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On Hyper-Chordal graphs

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ABSTRACT. Triangulated graphs have many interesting properties (perfection, recognition algorithms, combinatorial optimization algorithms with linear complexity). Hyper-triangulated graphs are those where each induced subgraph has a hyper-simplicial vertex. In this paper we give the characterizations of hyper-triangulated graphs using an ordering of vertices and the weak decomposition. We also offer a recognition algorithm for the hyper-triangulated graphs, the inclusions between the triangulated graphs generalizations and we show that any hyper-triangulated graph is perfect.

1. INTRODUCTION

Due to the properties of the triangulated (chordal) graphs (i.e., perfection, recognition algorithms and ability to solve some combinatorial optimization problems (determining the stability number and minimum number of covering cliques) with linear complexity algorithms), various ways of generalizing this notion were introduced. Among these, we quote the class of weakly triangulated graphs (R. Hayward [6], see [3]. A graph is weakly triangulated if neither it nor its complement contains a chordless cycle with five or more vertices (ie $\{C_k, \overline{C}_k\} \not\subseteq G, \forall k \geq 5$)).

Another generalization is the class of slightly triangulated graphs (F. Maire [8]. A graph G is slightly triangulated if it satisfies the two following conditions: 1. G contains no chordless cycle with at least 5 vertices. 2. For every induced subgraph H of G, there is a vertex in H with the neighborhood in H contains no chordless path of 4 vertices.).

Another generalization is the class of quasi-triangulated graphs (Ion Gorgos, Chinh T. Hoang, and Vitaly Voloshin [4]. A graph is quasi-triangulated if each of its induced subgraphs has a vertex which is either simplicial (its neighbors form a clique) or cosimplicial (its nonneighbors form an independent set). The authors prove that a graph G is quasi-triangulated if and only if each induced subgraph H of G contains a vertex that does not lie in a hole, or an antihole (A hole is an induced subgraf that is a chordless cycle with at least 4 vertices and an antihole is complementary to a hole). In addition, the authors present an algorithm that recognizes a quasi-triangulated graph in $O(nm)$ time).

A graph G is hyper-triangulated if each of its induced subgraphs has a vertex which is hyper-simplicial (ie, the connected components of its neighbors in the complement of G ($\overline{G}$) are modules).

2. PRELIMINARIES

According to ([2]), $G = (V, E)$ is a connected, finite and undirected graph, without loops and multiple edges, having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ (or *co-G*) is the complement of G. If $U \subseteq V$, by $G(U)$ (or $[U]_G$, or $[U]$) we denote the subgraph of G induced by U. By $G - X$ we mean the subgraph $G(V - X)$, whenever $X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$. If $e = xy$ is an edge of a

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graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy \in E$, we also use the property $x \sim y$, and $x \not\sim y$ whenever x, y are not adjacent in G. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A, B are *totally adjacent* and we denote by $A \sim B$, while by $A \not\sim B$ we mean that no edge of G joins some vertex of A to a vertex from B and, in this case, we say A and B are *totally non-adjacent*. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' *isomorphic*, and write $G \simeq G'$, if there exists a bijection $\varphi : V \to V'$ with $xy \in E$ if and only if $\varphi(x)\varphi(y) \in E'$ for all $x, y \in V$.

The neighborhood of the vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$, while $N_G[v] = \{v\} \cup N_G(v)$; we denote$N(v)$ and $N[v]$, when G appears clearly from the context. The degree of v in G is $d_G(v) = |N_G(v)|$. The neighborhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$. If the degree of v in G is $d_G(v) = 0$ then v is *isolated* vertex.

The neighborhood of $S \subset V$ is the set $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = S \cup N(S)$. A graph is complete if every pair of distinct vertices is adjacent.

A clique is a subset Q of V with the property that $G(Q)$ is complete. A stable set is a subset X of vertices where every two vertices are not adjacent.

By P_n, C_n, K_n we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

A hole is a chordless cycle with at least four vertices, and an antihole is the complement of a hole.

Let F denote a family of graphs. A graph G is called F *- free* if none of its subgraphs are in F.

