right)\dot{e}_{ij} + \left(\rho\frac{\partial\mathcal{E}}{\partial \mu_{ij}} - \sigma_{ij}\right)\dot{\mu}_{ij} + \left(\rho\frac{\partial\mathcal{E}}{\partial \chi_{ijk}} - \gamma_{ijk}\right)\dot{\chi}_{ijk} + \rho\left(\frac{\partial\mathcal{E}}{\partial \vartheta} + S\right)\dot{\vartheta} +$$
$$+\left(\rho\frac{\partial\mathcal{E}}{\partial \varphi} + g\right)\dot{\varphi} + \left(\rho\frac{\partial\mathcal{E}}{\partial \varphi_{,i}} + h_i\right)\dot{\varphi}_{,i} + \rho\left(\frac{\partial\mathcal{E}}{\partial \vartheta_{,i}}\right)\dot{\vartheta}_{,i} + \rho\left(\frac{\partial\mathcal{E}}{\partial q_i}\right)\dot{q}_i - \frac{\vartheta_{,i}}{\vartheta}q_i \le 0. \tag{3.17}$$

On the other hand, if we substitute E from (2.8) in (2.5), by taking into account the total differentiation respect to t, we deduce

$$\left(\rho\frac{\partial\mathcal{E}}{\partial e_{ij}} - \tau_{ij}\right)\dot{e}_{ij} + \left(\rho\frac{\partial\mathcal{E}}{\partial \mu_{ij}} - \sigma_{ij}\right)\dot{\mu}_{ij} + \left(\rho\frac{\partial\mathcal{E}}{\partial \chi_{ijk}} - \gamma_{ijk}\right)\dot{\chi}_{ijk} + \rho\left(\frac{\partial\mathcal{E}}{\partial \vartheta} + S\right)\dot{\vartheta}$$
$$+\left(\rho\frac{\partial\mathcal{E}}{\partial \varphi}+g\right)\dot{\varphi}+\left(\rho\frac{\partial\mathcal{E}}{\partial \varphi_{,i}}+h_i\right)\dot{\varphi}_{,i}+\rho\left(\frac{\partial\mathcal{E}}{\partial \vartheta_{,i}}\right)\dot{\vartheta}_{,i}+\rho\left(\frac{\partial\mathcal{E}}{\partial q_i}\right)\dot{q}_i+\rho\left(\dot{S}\vartheta - r\right)-q_{i,i}=0. \tag{3.18}$$

We can require that the inequality (3.17) and the equality (3.18) should hold for arbitrary $\dot{e}_{ij}$, $\dot{\mu}_{ij}$, $\dot{\chi}_{ijk}$, $\dot{\vartheta}$ and $\dot{\vartheta}_{,i}$. Furthermore, we can suppose that τ_{ij} does not depend on $\dot{e}_{ij}$, σ_{ij} does not depend on $\dot{\mu}_{ij}$, γ_{ijk} does not depend on $\dot{\chi}_{ijk}$, g does not depend on $\dot{\varphi}$ and S does not depend on $\dot{\vartheta}$. In this way, we are led to the following relations:

$$\tau_{ij} = \rho\frac{\partial\mathcal{E}}{\partial e_{ij}},\ \sigma_{ij} = \rho\frac{\partial\mathcal{E}}{\partial \mu_{ij}},\ \gamma_{ijk} = \rho\frac{\partial\mathcal{E}}{\partial \chi_{ijk}}, S = -\frac{\partial\mathcal{E}}{\partial \vartheta},\ g = -\rho\frac{\partial\mathcal{E}}{\partial \varphi},\ h_i = \rho\frac{\partial\mathcal{E}}{\partial \varphi_{,i}}; \tag{3.19}$$

$$\frac{\partial\mathcal{E}}{\partial q_i}\dot{q}_i + \dot{\eta}\vartheta - r - \frac{1}{\rho}q_{i,i} = 0; \tag{3.20}$$

$$\frac{\partial\mathcal{E}}{\partial \vartheta_{,i}} = 0,\ \ \rho\frac{\partial\mathcal{E}}{\partial q_i}\dot{q}_i - \frac{q_i}{\vartheta}\vartheta_{,i} \le 0. \tag{3.21}$$

