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An iterative regularization method for variational inequalities in Hilbert spaces

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ABSTRACT. We consider an iterative method for regularization of a variational inequality (VI) defined by a Lipschitz continuous monotone operator in the case where the set of feasible solutions is decomposed to the intersection of finitely many closed convex subsets of a Hilbert space. We prove the strong convergence of the sequence generated by our algorithm. It seems that this is the first time in the literature to handle iterative solution of ill-posed VIs in the domain decomposition case.

1. INTRODUCTION

We are concerned with an iterative method for the variational inequality (VI): Find a point $\hat{x} \in \Omega$ with the property

$$\langle U\hat{x}, x - \hat{x}\rangle \geq 0, \quad x \in \Omega, \tag{1.1}$$

where Ω is a nonempty closed convex subset of a Hilbert space H and U is a (single-valued) monotone operator in H (with domain containing Ω).

It is known that VI (1.1) is equivalent to the fixed point problem

$$P_\Omega(I - \lambda U)\hat{x} = \hat{x}, \tag{1.2}$$

where P_Ω is the projector from H onto Ω, and $\lambda > 0$ is any constant.

It is also known that if U is strongly monotone and Lipschitz continuous, then (1.1) has a unique solution and is well posed (the mapping $P_\Omega(I - \lambda U)$ is a contraction for small enough $\lambda > 0$). However, if U is not strongly monotone, VI (1.1) is ill-posed (i.e., not well-posed) and regularization is needed. The traditional Tikhonov regularization uses εI as a regularizer of U; namely, we consider the following regularized VI:

$$\bar{x}_\varepsilon \in \Omega, \quad \langle U\bar{x}_\varepsilon + \varepsilon\bar{x}_\varepsilon, x - \bar{x}_\varepsilon\rangle \geq 0, \quad x \in \Omega. \tag{1.3}$$

Since the regularized monotone operator $U + \varepsilon I$ is Lipschitzian and strongly monotone, VI (1.3) has a unique solution which we denote by $\bar{x}_\varepsilon \in \Omega$.

More generally, we may use εV to regularize U and consider the regularized VI:

$$\hat{x}_\varepsilon \in \Omega, \quad \langle U\hat{x}_\varepsilon + \varepsilon V\hat{x}_\varepsilon, x - \hat{x}_\varepsilon\rangle \geq 0, \quad x \in \Omega. \tag{1.4}$$

Here V is a Lipschtz and strongly monotone operator. In [14, Theorem 2.2], it is proved that the net of solutions $(\hat{x}_\varepsilon)$ strongly converges as $\varepsilon \to 0$ to a solution $s^\dagger$ of VI (1.1) (assuming $S_\Omega(U) \neq \emptyset$) which is the unique solution to the VI:

$$s^\dagger \in S_\Omega(U), \quad \langle Vs^\dagger, s - s^\dagger\rangle \geq 0, \quad s \in S_\Omega(U). \tag{1.5}$$

In particular, if $V = I$, then $(\hat{x}_\varepsilon)$ (i.e., $(\bar{x}_\varepsilon)$ defined by (1.3)) strongly converges to $s^\dagger = \arg\min\{\|s\| : s \in S_\Omega(U)\}$, the minimum-norm element of $S_\Omega(U)$.

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Iterative methods for finding $s^\dagger$ are introduced in [2] and [14]. These methods generate a sequence (x_n) through the following iteration manner:

$$x_{n+1} = P_Q(x_n - \lambda_n(Ux_n + \varepsilon_n V x_n)), \quad n = 0, 1, \cdots, \tag{1.6}$$

where the initial point $x_0 \in \Omega$ is chosen arbitrarily.

Under certain conditions imposed on the sequences (λ_n) and (ε_n), [2, Theorem 1] (in the case where $V = I$) and [14, Theorem 3.1] prove the strong convergence to $s^\dagger$ of the sequence (x_n) defined by (1.6).