The *sum* of the graphs G_1, G_2 is the graph $G = G_1 + G_2$ having:
$V(G) = V(G_1) \cup V(G_2)$,
$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

A class $\mathcal{H}$ of graphs is called *hereditary* if every induced subgraph of a graph in $\mathcal{H}$ is in $\mathcal{H}$.

3. RESULTS ON HYPER-CHORDAL GRAPHS

3.1. **The weak decomposition of a graph.** We recall a characterization of the weak decomposition of a graph.

Definition 3.1. ([10,11]) A set $A \subset V(G)$ is called a weak set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weak set, maximal with respect to set inclusion, then $G(A)$ is called a weak component. For simplicity, the weak component $G(A)$ will be denoted with A.

Definition 3.2. ([10,11]) Let $G = (V, E)$ be a connected and non-complete graph. If A is a weak set, then the partition $\{A, N(A), V - A \cup N(A)\}$ is called a weak decomposition of G with respect to A.

The name of *weak component* is justified by the following result.

Theorem 3.1. ([11]) *Every connected and non-complete graph $G = (V, E)$ admits a weak component A such that $G(V - A) = G(N(A)) + G(\overline{N}(A))$.*

Theorem 3.2. ([11]) *Let $G = (V, E)$ be a connected and non-complete graph and $A \subset V$. Then A is a weak component of G if and only if $G(A)$ is connected and $N(A) \sim \overline{N}(A)$.*

The next result, that follows from Theorem 3.2 , ensures the existence of a weak decomposition in a connected and non-complete graph .

Corollary 3.1. *If $G = (V, E)$ is a connected and non-complete graph, then V admits a weak decomposition (A, B, C), such that $G(A)$ is a weak component and $G(V - A) = G(B) + G(C)$.*

Theorem 3.2 provides an $O(n+m)$ algorithm for building a weak decomposition for a non-complete and connected graph.

Algorithm for the weak decomposition of a graph ([10], see [11])

Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.

Output: A partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A)$,
$R = \overline{N}(A)$ and $A \not\sim R$.

Begin
$A :=$ any set of vertices such that $A \cup N(A) \neq V$
$N := N(A)$
$R := V - A \cup N(A)$
While ($\exists$n$\in N$, $\exists$r$\in R$ such that nr$\notin E$) *do*
Begin
$A := A \cup \{n\}$
$N := (N - \{n\}) \cup (N(n) \cap R)$
$R := R - (N(n) \cap R)$
end
end

3.2. **Hyper-triangulated graphs.**

3.2.1. *Definition of hyper-triangulated graphs.*

Definition 3.3. A non-empty subset A of $V(G)$ is called a module if $\forall x \in V(G) - A$ either $x \sim A$ or $x \not\sim A$. If A is a proper subset of $V(G)$ with at least two elements, then A is called a homogeneous set.

Definition 3.4. A vertex x of G is called simplicial if $[N(x)]_G$ is complete.

Remark 3.1. If x is a simplicial vertex of G then the connected components of $[N(x)]_{\overline{G}}$ are modules.

Since $N_{\overline{G}}(x)$ is a stable set.

Definition 3.5. ([9]) Let H be a graph, $x \in V(H)$ and let $[N_1]_{\overline{H}}, \cdots, [N_p]_{\overline{H}}$ $(p \geq 1)$ be the connected components of $[N(x)]_{\overline{H}}$. The vertex x is called a hyper-simplicial vertex in H if all sets $N_1, \cdots, N_p$ are modules in H. $N_1, \cdots, N_p$ are called corresponding components of x.

The definition is based on the observation according to which for a simplicial vertex, the connected components of the subgraph induced by its neighborhood in $\overline{G}$ are modules.

3.2.2. *Inclusions between the generalizations of the triangulated graphs.*

Definition 3.6. We call hyper-triangulated graph, a graph G so that any induced subgraph has a hyper-simplicial vertex.

Remark 3.2. Any simplicial vertex is a hyper-simplicial vertex.

If x is simplicial in H then the connected components of $[N(x)]_{\overline{H}}$ are isolated vertices.

Definition 3.7. A vertex x which is not interior to any P_4 in H is called $\overline{P}_4 - free$.