Let us observe from $(3.21)_1$ that the free energy function does not depend on the temperature gradient. Also, the inequality $(3.21)_2$ introduces two restrictions on the constitutive equations:

$$\frac{1}{c}S_{kl}q_kq_l \geq 0,\ \vartheta_0 S_{mn}\kappa_{ni} = c\delta_{mi}, \tag{3.22}$$

in which c is a prescribed positive constant, ϑ_0 is the temperature in the initial state and δ_{ij} is the Kronecker's delta.
Now the proof of the theorem is complete because Eqs. (3.19) are the constitutive equations and Eq. (3.20) is the energy equation. □

Corollary 3.1. *The explicit form for the constitutive equations is*

$$\begin{aligned}
&\tau_{ij} = A_{ijmn}e_{mn} + G_{mnij}\mu_{mn} + F_{mnrij}\chi_{mnr} + B_{ij}\varphi + d_{ijm}\varphi_{,m} + P_{ijm}q_m - a_{ij}\vartheta,\\
&\sigma_{ij} = G_{ijmn}e_{mn} + B_{ijmn}\mu_{mn} + D_{ijmnr}\chi_{mnr} + C_{ij}\varphi + e_{ijm}\varphi_{,m} + Q_{ijm}q_m - b_{ij}\vartheta,\\
&\gamma_{ijk} = F_{ijkmn}e_{mn} + D_{mnijk}\mu_{mn} + A_{ijkmnr}\chi_{mnr} + D_{ijk}\varphi + f_{ijkm}\varphi_{,m} + R_{ijkm}q_m - c_{ijk}\vartheta,\\
&g = -B_{ij}e_{ij} - C_{ij}\mu_{ij} - D_{ijk}\chi_{ijk} - \xi\varphi - d_i\varphi_{,i} - m_iq_i + m\vartheta,\\
&h_k = d_{ijk}e_{ij} + e_{ijk}\mu_{ij} + f_{ijmk}\chi_{ijm} + d_k\varphi + A_{ik}\varphi_{,i} + m_{ik}q_i - a_k\vartheta + b_k,\\
&\rho S = a_{ij}e_{ij} + b_{ij}\mu_{ij} + c_{ijk}\chi_{ijk} + m\varphi + a_i\varphi_{,i} + \xi_iq_i + a\vartheta + c.
\end{aligned} \tag{3.23}$$

Proof. All equations from (3.23) can be obtained by direct calculations of derivatives from (3.15), after we substitute the energy function $\mathcal{E}$ defined in (3.12). □

For the sake of simplifying the calculations, we will make the following two hypothesis:

(H_1) the reference state of the body is free of heat flux and stress;
(H_2) the initial state of the medium is not affected by any intrinsic equilibrated force and its uniform temperature is the constant ϑ_0.

Based on these two hypotheses, the MacLaurin series developments of the functions $\mathcal{E}$ and Q_i (from (3)), in which we retain only the necessary terms in linear theory, have some simpler expressions, namely:

$$\begin{aligned}
&\rho\mathcal{E}\left(e_{ij},\mu_{ij},\chi_{ijk},\varphi,\varphi_{,i},\vartheta+\vartheta_0,q_i\right) = \rho\mathcal{E}\left(0,0,0,0,0,\vartheta_0,0\right) - b\varphi - b_i\varphi_{,i} - c\vartheta - c_iq_i +\\
&+\frac{1}{2}A_{ijmn}e_{ij}e_{mn} + G_{ijmn}e_{ij}\mu_{mn} + F_{ijmnr}e_{ij}\chi_{mnr} + \frac{1}{2}B_{ijmn}\mu_{ij}\mu_{mn} + D_{ijmnr}\mu_{ij}\chi_{mnr} +\\
&+\frac{1}{2}C_{ijkmnr}\chi_{ijk}\chi_{mnr} + B_{ij}e_{ij}\varphi + C_{ij}\mu_{ij}\varphi + D_{ijk}\chi_{ijk}\varphi + \frac{1}{2}\xi\varphi^2 + d_{ijm}e_{ij}\varphi_{,m} + e_{ijm}\mu_{ij}\varphi_{,m}\\
&+f_{ijkm}\chi_{ijk}\varphi_{,i} + \frac{1}{2}A_{ij}\varphi_{,i}\varphi_{,j} + d_i\varphi\varphi_{,i} + m_{ij}q_i\varphi_{,j} + m_iq_i\varphi + P_{ijm}e_{ij}q_m + Q_{ijm}\mu_{ij}q_m +\\
&+R_{ijkm}\chi_{ijk}q_m + \frac{1}{2}S_{ij}q_iq_j - b_{ij}e_{ij}\vartheta - c_{ij}\mu_{ij}\vartheta - d_{ijk}\chi_{ijk}\vartheta - a_i\varphi_{,i}\vartheta - \xi_iq_i\vartheta - m\varphi\vartheta - \frac{1}{2}a\vartheta^2,
\end{aligned} \tag{3.24}$$