In this paper we will continue to study iterative methods for the regularization of VI (1.1). However we will consider the case where Ω is decomposed to the intersection of finitely many closed convex subsets $\{\Omega_i\}_{i=1}^N$ of H, that is,

$$\Omega = \bigcap_{i=1}^{N} \Omega_i \tag{1.7}$$

where $N \geq 1$ is an integer. We denote by $S_\Omega(U)$ the set of solutions of VI (1.1) and always assume that $S_\Omega(U)$ is nonempty (i.e., (1.1) is solvable). To the best of our knowledge, this seems to be the first time in the literature to seek an iterative solution of VI (1.1) via regularization in the domain decomposition case (1.7). [Note: Theorem 3.6 of [11] looks iteratively for the solution $\bar{x}_\varepsilon$ of the regularized VI (1.4) (not a solution of VI (1.1)). It is indeed an open question raised in Remark 3.9 of [11] whether an iterative method can be found to converge in the norm topology to a solution of VI (1.1) in the case of Ω being defined by (1.7).]

The aim of this paper is to introduce an iterative method for the regularization of VI (1.1) with Ω of form (1.7) which generates a sequence convergent in the norm topology to a solution of VI (1.1). The convergence result will be presented in section 3, prior to which we will discuss in section 2 some basic tools useful to the proof of the main result (i.e., Theorem 3.3) in section 3. An application on a constrained linear inverse problem is given in the final section 4.

2. Preliminaries

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and let Ω be a nonempty closed convex subset of H. We use P_Ω to denote the projection of H onto Ω. Namely, for each $x \in \Omega$,

$$P_\Omega x = \arg\min_{y \in \Omega} \|y - x\|^2.$$

The following two properties of a projection are pertinent to our argument in this paper:

- $\langle x - P_\Omega x, y - P_\Omega x \rangle \leq 0$ for $x \in H$ and $y \in \Omega$,
- P_Ω is nonexpansive, i.e., $\|P_\Omega x - P_\Omega z\| \leq \|x - z\|$ for $x, z \in H$.

Recall that a (single-valued) operator A with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in H is said to be monotone if

$$\langle x_2 - x_1, Ax_2 - Ax_1 \rangle \geq 0, \quad x_1,\, x_2 \in \mathcal{D}(A).$$

Recall also that a monotone operator A is said to be ν-strongly monotone if, for some constant $\nu > 0$,

$$\langle x_2 - x_1, Ax_2 - Ax_1 \rangle \geq \nu \|x_2 - x_1\|^2, \quad x_1,\, x_2 \in \mathcal{D}(A).$$

In the subsequent argument, we need the dual version of IV (1.1) which is the VI:

$$\hat{x} \in \Omega, \quad \langle Ux, x - \hat{x} \rangle \geq 0, \quad x \in \Omega. \tag{2.8}$$

The lemma below shows an equivalence between the primal VI (1.1) and its dual VI (2.8).

Lemma 2.1. [4] *Suppose $U : \Omega \to H$ is monotone and weakly continuous along segments (i.e., $U((1-\tau)x + \tau y) \to Ux$ weakly, as $\tau \to 0$, for $x, y \in \Omega$). Then the dual VI* (2.8) *is equivalent to the primal VI* (1.1).

The lemma below plays an important role in strong convergence analysis of iterative methods.

Lemma 2.2. [13] *Assume (σ_n) is a sequence of nonnegative real numbers satisfying the condition:*

$$\sigma_{n+1} \le (1-\tau_n)\sigma_n + \tau_n\beta_n + \delta_n, \quad n \ge 0, \tag{2.9}$$

where (τ_n) and (δ_n) are sequences in (0,1) and (β_n) is a sequence in $\mathbb{R}$. Assume

(i) $\sum_{n=1}^{\infty} \tau_n = \infty$,
(ii) $\limsup_{n\to\infty} \beta_n \le 0$, *and*
(iii) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\lim_{n\to\infty} \sigma_n = 0$.

