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Univalence criteria for a general integral operator

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ABSTRACT. The main object of this paper is to give sufficient conditions for the general integral operator $\mathcal{T}_n$, to be univalent in the open disk $\mathbb{U}$, when $g_i, h_i, k_i \in \mathcal{G}_{b_i}$ for all $i = \overline{1, n}$. This general integral operator was considered in a recent work [Bărbatu, C. and Breaz, D., *Classes of an univalent integral operator*, Studia Univ. Babeş-Bolyai Math., accepted]. The results derived in this paper are shown to follow upon specializing the parameters involved in our results. Several corollaries of the main results are also considered.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of the functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} :| z |< 1\}$$

and satisfy the following usual normalization conditions:

$$f(0) = f^{'}(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

Let $0 < b \leq 1$. In [19] Silverman introduced the class $\mathcal{G}_b$ defined by

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < b \left| \frac{zf'(z)}{f(z)} \right|, \quad z \in \mathbb{U} \right\}. \tag{1.2}$$

We consider the integral operator

$$\mathcal{T}_n(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} (g_i'(t))^{\beta_i} \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\delta_i} \right] \mathrm{dt} \right\}^{\frac{1}{\delta}}, \tag{1.3}$$

where f_i, g_i, h_i, k_i are analytic in $\mathbb{U}$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$ for all $i = \overline{1, n}$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\mathrm{Re}\delta > 0$.

Remark 1.1. The integral operator $\mathcal{T}_n$ defined by (1.3), introduced by Bărbatu and Breaz in the paper [1] is a general integral operator of Pfaltzgraff, Kim-Merkes and Ovesea types which extends also the other operators as follows:

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i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$ and $\beta_1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [7],

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha \mathrm{dt}.$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [18],

$$\mathcal{G}_\alpha(z) = \int_0^z (f'(t))^\alpha \mathrm{dt}.$$

iii) For $\alpha_i - 1 \equiv \alpha_i$ and $\beta_i = \gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [2],

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Pascu and Pescar [14].

iv) For $\alpha_i - 1 = \gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [3],

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n [f_i'(t)]^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Pescar and Owa in [17] .

v) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by Frasin [6],

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left(f_i{}'(t)\right)^{\beta_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Ovesea in [10].

vi) For $\alpha_i - 1 = \beta_i = 0$ we obtain the integral operator which was defined and studied by Pescar [15],

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{g_i(t)}\right)^{\gamma_i} \left(\frac{f_i'(t)}{g_i'(t)}\right)^{\delta_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

Thus, the integral operator $\mathcal{T}_n$, introduced here by the formula (1.3), can be considered as an extension and a generalization of these operators above mentioned.

In the present paper, we derive the univalence conditions for the integral operator $\mathcal{T}_n$, when $g_i, h_i, k_i \in G_{b_i}$ and $f_i \in \mathcal{A}$ for all $i = \overline{1,n}$, by using three univalence criteria, which were derived by Pascu [11], [12], and Pescar [14], respectively.

Theorem 1.1. (Pascu [11]) *Let $f \in \mathcal{A}$ and $\gamma \in \mathbb{C}$. If $Re\gamma > 0$ and*

$$\frac{1 - |z|^{2Re\gamma}}{Re\gamma} \left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, *then the integral operator*

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)dt\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Theorem 1.2. (Pascu [12]) *Let* $\delta \in \mathbb{C}$ *with* $Re\delta > 0$. *If* $f \in \mathcal{A}$ *satisfies*

$$\frac{1-|z|^{2Re\delta}}{Re\delta}\left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, *then, for any complex* γ *with* $Re\gamma \geq Re\delta$, *the integral operator*

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)dt\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Pescar [14], on the other hand, proved another univalent condition asserted by Theorem 1.3.

Theorem 1.3. (Pescar [14]) *Let* γ *be complex number,* $Re\gamma > 0$ *and* c *a complex number,* $|c| \leq 1$, $c \neq -1$, *and* $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \dots$. *If*

$$\left|c\,|z|^{2\gamma} + \left(1-|z|^{2\gamma}\right)\frac{zf''(z))}{\gamma f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, *then the integral operator*

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)dt\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Finally, in our present investigation, we shall also need th familiar Schwarz Lemma [8].