Theorem 3.3. ([9], the result is also used and in [10]) *A vertex x of a graph H is hyper-simplicial of H if and only if it is* $\overline{P}_4 - free$.

Corollary 3.2. *If x is hyper-simplicial in H then x is hyper-simplicial in any induced subgraph that contains x.*

Ifx is $\overline{P}_4 - free$ in H then x is $\overline{P}_4 - free$ in any induced subgraph.

Definition 3.8. A graph G is called sequential $C_k - free$ if $\forall H \subseteq G$ as induced subgraph, H has the property that $\exists x \in V(H)$ so that $x \notin C_k$ $(k \geq 5)$. A graph G is called sequential $\overline{C}_k - free$ if $\forall H \subseteq G$ as induced subgraph, H has the property that $\exists x \in V(H)$ so that $x \notin \overline{C}_k$ $(k \geq 5)$.

We consider:
T: the class of triangulated graphs;
T_{HT}: the class of hyper-triangulated graphs;
T_{ST}: the class of weakly triangulated graphs;
$T(C_k)$: the class of sequentially $C_k - free$ $(k \geq 5)$ graphs;
$T(\overline{C}_k)$: the class of sequentially $\overline{C}_k - free$ $(k \geq 5)$ graphs;
$T_C = T(C_k) \cap ((\overline{C}_k)$.

Theorem 3.4. *$T \subseteq T_{HT} \subseteq T_C = T_{ST}$ holds.*

Proof. I) Let $G \in T$ be and H be an induced subgraph of G. H is triangulated, so H has a simplicial vertex v. Since all connected components of $N_H(v)$ in $\overline{H}$ are isolated vertices, it follows that each connected component of $N_H(v)$ in $\overline{H}$ is a module in H. Therefore, v is a hyper-simplicial vertex of H. So $G \in T_{HT}$.

II) Let $G \in T_{HT}$. A) We show $G \in T(C_k)$. So, $\forall H \subseteq G$, H has at least one hyper-simplicial vertex, i.e. $\exists x \in V(H)$, $x\,\overline{P}_4 - free$ in H, i.e. x is not the inside vertex of any P_4 of H. If $G \notin T(C_k)$ then $\exists H \subseteq G$ such that$H \simeq C_k (k \geq 5)$. Any vertex t of H is an inside vertex of an induced P_4, made of four consecutive vertices during a traversal of C_k which contained t inside. So, $\forall t \in V(H)$ is not $\overline{P}_4 - free$, i.e. not is hyper-simplicial, contrary to the assumption that $G \in T_{HT}$. Therefore $G \in T(C_k)$.

B) We show that $T_{HT} \subseteq T(\overline{C}_k)$. Suppose $G \notin T(\overline{C}_k)$. There is H an induced subgraph of G so that $\forall x \in V(H)$: $x \in \overline{C}_k$ $(k \geq 5)$, i.e. $H \simeq \overline{C}_k$ $(k \geq 5)$. For $H = \overline{C}_k$ $(k \geq 5)$ with the consecutive vertices $1, 2, ..., k$, where any two consecutive vertices are not adjacent, we have: the vertex $i(i = 1, ..., k)$ is an inside vertex of $P_4 : i-2, i, i-3, i-1, i = 1, ..., k$ modulo k. So, in H any vertex v is an inside vertex of P_4, i.e. v is not hyper-simplicial. So, G is not hyper-triangulated, a contradiction.

III) Obviously $T_{ST} \subseteq T_C$. We show that $T_C \subseteq T_{ST}$. Let $G \in T_C$. If $G \notin T_{ST}$ then there is H, an induced subgraph of G, so that $H \simeq C_k$ or $H \simeq \overline{C}_k$. But then, either for $\forall x \in V(H)$, $x \in C_k$ or for $\forall x \in V(H)$, $x \in \overline{C}_k$, a contradiction. □

Below we shall some remarks showing the inclusion between the generalizations of the triangulated graphs.

Remark 3.3. Let G be $\{C_4, \overline{C}_4\} - free$ graph. If G is hyper-triangulated graph then G is quasi-triangulated graph.