3. Strong Convergence Analysis of an Iterative Method

Let Ω be a nonempty closed convex subset of a Hilbert space H which is of form (1.7). We consider the monotone VI (1.1), where U is a monotone operator with domain $\mathcal{D}(U) \supset \cup_{i=1}^{N}\Omega_i$. Assume that U is Lipschitz continuous with coefficient $L \ge 0$:

$$\|Ux - Uy\| \le L\|x-y\|, \quad x, y \in \mathcal{D}(U). \tag{3.10}$$

Since we do not assume strong monotonicity of U, VI (1.1) is not well posed in general; regularization is needed. We will use an ℓ-Lipschitz and ν-strongly monotone operator V to regularize U and consider the regularized VI:

$$\langle Uz_\varepsilon + \varepsilon Vz_\varepsilon, z - z_\varepsilon\rangle \ge 0, \quad z \in \Omega. \tag{3.11}$$

Note that the regularized VI (3.11) has a unique solution that is denoted as z_ε for each $\varepsilon > 0$. Note also that when $V = I$, this is reduced to the traditional Tikhonov regularization (1.3).

Observe that the fixed point equivalence of VI (3.11) is as follows:

$$T_\varepsilon z_\varepsilon = z_\varepsilon, \quad \text{with} \quad T_\varepsilon x := P_\Omega(I - \lambda(U + \varepsilon V))x, \quad x \in \Omega, \tag{3.12}$$

where $\lambda > 0$, and P_Ω is the projection from H to Ω.

Lemma 3.3. [14, Lemma 2.1] *The mapping T_ε defined by (3.12) is a $(1 - \frac{1}{2}\nu\lambda\varepsilon)$-contraction provided*

$$0 < \lambda < \frac{\nu\varepsilon}{(L^2 + \varepsilon\ell)^2}. \tag{3.13}$$

Theorem 3.1. [14, Theorem 2.2] *Suppose U is L-Lipschitz and monotone, and V is ℓ-Lipschitz and ν-strongly monotone. Assume* (3.13). *Then (z_ε) converges in norm, as $\varepsilon \to 0$, to $z^\dagger \in S_\Omega(U)$ that solves the VI:*

$$\langle Vz^\dagger, z - z^\dagger\rangle \ge 0, \quad z \in S_\Omega(U). \tag{3.14}$$

Discretization of the implicit fixed point method (3.12) induces an iterative algorithm that generates a sequence (x_n) as follows:

$$x_{n+1} = P_\Omega(x_n - \lambda_n(Ux_n + \varepsilon_n Vx_n)), \quad n \ge 0, \tag{3.15}$$

where the initial guess $x_0 \in \Omega$.

The convergence of (3.15) is proved as follows.

Theorem 3.2. [14, Theorem 3.1] (see also [2, Theorem 1]) *Suppose U is L-Lipschtiz and monotone, and V is ℓ-Lipschtiz and ν-strongly monotone. Assume the following conditions are satisfied:* (i) $0 < \lambda_n < \nu\varepsilon_n/[(L+\varepsilon_n\ell)^2 + (\varepsilon_n^2/4)]$, (ii) $\varepsilon_n \to 0$, (iii) $\sum_{n=1}^{\infty} \varepsilon_n\lambda_n = \infty$, (iv) $(|\lambda_n - \lambda_{n-1}| + |\varepsilon_n\lambda_n - \varepsilon_{n-1}\lambda_{n-1}|)/(\varepsilon_n\lambda_n)^2 \to 0$.

Then the sequence (x_n) generated by the algorithm (3.15) *converges strongly to the unique solution $s^\dagger$ of VI* (1.5).

We here consider an iterative method for regularization of VI (1.1) in the case where the set Ω of feasible solutions is decomposed into the intersection (1.7) of $N \geq 1$ closed convex subsets of H. One motivation, as pointed out in [2], is that the set Ω is usually determined by a system of inequalities:

$$\Omega = \{x \in H : f_i(x) \leq 0, \quad i = 1, 2, \cdots, N\},$$

where each $f_i : H \to \mathbb{R}$ is a convex function. In this case, we have $\Omega_i = \{x \in H : f_i(x) \leq 0\}$ for $1 \leq i \leq N$.

We notice that decomposition methods are popular in nonexpansive mappings and their applications in optimization [3, 9, 10, 12, 7].