and, respectively

$$\dot{q}_m = w_{mi}\left[\alpha_{ijk}e_{jk} + \beta_{ijk}\mu_{jk} + \delta_{ijkn}\chi_{jkn} + \kappa_{ij}\vartheta_{,j} + r_{ij}\varphi_{,j} - q_i\right]. \tag{3.25}$$

Theorem 3.2.. *If hypotheses H_1 and H_2 are satisfied, then the constitutive equations received the form:*

$$\begin{aligned}
&\tau_{ij} = A_{ijmn}e_{mn} + G_{mnij}\mu_{mn} + F_{mnrij}\chi_{mnr} + B_{ij}\varphi + d_{ijm}\varphi_{,m} - a_{ij}\vartheta,\\
&\sigma_{ij} = G_{ijmn}e_{mn} + B_{ijmn}\mu_{mn} + D_{ijmnr}\chi_{mnr} + C_{ij}\varphi + e_{ijm}\varphi_{,m} - b_{ij}\vartheta,\\
&\gamma_{ijk} = F_{ijkmn}e_{mn} + D_{mnijk}\mu_{mn} + A_{ijkmnr}\chi_{mnr} + D_{ijk}\varphi + f_{ijkm}\varphi_{,m} - c_{ijk}\vartheta,\\
&g = -B_{ij}e_{ij} - C_{ij}\mu_{ij} - D_{ijk}\chi_{ijk} - \xi\varphi - d_i\varphi_{,i} - m_i q_i + m\vartheta,\\
&h_k = d_{ijk}e_{ij} + e_{ijk}\mu_{ij} + f_{ijmk}\chi_{ijm} + d_k\varphi + A_{ik}\varphi_{,i} - a_k\vartheta + b_k,\\
&\rho S = a_{ij}e_{ij} + b_{ij}\mu_{ij} + c_{ijk}\chi_{ijk} + m\varphi + a_i\varphi_{,i} + a\vartheta + c.
\end{aligned} \tag{3.26}$$

Furthermore, the energy equation receives the form:

$$\alpha\dot{q}_i + q_i = k_{ij}\vartheta_{,j}. \tag{3.27}$$

Proof. By combining relations (3.24) and (3.25) into inequality $(3.21)_2$, we find that the following conditions are necessary

$$\begin{aligned}
&P_{ijm} = 0,\ Q_{ijm} = 0,\ R_{ijkm} = 0,\ c_i = 0,\ \xi_i = 0,\ n = 0,\ r_{ij} = 0,\ \alpha_{ijk} = 0,\ \beta_{ijk} = 0,\\
&\delta_{ijkn} = 0,\ m_i = 0,\ m_{ij} = 0,\ b_{im}w_{mn}\kappa_{nj} = -\frac{1}{\vartheta_0}\delta_{ij},\ b_{ij}w_{im}q_jq_m \geq 0.
\end{aligned} \tag{3.28}$$

On the other hand, we supposed that the initial state of the medium is free of any stress, of heat flux and of intrinsic equilibrated force. In these circumstances, the following conditions must be met:

$$a = 0,\ b = 0,\ b_i = 0. \tag{3.29}$$

Based on condition (2.4) we can deduce that there is constant α so that

$$w_{ij} = \frac{1}{\alpha}\delta_{ij}. \tag{3.30}$$

In $(3.28)_{13}$ and (3.30) we noted with δ the Kronecker symbol.
Finally, we consider relations (3.28)-(3.30) such that from Eqs. (3.24) and (3.25) we obtain the desired equations (3.26) and (3.27). □

In the following theorem we will show that the introduction of a new vector for the heat flux among the independent variables does not affect the solvability of the system of differential equations that govern the evolution of the dipolar thermoelastic bodies with voids.