Proof. Let G be a hyper-triangulated graph. So $\forall H \subseteq G$, the H induced subgraph has at least one hyper-simplicial vertex, i.e. $\exists x \in V(H)$, $x\,\overline{P}_4 - free$ in H, i.e. x is not an inside vertex of any P_4 of H.

If G is not quasi-triangulated, as G is $\{C_4, \overline{C}_4\} - free$, $\exists H_0 \subseteq G$ so that $H_0 \simeq C_k (k \geq 5)$ or $H_0 \simeq \overline{C}_k (k \geq 5)$. Let $H_0 \simeq C_k (k \geq 5)$. Any vertex t of H_0 is an inside vertex of an induced P_4 made of four consecutive vertices during a traversal of C_k which contained t in inside. So, $\forall t \in V(H_0)$ is not $\overline{P}_4 - free$, i.e. not is hyper-simplicial, contrary to the assumption that G is hyper-triangulated. For $H_0 \simeq \overline{C}_k(k \geq 5)$ the consecutive vertices $1, 2, ..., k$, where any two consecutive vertices are not adjacent is checked : the vertex $i(i = 1, ..., k)$ is the inside vertex of $P_4 : i-2, i, i-3, i-1, i = 1, ..., k$ modulo k . So, in H_0, any vertex v is interior of P_4, i.e. v is not hyper-simplicial. So, G is not hyper-triangulated which is a contradiction. □

Remark 3.4. There are hyper-triangulated graphs that are not slightly triangulated graphs.

Let G be the graph consisting of two $P_4 : x_1, x_2, x_3, x_4$ and $P_4 : x_5, x_6, x_7, x_8$ so that any vertex of a P_4 is adjacent to any vertex of the other P_4. The neighborhood of any vertex contains P_4, so G is not slightly triangulated. On the other hand, there is the sequence $[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$ so that the vertex x_i is hyper-simplicial in $G_i = [x_i, \cdots, x_8]_G$, $i = 1, \cdots, 8$. So G is hyper-triangulated (see Theorem 3.6).

Remark 3.5. There are slightly triangulated graphs that are not hyper-triangulated graphs.

Let G be the graph obtained from $C_9 : x_1, \cdots, x_9$ to which we add the edges x_2x_5, x_5x_8, x_8x_2. Since $C_k \not\subset G, (k \geq 5)$ and the neighborhood of any vertex in any induced subgraph of G does not contain P_4, it follows that G is slightly triangulated. For $H = G - \{x_3, x_4\}$, as each vertex of H is inside a P_4, it follows that G is not hyper-triangulated.

Remark 3.6. Let G be a $C_4 - free$ graph. G is a triangulated graphs if and only if G is hyper - triangulated graphs.

If G is triangulated, according to the Theorem 3.4 it follows that G is hyper-triangulated. If G is hyper-triangulated then according to the Theorem 3.4, G is $C_k - free$, $(k \geq 5)$. Since G is $C_4 - free$, it results that G is triangulated.

Remark 3.7. Let G be a $C_4 - free$ graph. G is a slightly triangulated graph if and only if G is a hyper- triangulated graph.

Let G be hyper-triangulated and $C_4 - free$. From the Remark 3.6, it follows that G is triangulated. So, $\forall k \geq 4$, $C_k \not\subset G$ and $\forall H \subseteq G$, $\exists x \in V(H)$ so that $\overline{P_2} \not\subset N_H(x)$. Therefore, $\forall k \geq 5$, $C_k \not\subset G$ and $\forall H \subseteq G$, $\exists x \in V(H)$ so that $\overline{P_4} \not\subset N_H(x)$, i.e. G is slightly triangulated.

Let G be slightly triangulated and $C_4 - free$. Then $\forall k \geq 4$, $C_k \not\subset G$ and $\forall H \subseteq G$, $\exists x \in V(H)$ so that $\overline{P_4} \not\subset N_H(x)$. Thus, G is triangulated. According to the Theorem 3.4, it follows that G is hyper-triangulated.