We now introduce our iterative algorithm for the regularization of VI (1.1): beginning with an initial guess $x_0 \in \Omega$, iterations are updated by the following sequential projection algorithm:

$$\begin{aligned}
x_1 &= P_{\Omega_1}(x_0 - \lambda_0(Ux_0 + \varepsilon_0 Vx_0)), \\
x_2 &= P_{\Omega_2}(x_1 - \lambda_1(Ux_1 + \varepsilon_1 Vx_1)), \\
&\vdots \\
x_N &= P_{\Omega_N}(x_{N-1} - \lambda_{N-1}(Ux_{N-1} + \varepsilon_{N-1} Vx_{N-1})), \\
x_{N+1} &= P_{\Omega_1}(x_N - \lambda_N(Ux_N + \varepsilon_N Vx_N)), \\
&\vdots
\end{aligned}$$

where (λ_n) and (ε_n) are sequences of positive real numbers.

We can rewrite the algorithm in a more compact form:

$$x_{n+1} = P_{\Omega_{[n]}}(x_n - \lambda_n(Ux_n + \varepsilon_n Vx_n)), \quad n \geq 0, \tag{3.16}$$

where $[n] = (n \mod N) + 1$. Set

$$T_n := P_{\Omega_{[n]}}(I - \lambda_n(U + \varepsilon_n V)) = P_{\Omega_{[n]}}(I - \lambda_n W_n)), \quad W_n := U + \varepsilon_n V.$$

Then the algorithm (3.16) can be rewritten as

$$x_{n+1} = T_n x_n. \tag{3.17}$$

We will always assume that λ_n and ε_n satisfy the relation (3.13) so that T_n is $(1 - \frac{1}{2}\nu\lambda_n\varepsilon_n)$-contraction for each $n \geq 0$. Consequently, T_n has a unique fixed point which is denoted as z_n; that is,

$$z_n = T_n z_n = P_{\Omega_{[n]}}(I - \lambda_n(U + \varepsilon_n V))z_n. \tag{3.18}$$

To analyze the convergence of the (explicit) algorithm (3.16), we need the convergence analysis of the implicit scheme (3.18). Towards this we need two technical assumptions:

- (A1) $S^* := \bigcap_{i=1}^N S_{\Omega_i}(U) \neq \emptyset$.
- (A2) For each bounded set Q of H, there exists a constant $K \geq 0$ such that $\|P_{\Omega_i}x - P_{\Omega_j}y\| \leq K\|x - y\|$ for all $x, y \in Q$.

Note that $S_\Omega(U) \supset S^*$. Note also that (A1) and (A2) hold trivially for the case of $N = 1$.

Lemma 3.4. *Assume (A1) and $\varepsilon_n \to 0$. Then (z_n) converges strongly to the unique solution $z^\dagger$ of the variational inequality:*

$$\langle Vz^\dagger, z - z^\dagger\rangle \geq 0, \quad z \in S^*. \tag{3.19}$$

Proof. Since z_n $(= T_n z_n)$ is the projection of $z_n - \lambda_n(Uz_n + \varepsilon_n V z_n)$ on $\Omega_{[n]}$, we obtain

$$\langle Uz_n + \varepsilon_n V z_n, z - z_n\rangle \geq 0, \quad z \in \Omega_{[n]} \supset \Omega. \tag{3.20}$$

Since U is monotone and V is ν-strongly monotone, we have, for each $\hat{x}$,

$$\langle U\hat{x}, \hat{x} - z_n\rangle \geq \langle Uz_n, \hat{x} - z_n\rangle, \quad \langle V\hat{x}, \hat{x} - z_n\rangle \geq \langle Vz_n, \hat{x} - z_n\rangle + \nu\|z_n - \hat{x}\|^2.$$

It turns out from (3.20) that $\langle U\hat{x}, \hat{x} - z_n\rangle + \varepsilon_n\langle V\hat{x}, \hat{x} - z_n\rangle - \nu\varepsilon_n\|z_n - \hat{x}\|^2 \geq 0$. Now assume $\hat{x} \in S^*$. Then $\langle U\hat{x}, \hat{x} - z_n\rangle \leq 0$ (as $\hat{x} \in S_{\Omega_{[n]}}(U)$ and $z_n \in \Omega_{[n]}$) and we get

$$\|z_n - \hat{x}\|^2 \leq \frac{1}{\nu}\langle V\hat{x}, \hat{x} - z_n\rangle. \tag{3.21}$$

It follows that $\|z_n - \hat{x}\| \leq \frac{1}{\nu}\|V\hat{x}\|$; in particular, (z_n) is bounded.