Theorem 3.3. *We assume that the hypotheses H_1 and H_2 are satisfied. Then the motion equations (2.10), the balance of equilibrated force (2.11) and the energy equation (3.27) form the following system of equations in the independent variables v_i, ϕ_{ij}, φ and ϑ:*

$$\begin{aligned}
&[(A_{ijmn} + G_{ijmn})\, v_{n,m} + (G_{mnij} + B_{ijmn})\,(v_{n,m} - \phi_{mn}) + (F_{mnrij} + D_{ijmnr})\,\phi_{nr,m} +\\
&\quad (B_{ij} + C_{ij})\,\varphi + (d_{ijm} + e_{ijm})\,\varphi_{,m} - (a_{ij} + b_{ij})\,\vartheta]_{,j} + \rho F_i = \rho\ddot{v}_i.
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
&[F_{ijkmn}v_{n,m} + D_{mnijk}\,(v_{n,m} - \phi_{mn}) + C_{ijkmnr}\phi_{nr,m} + D_{ijk}\varphi + f_{ijkm}\varphi_{,m} - c_{ijk}\vartheta]_{,i}\\
&\quad + G_{jkmn}v_{m,n} + B_{jkmn}\,(v_{n,m} - \phi_{mn}) + D_{jkmnr}\phi_{nr,m} + C_{jk}\varphi +\\
&\quad + e_{jkm}\varphi_{,m} - b_{jk}\vartheta + \rho G_{jk} = I_{kr}\ddot{\phi}_{jr}.
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
&[d_{ijk}e_{ij} + e_{ijk}\mu_{ij} + f_{ijmk}\chi_{ijm} + d_k\varphi + A_{ik}\varphi_{,i} - a_k\vartheta]_{,k} - B_{ij}e_{ij} -\\
&\quad - C_{ij}\mu_{ij} - D_{ijk}\chi_{ijk} - \xi\varphi - d_i\varphi_{,i} - m_i q_i + m\vartheta + \rho L = \rho\kappa\ddot{\varphi}.
\end{aligned} \tag{3.33}$$

$$\rho\left(1+\alpha\frac{\partial}{\partial t}\right)S-\vartheta_0\left(1+\alpha\frac{\partial}{\partial t}\right)\left[a_{ij}\dot{u}_{i,j}+b_{ij}\left(\dot{u}_{j,i}-\dot{\varphi}_{ij}\right)+\right.$$
$$\left.+c_{ijk}\dot{\varphi}_{ij,k}+m\dot{\varphi}+a_i\dot{\varphi}_{,i}+a\dot{\vartheta}\right]+k_{ij}\vartheta_{,ji}=0. \tag{3.34}$$

Furthermore, in the heat-flux dependent theory of thermoelasticity for dipolar bodies with pores the number of the basic equations is the same as the number of internal variables.

Proof. Clearly, the purpose of this theorem is to deduce more explicit expressions for the basic balances (2.10), (2.11) and (3.27), in which the involvement of independent variables can be seen.
So, if we take into account the geometric equations (2.2) and substitute the constitutive equations $(3.26)_1$ and $(3.26)_2$ in equation $(2.10)_1$, we obtain Eqs. (3.31).
Also, if we take into account the geometric equations (2.2) and substitute the constitutive equations $(3.26)_2$ and $(3.26)_3$ in equation $(2.10)_2$, we obtain Eqs. (3.32).
Similarly, we consider the geometric equations (2.2) and substitute the constitutive equations $(3.26)_4$ and $(3.26)_5$ in equation (2.11), we are led to Eq. (3.33).
Finally, we use the geometric equations (2.2) and substitute the constitutive equation $(3.26)_6$ in equation (3.27), so that we deduce Eqs. (3.34).
As for the second statement of the theorem, it is sufficient to analyze the equations (3.31)-(3.34) to find a number of fourteen differential equations in which the fourteen independent variables v_i, ϕ_{ij}, φ and ϑ are involved. In this way, the proof of the theorem is finished. □