Lemma 1.1. (General Schwarz Lemma [8]) *Let* f *be the function regular in the disk* $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R, R > 0\}$ *with* $|f(z)| < M$ *for a fixed number* $\mathrm{M} > 0$ *fixed. If* $f(z)$ *has one zero with multiplicity order bigger than a positive integer* m *for* $z = 0$, *then*

$$|f(z)| \leq \frac{M}{R^m} z^m, \quad z \in \mathbb{U}_R.$$

The equality for $z \neq 0$ *can hold only if*

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ *is constant.*

2. The main Univalence Criterion

Our main results give sufficient conditions for the general integral operator $\mathcal{T}_n$ defined by (1.3) to be univalent in the open disk $\mathbb{U}$.

Theorem 2.4. *Let* $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ *be complex numbers,* $c = Re\gamma > 0$, *with*

$$c \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1)|\beta_i| + 2|\gamma_i| + (4b_i + 2)|\delta_i|\right] \tag{2.4}$$

If for all $i=\overline{1,n}$, $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0<b_i\le 1$, $f_i \in \mathcal{A}$ *and*

$$\left|\frac{zf_i'(z)}{f_i(z)}-1\right|<1,\quad \left|\frac{zg_i'(z)}{g_i(z)}-1\right|<1,\quad \left|\frac{zh_i'(z)}{h_i(z)}-1\right|<1,\quad \left|\frac{zk_i'(z)}{k_i(z)}-1\right|<1, \tag{2.5}$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$, *then the integral operator* $\mathcal{T}_n$, *defined by (1.3) is in the class* $\mathcal{S}$.

Proof. We define the function

$$H_n(z)=\int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}(g_i'(t))^{\beta_i}\left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\left(\frac{h_i{}'(t)}{k_i{}'(t)}\right)^{\delta_i}\right]\mathrm{dt},$$

for all $i=\overline{1,n}$, $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0<b_i\le 1$, $f_i\in\mathcal{A}$ so, that obviously

$$H_n{}'(z)=\prod_{i=1}^n \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}(g_i'(t))^{\beta_i}\left(\frac{h_i(z)}{k_i(z)}\right)^{\gamma_i}\left(\frac{h_i'(z)}{k_i'(z)}\right)^{\delta_i}\right],$$

and

$$\begin{aligned}\frac{zH_n''(z)}{H_n'(z)}&=\sum_{i=1}^n\left[(\alpha_i-1)\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\beta_i\frac{zg_i''(z)}{g_i'(z)}+\gamma_i\left(\frac{zh_i'(z)}{h_i(z)}-\frac{zk_i'(z)}{k_i(z)}\right)\right]+\\&+\sum_{i=1}^n\delta_i\left(\frac{zh_i''(z)}{h_i'(z)}-\frac{zk_i''(z)}{k_i'(z)}\right)=\sum_{i=1}^n\left[(\alpha_i-1)\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\beta_i\left(\frac{zg_i'(z)}{g_i(z)}-1\right)\right]+\\&+\sum_{i=1}^n\left[\beta_i\left(\frac{zg_i''(z)}{g_i'(z)}-\frac{zg_i'(z)}{g_i(z)}+1\right)+\gamma_i\left(\frac{zh_i'(z)}{h_i(z)}-1\right)-\gamma_i\left(\frac{zk_i'(z)}{k_i(z)}-1\right)\right]+\\&+\sum_{i=1}^n\left[\delta_i\left(\frac{zh_i''(z)}{h_i'(z)}-\frac{zh_i'(z)}{h_i(z)}+1\right)-\delta_i\left(\frac{zk_i''(z)}{k_i'(z)}-\frac{zk_i'(z)}{k_i(z)}+1\right)\right]+\\&+\sum_{i=1}^n\left[\delta_i\left(\frac{zh_i'(z)}{h_i(z)}-1\right)-\delta_i\left(\frac{zk_i'(z)}{k_i(z)}-1\right)\right].\end{aligned}$$