Consider the dual VI of (3.20):

$$\langle Uz + \varepsilon_n Vz, z - z_n\rangle \geq 0, \quad z \in \Omega_{[n]}. \tag{3.22}$$

Now if (z_{n_j}) is a subsequence of (z_n) weakly convergent to a point $\tilde{z}$, then (3.22) implies that

$$\langle Uz, z - \tilde{z}\rangle \geq 0, \quad z \in \Omega.$$

This is the dual VI of (1.1); hence, $\tilde{z} \in S_\Omega(U)$. Moreover, from (3.21) it follows that $\langle V\hat{x}, \hat{x} - z_{n_j}\rangle \geq 0$ for all j. Taking the limit as $j \to \infty$ yields $\langle V\hat{x}, \hat{x} - \tilde{z}\rangle \geq 0$ for every $\hat{x} \in S^*$. This is the dual VI to the VI: $\langle V\tilde{z}, \hat{x} - \tilde{z}\rangle \geq 0$, which has a unique solution due to the strong monotonicity of V. It turns out that the full sequence (z_n) is weakly convergent to a point $\tilde{z} \in S_\Omega(U)$.

Fix $1 \leq i \leq N$ and consider a subsequence (z_{n_k}) of (z_n) such that $[n_k] = i$ for all k. Then VI (3.22) is reduced to

$$\langle Uz + \varepsilon_{n_k} Vz, z - z_{n_k}\rangle \geq 0, \quad z \in \Omega_i.$$

Taking the limit as $k \to \infty$ gives that

$$\langle Uz, z - \tilde{z}\rangle \geq 0, \quad z \in \Omega_i.$$

Since $\tilde{z} \in \Omega \subset \Omega_i$, we find that $\tilde{z} \in S_{\Omega_i}(U)$ for each $1 \leq i \leq N$. It turns out that $\tilde{z} \in \cap_{i=1}^N S_{\Omega_i}(U) = S^* \subset S_\Omega(U)$. Consequently, we can replace the $\hat{x}$ in (3.21) with $\tilde{z}$ to get

$$\|z_n - \tilde{z}\|^2 \leq \frac{1}{\nu}\langle V\tilde{z}, \tilde{z} - z_n\rangle.$$

Now the weak convergence to $\tilde{z}$ of (z_n) ensures that the right side of the last relation tends to zero, hence $z_n \to \tilde{z}$ strongly. □

We are now in a position to state and prove the main result of this manuscript.

Theorem 3.3. *Assume (A1) and (A2). Assume that the sequences (λ_n) and (ε_n) satisfy the relation* (3.13) *and the conditions (ii)-(iv) of Theorem 3.2. Then the sequence (x_n) generated by the sequential projection method* (3.16) *converges strongly to the unique solution $z\dagger$ of VI* (3.19).

Proof. It suffices to prove $\|x_{n+1}-z_n\| \to 0$, where (z_n) is defined by (3.18) and converges in norm to $z^\dagger$ by Lemma 3.4. Since $x_{n+1} = T_n x_n$ and $z_n = T_n z_n$ and since T_n is $(1-\frac{1}{2}\nu\lambda_n\varepsilon_n)$-contraction, we obtain

$$\begin{aligned}\|x_{n+1} - z_n\| = \|T_n x_n - T_n z_n\| &\leq (1 - \frac{1}{2}\nu\lambda_n\varepsilon_n)\|x_n - z_n\| \\ &\leq (1 - \frac{1}{2}\nu\lambda_n\varepsilon_n)\|x_n - z_{n-1}\| + \|z_n - z_{n-1}\| \end{aligned} \tag{3.23}$$

and

$$\begin{aligned}\|z_n - z_{n-1}\| = \|T_n z_n - T_{n-1}z_{n-1}\| &\leq \|T_n z_n - T_n z_{n-1}\| + \|T_n z_{n-1} - T_{n-1}z_{n-1}\| \\ &\leq (1 - \frac{1}{2}\nu\lambda_n\varepsilon_n)\|z_n - z_{n-1}\| + \|T_n z_{n-1} - T_{n-1}z_{n-1}\|.\end{aligned}$$