In the last theorem of our study we intend to prove the consistency of the above theory, by proving a theorem regarding the uniqueness of solution of the mixed initial-boundary value problem in our context.
But first, we will build this mixed problem. Thus, in addition to the basic equations (3.31)-(3.34), which take place in the cylinder $D\times[0,\infty)$, we add the following initial data, which involve an initial date for the heat-flux q_i, as well:

$$v_i\left(x_k,0\right)=0,\ \phi_{ij}\left(x_k,0\right)=0,\ \varphi\left(x_k,0\right)=0,\ \vartheta\left(x_k,0\right)=0,\ \forall x=\left(x_k\right)\in D,$$
$$\dot{v}_i\left(x_k,0\right)=0,\ \dot{\phi}_{ij}\left(x_k,0\right)=0,\ \dot{\varphi}\left(x_k,0\right)=0,\ q_i\left(x_k,0\right)=0,\ \forall x=\left(x_k\right)\in D. \tag{3.35}$$

The mixed problem is complete if we consider next boundary conditions, which also involve a boundary date for the heat-flux q_i, as well:

$$v_i(x_k,0)=v_i^0,\ (x_k,0)\in S_1\times[0,t_0),\ (\tau_{ij}+\sigma_{ij})\left(x_k,0\right)n_j=t_i^0,\ (x_k,0)\in S_1^c\times[0,t_0),$$
$$\phi_{ij}(x_k,0)=\phi_{ij}^0,\ (x_k,0)\in S_2\times[0,t_0),\ \gamma_{ijm}\left(x_k,0\right)n_i=m_{jm}^0,\ (x_k,0)\in S_2^c\times[0,t_0),$$
$$\varphi(x_k,0)=\varphi^0,\ (x_k,0)\in S_3\times[0,t_0),\ h_i\left(x_k,0\right)n_i=h^0,\ (x_k,0)\in S_3^c\times[0,t_0), \tag{3.36}$$
$$\vartheta(x_k,0)=\vartheta^0,\ (x_k,0)\in S_4\times[0,t_0),\ q_i\left(x_k,0\right)n_i=q^0,\ (x_k,0)\in S_4^c\times[0,t_0).$$

Here the instant of time t_0 can be infinite and $n=(n_k)$ is the vector of the unit normal outward to the surface ∂D. The above functions v_i^0, t_i^0, ϕ_{jk}^0, m_{jk}^0, ϑ^0 and q^0 are prescribed and enough regular functions in the domains of their definition. Also, we denoted by S_i some parts of the surface ∂D and by S_i^c its complements, satisfying the following relations:

$$S_1\cup S_1^c=S_2\cup S_2^c=S_3\cup S_3^c=S_4\cup S_4^c=\partial D,$$
$$S_1\cap S_1^c=S_2\cap S_2^c=S_3\cap S_3^c=S_4\cap S_4^c=\emptyset.$$

We will call a solution of the mixed problem defined by above equations and conditions for the heat flux thermoelasticity of a dipolar porous solid in the domain $D\times[0,t_0)$, denoted by $\mathcal{P}$, a state of deformation $(v_i,\phi_{ij},\varphi,\vartheta)$ satisfying the basic equations (3.31)-(3.34),

the initial data (3.35) and the condition to the limit (3.36).
In the following theorem we will prove the uniqueness of the solution of problem $\mathcal{P}$.