Theorem 3.5. *Any quasi-triangulated graph is weakly triangulated.*

Let G be a quasi-triangulated graph and $\forall H$, induced subgraph. Then $\exists x \in V(H)$ so that $x \notin C_k(k \geq 4)$ and $x \notin \overline{C}_k(k \geq 4)$. According to the Theorem 3.4, it follows that $T_{ST} = T_C$, where $T_C = T(C_k) \cap T(\overline{C}_k), (k \geq 5)$. Since, if $x \notin C_k(k \geq 4)$ then $x \notin C_k(k \geq 5)$ and if $x \notin \overline{C}_k(k \geq 4)$ then $x \notin \overline{C}_k(k \geq 5)$, it follows that G is weakly triangulated.

3.2.3. *Characterizatios of the hyper-triangulated graphs by using an ordering of vertices set.*

Definition 3.9. Let G= (V,E) be a graph. A proper subset $A \subset V$ is called hyper-stable if the connected components of $[A]_G$ are modules. A proper subset $A \subset V$ is called a hyper-clique if A is hyper-stable in $\overline{G}$.

Remark 3.8. Any clique is hyper-clique.

Remark 3.9. A vertex v is hyper-simplicial if and only if the neighborhood of $v, N(v)$, is a hyper-clique.

Theorem 3.6. *$AgraphG = (V, E)$ is hyper-triangulated if and only if there is an ordering of vertices $x_1, x_2, \cdots, x_n$ so that the vertex x_i is hyper-simplicial in the graph G_i induced in G by the set $\{x_i, \cdots, x_n\}$ for $i = 1, \ldots, n$.*

Proof. 1. If G is a hyper-triangulated graph then we note with x_i the hyper-simplicial vertex of G_i=G-$\{x_1, \cdots, x_{i-1}\}$, for $i = 1, \ldots, n$. The order $x_1, x_2, \cdots, x_n$ thus obtained satisfies the condition from the enunciation.

2. We consider $x_1, x_2, \cdots, x_n$ an ordering of vertices of G satisfying the condition from the enunciation, and H an induced subgraph of G. The order of the vertices in G induces an order of vertices in H. We consider $x \in V(H)$, the first element in the order of vertices of H, induced by the order of G. If $x = x_j$ then x is hyper-simplicial in G_j. According to the choice of x, we have $V(H) \subseteq V(G_j)$. Then x is hyper-simplicial in H, since x being $\overline{P}_4 - free$ in G_j is $\overline{P}_4 - free$ in H. □

3.2.4. *Characterizatios of the hyper-triangulated graphs by using weak decomposition.*

Remark 3.10. If A is hyper-clique in G then A is hyper-clique in any induced subgraph H of G with $A \subseteq V(H)$.

Theorem 3.7. ([10]) *$LetG = (V, E)$ be a connected and incomplete graph. G is triangulated if and only if, for any weak decomposition $(A; N; R)$ with $G(A)$ weak component:*
1) N is a clique
2) $G(R)$ and $G(V - R)$ are triangulated.

A result similar to the triangulated graphs, from Theorem 3.7, it follows and for hyper-triangulated graphs, presented below.

Theorem 3.8. *Let $G = (V, E)$ be a connected and incomplete graph. G is hyper-chordal if and only if for any weak decomposition (A, N, R) with $G(A)$ weak component:*

(1) N is hyper-clique
(2) $G(R)$ and $G(V - R)$ are hyper-chordal.

Proof. I). Let G be a hyper-triangulated graph. Then $G(R)$ and $G(V - R)$ are hyper-triangulated graphs. So, for every induced subgraph, F of $G(R)$, there is $x \in V(F)$, x hyper-simplicial vertex. Thus, $N(x)$ is hyper-clique. Since $x \in V(F) \subseteq R$ and $R \sim N$, it follows that $N \subseteq N(x)$. So, N is the hyper-clique.

II) Suppose (1) and (2) hold. We show that G is a hyper-triangulated graph. If G is not a hyper-triangulated graph, $\exists H \subseteq G$, H as an induced subgraph, so that $\forall x \in V(H)$, x is not hyper-simplicial vertex in H, i.e. $\exists P_4$ in H so that x is an interior vertex of P_4.