It follows that

$$\|z_n - z_{n-1}\| \leq \frac{2}{\nu\lambda_n\varepsilon_n}\|T_n z_{n-1} - T_{n-1}z_{n-1}\|. \tag{3.24}$$

We have by virtue of (A2)

$$\begin{aligned}\|T_n z_{n-1} - T_{n-1}z_{n-1}\| &= \|P_{\Omega_{[n]}}(z_{n-1} - \lambda_n W_n z_{n-1}) - P_{\Omega_{[n-1]}}(z_{n-1} - \lambda_{n-1}W_{n-1}z_{n-1})\| \\ &\leq K\|(z_{n-1} - \lambda_n W_n z_{n-1}) - (z_{n-1} - \lambda_{n-1}W_{n-1}z_{n-1})\| \\ &= K\|(\lambda_n - \lambda_{n-1})Uz_{n-1} + (\lambda_n\varepsilon_n - \lambda_{n-1}\varepsilon_{n-1})Vz_{n-1}\| \\ &\leq M(|\lambda_n - \lambda_{n-1}| + |\lambda_n\varepsilon_n - \lambda_{n-1}\varepsilon_{n-1}|).\end{aligned} \tag{3.25}$$

Here M is a constant such that $M \geq K \cdot \max\{\|Uz_n\|, \|Vz_n\|\}$ for all n.

Substituting (3.24) and (3.25) into (3.23) yields

$$\|x_{n+1} - z_n\| \leq (1 - \frac{1}{2}\nu\lambda_n\varepsilon_n)\|x_n - z_{n-1}\| + \frac{2M}{\nu\lambda_n\varepsilon_n}(|\lambda_n - \lambda_{n-1}| + |\lambda_n\varepsilon_n - \lambda_{n-1}\varepsilon_{n-1}|).$$

This can equivalently be rewritten as

$$\sigma_{n+1} \leq (1 - \tau_n)\sigma_n + \tau_n\beta_n, \tag{3.26}$$

where $\sigma_n = \|x_n - z_{n-1}\|$, $\tau_n = \frac{1}{2}\nu\lambda_n\varepsilon_n$, and $\beta_n = \frac{4M}{\nu^2} \cdot \frac{|\lambda_n - \lambda_{n-1}| + |\lambda_n\varepsilon_n - \lambda_{n-1}\varepsilon_{n-1}|}{(\lambda_n\varepsilon_n)^2}$. By the conditions (iii)-(iv), we see that the following conditions are satisfied:

a) $\sum_{n=1}^{\infty} \lambda_n\varepsilon_n = \infty$, b) $\lim_{n\to\infty} \beta_n = 0$.

Applying Lemma 2.2 (with $\delta_n = 0$ for all n) immediately yields $\sigma_n \to 0$. □

Remark 3.1. It is easy to verify that the choices $\lambda_n := \frac{c}{(1+n)^a}$, $\varepsilon_n := \frac{1}{(1+n)^b}$ satisfy the relation (1.4) and the conditions (ii)-(iv) of Theorem 3.2, where $a, b > 0$ are such that $a \geq b$ and $a + 2b < 1$, and $0 < c < \nu(L + \ell)^{-2}$.

Remark 3.2. The way of the proof of Theorem 3.3 given above seems to be indirect for the reason that we prove $\|x_{n+1} - z_n\| \to 0$; thus the convergence to $z^\dagger$ of (x_n) follows from that of (z_n). It is an open question whether one can prove the convergence to $z^\dagger$ of (x_n) in a direct way by estimating $\|x_n - z^\dagger\|$ and then proving $\|x_n - z^\dagger\| \to 0$. If the answer is positive, then we conjecture that the square raised to the denominator $(\lambda_n\varepsilon_n)$ in the condition (iv) can be removed. It was recently proved [1] that a direct proof of strong convergence works for the Krasnoselskii-Mann viscosity approximation method for fixed points of nonexpansive mappings.