Theorem 3.4.. *If we suppose that the coefficients* $\rho, \kappa, \xi, \vartheta_0, a$ *and* α *are positive constants, then the mixed problem* $\mathcal{P}$ *admits only one solution.*

Proof. We will use the well-known procedure of reducing to absurdity. According to this, we assume our problem $\mathcal{P}$ admits two different solutions. Of course, their difference is a new solution of problem $\mathcal{P}$, because of linearity. But this solution corresponds to null charges, namely $F_i = 0,\ G_{ij} = 0,\ L = 0$ and $S = 0$.
We will use Eq. (3.31), in the case $F_i = 0$ and multiply it by $\dot{v}_i$ so that we are led to the equation:

$$\begin{aligned}&\{[(A_{ijmn} + G_{ijmn}) v_{n,m} + (G_{mnij} + B_{ijmn}) (v_{n,m} - \phi_{mn}) + (F_{mnrij} + D_{ijmnr}) \phi_{nr,m}+\\&+ (B_{ij} + C_{ij}) \varphi + (d_{ijm} + e_{ijm}) \varphi_{,m} - (a_{ij} + b_{ij}) \vartheta] \dot{v}_i\}_{,j} - [(A_{ijmn} + G_{ijmn}) v_{n,m}+\\&+ (G_{mnij} + B_{ijmn}) (v_{n,m} - \phi_{mn}) + (F_{mnrij} + D_{ijmnr}) \phi_{nr,m} + (B_{ij} + C_{ij}) \varphi +\\&+ (d_{ijm} + e_{ijm}) \varphi_{,m} - (a_{ij} + b_{ij}) \vartheta] \dot{v}_{i,j} = \rho \ddot{v}_i \dot{v}_i.\end{aligned} \tag{3.37}$$

We consider now Eq. (32), for $G_{jk} = 0$, and multiply it by $\dot{\phi}_{jk}$ and deduce

$$\begin{aligned}&\Big\{[F_{ijkmn}v_{n,m}+D_{mnijk} (v_{n,m}-\phi_{mn})+C_{ijkmnr}\phi_{nr,m}+D_{ijk}\varphi+f_{ijkm}\varphi_{,m}- c_{ijk}\vartheta] \dot{\phi}_{jk}\Big\}_{,i}-\\&- [F_{ijkmn}v_{n,m}+D_{mnijk} (v_{n,m}-\phi_{mn})+C_{ijkmnr}\phi_{nr,m}+D_{ijk}\varphi+f_{ijkm}\varphi_{,m}-c_{ijk}\vartheta] \dot{\phi}_{jk,i}+\\&+ [G_{jkmn}v_{m,n} + B_{jkmn} (v_{n,m} - \phi_{mn}) + D_{jkmnr}\phi_{nr,m} + C_{jk}\varphi+\\&+e_{jkm}\varphi_{,m} - b_{jk}\vartheta + \rho G_{jk}] \dot{\phi}_{jk} = I_{kr}\ddot{\phi}_{jr}\dot{\phi}_{jk}.\end{aligned} \tag{3.38}$$

We gather member by member the equalities (3.37) and (3.38) and then integrate the resulting equality over the domain D and the interval $[0, t]$. Then we use the divergence theorem and consider the boundary conditions (3.38), in their homogeneous form, so that we obtain:

$$\begin{aligned}&\int_D \Big(\rho\dot{v}_i\dot{v}_i + I_{jk}\dot{\phi}_{jr}\dot{\phi}_{kr} + A_{ijmn}e_{ij}e_{mn} + 2G_{ijmn}e_{ij}\mu_{mn} + 2F_{ijmnr}e_{ij}\chi_{mnr}+\\&+B_{ijmn}\mu_{ij}\mu_{mn}+2D_{ijmnr}\mu_{ij}\chi_{mnr}+C_{ijkmnr}\chi_{ijk}\chi_{mnr})dV+\int_0^t\int_D(d_{ijm}\dot{e}_{ij} + e_{ijm}\dot{\mu}_{ij}+\\&+f_{ijkm}\dot{\mu}_{ijk})\, \varphi_{,m}+(B_{ij}\dot{e}_{ij}+C_{ij}\dot{\mu}_{ij}+D_{ijk}\dot{\mu}_{ijk})+(b_{ij}\dot{e}_{ij}+c_{ij}\dot{\mu}_{ij}+d_{ijk}\dot{\mu}_{ijk})\, \vartheta dV ds=0.\end{aligned} \tag{3.39}$$