P_4 does not exist in $G(R)$ so that x is an interior vertex, P_4 does not exist in $G(V - R)$ so that x is an interior vertex. There could be P_4 with $V(P_4) \cap A \neq \phi$, $V(P_4) \cap N \neq \phi$ and $V(P_4) \cap R \neq \phi$. The only possibilities would be (with x an interior vertex) Case I: $P_4 : a, an, n, nr, r, rn', n'$, where $a \in A, n, n' \in N, r \in R$; Case II: $P_4 : a', a'a, a, an, n, nr, r$, where $a', a \in A, n \in N, r \in R$. Since N is the hyper-clique, according to the Remark 3.9, it results that $\exists v \in A \cup R$ so that v is hyper-simplicial vertex and $N \subseteq N(v)$. For v∈R, since v is a hyper-simplicial vertex, it follows that v is not inside any P_4, i.e. case I does not hold. For v∈A, since v is a hyper-simplicial vertex, it follows that v is not inside any P_4, i.e. case II not hold. □

3.2.5. *Recognition algorithm for hyper-triangulated graphs.*

The above results lead to the following recognition algorithm.

Consequence 1. *Recognition algorithm for hyper-triangulated graphs*
Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.
Output: An answer to the question: is G a *hyper-chordal* graph ?
begin
$L := \{G\}$;*//* L *a list of graphs*
While ($L \neq \phi$)
Extract an element F *from* L;
Find a weak decomposition (A, N, R) *for* F;
If (N *not hyper-clique in* F) *then*
Return: G *is not hyper-chordal*
else introduce in L *the connected components of* $G(R), G(V - R)$ *incomplete*
Return: G *is hyper-chordal*
end

The fact that N is a hyper-clique is proved by using Remark 3.9 and Theorem 3.3 (i.e. N is hyper-clique if and only if $\exists v \in A \cup R$ such that v is not inside any P_4). Since Theorem 3.2 provides an $O(n+m)$ algorithm for building a weak decomposition for a non-complete and connected graph and since the recognition of P_4-free graphs is performed in $O(n+m)$ ([5]) time and finding of small cycles in undirected graphs in $O(m^{1,63})$ ([1]) time, then, the algorithm is run in $O(n(n + m^{1,63}))$.

3.2.6. **Hyper-triangulated graphs are perfect.**

Using Theorem 3.4, we show that any hyper-triangulated graph is perfect.

A graph is perfect if for all induced subgraphs $H : \chi(H) = \omega(H)$, where χ is the chromatic number (the chromatic number of a graph is the minimum number of colours needed to label all its vertices in such a way that no two vertices with the same color are adjacent) and ω is the size of a maximum clique.

A graph G is minimal imperfect if and only if G is not perfect and any subgraph $G - v(v \in V(G))$ is perfect. A graph G is unbreakable if it has at least three vertices and neither G nor $\overline{G}$ no star cutset (A nonempty of vertices set T in the connected graph $G = (V, E)$ is called star cutset if $G - T$ is non-connected and there is a vertex v of T which is adjacent to all the remaining vertices of T).

In ([7]) the author show that every vertex in an unbreakable graph is in a disc, where a disc is a chordless cycle, or the complement of a chordless cycle, with at least five vertices. A corollary is that every vertex in a minimal imperfect graph is in a disc.

Theorem 3.9. *Any hyper-triangulated graph is perfect.*

Proof. Let G be a hyper-triangulated graph. If G is minimal imperfect then G is unbreakable ([7]). So, any vertex of G is a disc ([7]). So, $G \notin T(C_k)$ or $G \notin T(\overline{C_k})$ $(k \geq 5)$, contradicting Theorem 3.4. So G is perfect. □

4. Conclusions

In this paper, we have shown several characterizations of the hyper-chordal graphs as another generalization by using the ordering of vertices according to the Theorem 3.6 as well as the weak decomposition based on the Theorem 3.8. Their properties also emerge from the inclusions between the generalizations of the chordal graphs shown by several theorems and remarks,such as the Theorem 3.4, Remark 3.3... 3.7 and Theorem 3.5. The $O(n(n + m^{1,63}))$ recognition algorithm is a useful tool for identifying the hyper-chordal graphs according to the Consequence 1. In addition, we have proven that any hyper-chordal graph is perfect according to the Theorem 3.9.

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