4. An application

Consider the constrained linear inverse problem [5, 6, 8, 11]

$$Bx = f \quad \text{(subject to } x \in \Omega), \tag{4.27}$$

where B is a bounded linear operator on a Hilbert space H, $f \in H$ is a given element, and Ω is a nonempty closed convex subset of H. Let $S_\Omega(f)$ denote the set of solutions of (4.27) and assume it is nonempty. The least-squares method for solving (4.27) is the minimization problem

$$\min_{x \in \Omega} \Gamma(x) := \frac{1}{2}\|Bx - f\|^2. \tag{4.28}$$

The Tikhonov regularization of (4.28) is the minimization problem

$$\min_{x \in \Omega} \Gamma_\varepsilon(x) := \frac{1}{2}\|Bx - f\|^2 + \frac{1}{2}\varepsilon\|x\|^2. \tag{4.29}$$

Define the operators U and U_ε by

$$Ux := \nabla\Gamma(x) = B^*(Bx - f), \quad U_\varepsilon x := \nabla\Gamma_\varepsilon(x) = B^*(Bx - f) + \varepsilon x = Ux + \varepsilon x,$$

where B^* is the adjoint of B.

Observe that the minimization problems (4.28) and (4.29) are equivalent to the variational inequalities

$$x^* \in \Omega, \quad \langle Ux^*, x - x^*\rangle \geq 0, \quad x \in \Omega \tag{4.30}$$

and, respectively,

$$x_\varepsilon \in \Omega, \quad \langle Ux_\varepsilon + \varepsilon x_\varepsilon, x - x_\varepsilon\rangle \geq 0, \quad x \in \Omega. \tag{4.31}$$

This shows that the algorithm (3.16) and Theorem 3.3 are applicable to the problem (4.27). Consequently, we have the following result.

Theorem 4.4. *Suppose Ω is of form* (1.7) *and the assumptions (A1) and (A2) hold. Assume (λ_n) and (ε_n) are sequences of positive real numbers satisfying the relations $\lambda_n < \varepsilon_n/(\|B\|^2 + \varepsilon_n)^2$ and the conditions (ii)-(iv) of Theorem 3.2. Let $x_0 \in \Omega$ and define a sequence (x_n) by the iteration process:*

$$x_{n+1} = P_{\Omega_{[n]}}\left((1 - \lambda_n\varepsilon_n)x_n - \lambda_n B^*B(x_n) + \lambda_n f\right), \quad n = 0, 1, 2, \cdots. \tag{4.32}$$

Then (x_n) converges strongly to the solution $z^\dagger$ of (4.27) *which is the minimum-norm element of S^*, namely, $z^\dagger = \arg\min\{\|z\| : z \in S^*\}$.*

Proof. Since $Ux = B^*(Bx - f)$ and $V = I$, it is easily seen that the algorithm (3.17) is reduced to (4.32). Then applying Theorem 3.3 yields the strong convergence to $x^\dagger$ of the sequence (x_n). Finally, we notice that when $V = I$, the solution $z^\dagger$ of VI (3.19) coincides with the minimum-norm element of S^* [15]. This completes the proof. □

5. Conclusion

We have studied an iterative method for the regularization of an ill-posed variational inequality in a Hilbert space H in the case where the set of feasible solutions is decomposed to the intersection of finitely many closed convex subsets. This seems to be the first time in the literature for iterative methods for ill-posed variational inequalities in the domain decomposition case. We have proved the strong convergence of the sequence generated by our iterative method. However our proof is indirect in the sense that the strong convergence of our (explicit) iterative method is proved through the strong convergence of an implicit method. It is unclear if a direct proof can be provided. If the answer is positive, then we guess that the conditions imposed on the sequences of parameters in our iterative method can be weakened (e.g., the square raised in the denominator of condition (iv) of Theorem 3.2 may be removed).

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