We now consider Eq. (3.33), for $L = 0$, and multiply it by $\dot{\varphi}$ to deduce

$$\begin{aligned}&\int_D \left(\rho\kappa\dot{\varphi}^2 + \xi\varphi^2 + d_i\varphi\varphi_{,i} + A_{ij}\varphi_{,i}\varphi_{,j}\right) dV - \int_0^t \int_D (a_i\varphi_{,i} + m\varphi + a\vartheta + m_iq_i)\, \vartheta dV ds+\\&+\int_0^t\int_D(d_{ijm}e_{ij}+e_{ijm}\mu_{ij}+f_{ijkm}\mu_{ijk})\, \dot{\varphi}_{,m}+(B_{ij}e_{ij}+C_{ij}\mu_{ij}+D_{ijk}\mu_{ijk})\, \dot{\varphi}dV ds=0.\end{aligned} \tag{3.40}$$

Analogous, we multiply both terms of the last basic equation (3.34) by θ and then integrate the resulting equality over the domain D and the interval $[0, t]$ and take into account Eq. (3.27). So, we arrive at the following equation:

$$\begin{aligned}&\frac{\vartheta_0}{2} \int_D a\vartheta^2 dV - \int_0^t \int_D \alpha\dot{q}_i\vartheta_{,i} dV ds + \int_0^t \int_D \kappa_{ij}\vartheta_{,j}\vartheta_{,j} dV ds +\\&+ \int_0^t \int_D (\alpha_{ij}\dot{e}_{ij} + \beta_{ij}\dot{\mu}_{ij} + \delta_{ijk}\dot{\chi}_{ijk} + a_i\dot{\varphi}_{,i} + m\dot{\varphi})\vartheta\, dV ds = 0.\end{aligned} \tag{3.41}$$

Based on the last two relations from (3.28) and Eq. (3.30) we can write:

$$k_{ij}b_{ik} = -\frac{\alpha}{\theta_0}\delta_{jk},\ \frac{1}{\alpha}b_{jk}q_jq_k \geq 0. \tag{3.42}$$

Inspired by equation (3.41), we can combine relations (3.27) and (3.42) so that we are led to the following equality

$$\begin{aligned}\int_D\int_0^t \dot{q}_i\vartheta_0\left(\alpha b_{ik}\dot{q}_k + b_{ik}q_k\right)dV ds = \alpha\vartheta_0\int_D\int_0^t b_{ik}\dot{q}_i\dot{q}_k dV ds +\\ +\frac{\vartheta_0}{2}\int_D b_{ik}q_iq_k dV = -\alpha\int_D\int_0^t \dot{q}_k\vartheta_{,k}dV.\end{aligned} \tag{3.43}$$

Considering equality (3.43) in respect to relation (3.41), the latter takes the following form

$$\begin{aligned}&\frac{\vartheta_0}{2}\int_D\left(a\vartheta^2 + b_{ik}q_iq_k\right)dV + \vartheta_0\int_0^t\int_D \alpha b_{ik}\dot{q}_i\dot{q}_k dV ds +\\ &+\int_0^t\int_D\left(\alpha_{ij}\dot{e}_{ij} + \beta_{ij}\dot{\mu}_{ij} + \delta_{ijk}\dot{\chi}_{ijk} + a_i\dot{\varphi}_{,i} + m\dot{\varphi}\right)\vartheta\, dV ds +\\ &+\int_0^t\int_D \kappa_{ij}\vartheta_{,i}\vartheta_{,j}\, dV ds = 0.\end{aligned} \tag{3.44}$$

Finally, we summing together the equalities (3.39), (3.40) and (3.44) and obtain the equality

$$\begin{aligned}&\int_D\left(\rho\dot{v}_i\dot{v}_i + I_{jk}\dot{\phi}_{jr}\dot{\phi}_{kr} + \rho\kappa\dot{\varphi}^2 + \xi\varphi^2 + \vartheta_0 a\vartheta^2 + d_i\varphi\varphi_{,i} + \vartheta_0 b_{ik}q_iq_k\right)dV +\\ &+\int_D\left(A_{ijmn}e_{ij}e_{mn} + 2G_{ijmn}e_{ij}\mu_{mn} + 2F_{ijmnr}e_{ij}\chi_{mnr} +\right.\\ &\left.+B_{ijmn}\mu_{ij}\mu_{mn} + 2D_{ijmnr}\mu_{ij}\chi_{mnr} + C_{ijkmnr}\chi_{ijk}\chi_{mnr}\right)dV +\\ &+\int_0^t\int_D\left[\left(d_{ijm}e_{ij} + e_{ijm}\mu_{ij} + f_{ijkm}\mu_{ijk}\right)\varphi_{,m} + \left(B_{ij}e_{ij} + C_{ij}\mu_{ij} + D_{ijk}\mu_{ijk}\right)\varphi\right]dV ds +\\ &+\int_0^t\int_D\left(A_{ij}\varphi_{,i}\varphi_{,j} + d_i\varphi\varphi_{,i} + \kappa_{ij}\vartheta_{,i}\vartheta_{,j}\right)\, dV + \vartheta_0\alpha\int_0^t\int_D b_{ik}\dot{q}_i\dot{q}_k dV ds = 0.\end{aligned} \tag{3.45}$$

According to the hypotheses of Theorem 4, the coefficients $\rho, \kappa, \xi, \vartheta_0, a$ and α are positive. If, in addition, we also use inequality $(3.42)_2$, we come to the conclusion that the following two inequalities appear:

$$\mathcal{P} \geq 0,\ \mathcal{P} + \mathcal{K} \leq 0, \tag{3.46}$$

where we used the notations:

$$\begin{aligned}\mathcal{P} &= \int_D\left(\rho\dot{v}_i\dot{v}_i + I_{jk}\dot{\phi}_{jr}\dot{\phi}_{kr} + \rho\kappa\dot{\varphi}^2 + \xi\varphi^2 + \vartheta_0 a\vartheta^2 + d_i\varphi\varphi_{,i} + \vartheta_0 b_{ik}q_iq_k\right)dV,\\ \mathcal{K} &= \int_D\left(A_{ijmn}e_{ij}e_{mn} + 2G_{ijmn}e_{ij}\mu_{mn} + 2F_{ijmnr}e_{ij}\chi_{mnr} +\right.\\ &\left.+B_{ijmn}\mu_{ij}\mu_{mn} + 2D_{ijmnr}\mu_{ij}\chi_{mnr} + C_{ijkmnr}\chi_{ijk}\chi_{mnr}\right)dV +\\ &+\int_D\left(d_{ijm}e_{ij} + e_{ijm}\mu_{ij} + f_{ijkm}\mu_{ijk} + A_{im}\varphi_{,i}\right)\varphi_{,m}\, dV +\\ &+\int_D\left(B_{ij}e_{ij} + C_{ij}\mu_{ij} + D_{ijk}\mu_{ijk} + d_i\varphi_{,i}\right)\varphi\, dV.\end{aligned} \tag{3.47}$$

Based on the assumptions, we have $\mathcal{K} \geq$ and then, from (3.46) we deduce that $\mathcal{P} \leq 0$, so that, because the coefficients $\rho, \kappa, \xi, \vartheta_0, a$ and α are positive, we deduce

$$\dot{v}_i = 0, \ \dot{\phi}_{ij} = 0, \ \varphi = 0, \ \vartheta = 0, \ q_i = 0, \tag{3.48}$$

so that, because the initial conditions are null for difference, we deduce $v_i = 0, \ \phi_{ij} = 0$, what was to be demonstrated. □

4. CONCLUSIONS

The new form of energy equation can be considered as an element of novelty of our paper. Unlike the energy balance of other studies, our equation involves the heat flux, of components q_i, as an internal variable, together with the other variables used so far. Also, our study is a natural generalization of the previous studies, for instance, in the transition from bodies with a micropolar structure and voids to the dipolar solids with pores. Another originality is our intention to solve the paradox of the conduction of the heat. An additional argument is that our mixed initial-boundary value problem is governed by a system of partial differential equations of the hyperbolic type. We anticipate that our heat flux dependent theory will become a predictive theory of shock wave structure, of sound dispersion, of light scattering, and so on.

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