Since $g_i, h_i, k_i\in\mathcal{G}_{b_i}$, $0<b_i\le1$ for all $i=\overline{1,n}$ from (1.2) and (1.5) we obtain:

$$\begin{aligned}\left|\frac{zH_n''(z)}{H_n'(z)}\right|&\le\sum_{i=1}^n\left[|\alpha_i-1|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+|\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}-\frac{zg_i'(z)}{g_i(z)}+1\right|+|\beta_i|\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\right]+\\&+\sum_{i=1}^n\left[|\gamma_i|\left|\frac{zh_i'(z)}{h_i(z)}-1\right|+|\gamma_i|\left|\frac{zk_i'(z)}{k_i(z)}-1\right|+|\delta_i|\left|\frac{zh_i''(z)}{h_i'(z)}-\frac{zh_i'(z)}{h_i(z)}+1\right|\right]+\\&+\sum_{i=1}^n\left[|\delta_i|\left|\frac{zk_i''(z)}{k_i'(z)}-\frac{zk_i'(z)}{k_i(z)}+1\right|+|\delta_i|\left|\frac{zh_i'(z)}{h_i(z)}-1\right|+|\delta_i|\left|\frac{zk_i'(z)}{k_i(z)}-1\right|\right]\le\\&\le\sum_{i=1}^n\left[|\alpha_i-1|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+|\beta_i|\,b_i\left|\frac{zg_i'(z)}{g_i(z)}\right|+|\beta_i|\left|\frac{zg_i'(z)}{g_i(z)}-1\right|+|\gamma_i|\left|\frac{zh_i'(z)}{h_i(z)}-1\right|\right]+\\&+\sum_{i=1}^n\left[|\gamma_i|\left|\frac{zk_i'(z)}{k_i(z)}-1\right|+|\delta_i|\,b_i\left|\frac{zh_i'(z)}{h_i(z)}\right|+|\delta_i|\,b_i\left|\frac{zk_i'(z)}{k_i(z)}\right|+|\delta_i|\left|\frac{zh_i'(z)}{h_i(z)}-1\right|\right]+\\&+\sum_{i=1}^n|\delta_i|\left|\frac{zk_i'(z)}{k_i(z)}-1\right|\le\sum_{i=1}^n\left[|\alpha_i-1|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+|\beta_i|\,b_i\left|\frac{zg_i'(z)}{g_i(z)}-1\right|+|\beta_i|\,b_i+\right]+\\&+\sum_{i=1}^n\left[|\beta_i|\left|\frac{zg_i'(z)}{g_i(z)}-1\right|+|\gamma_i|\left|\frac{zh_i'(z)}{h_i(z)}-1\right|+|\delta_i|\,b_i\left(\left|\frac{zh_i'(z)}{h_i(z)}-1\right|+1\right)\right]+\end{aligned}$$

$$+\sum_{i=1}^{n}\left[|\gamma_i|\left|\frac{zk_i'(z)}{k_i(z)}-1\right|+|\delta_i|\,b_i\left(\left|\frac{zk_i'(z)}{k_i(z)}-1\right|+1\right)+|\delta_i|\left|\frac{zh_i'(z)}{h_i(z)}-1\right|\right]+$$

$$+\sum_{i=1}^{n}|\delta_i|\left|\frac{zk_i'(z)}{k_i(z)}-1\right|=\sum_{i=1}^{n}\left[|\alpha_i-1|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+|\beta_i|\,b_i\left|\frac{zg_i'(z)}{g_i(z)}-1\right|+|\beta_i|\,b_i\right]+$$

$$+\sum_{i=1}^{n}\left[|\beta_i|\left|\frac{zg_i'(z)}{g_i(z)}-1\right|+|\gamma_i|\left|\frac{zh_i'(z)}{h_i(z)}-1\right|+|\gamma_i|\left|\frac{zk_i'(z)}{k_i(z)}-1\right|+|\delta_i|\,b_i\left|\frac{zh_i'(z)}{h_i(z)}-1\right|\right]+$$

$$+\sum_{i=1}^{n}\left[|\delta_i|\,b_i\left|\frac{zk_i'(z)}{k_i(z)}-1\right|+|\delta_i|\,2b_i+|\delta_i|\left|\frac{zh_i'(z)}{h_i(z)}-1\right|+|\delta_i|\left|\frac{zk_i'(z)}{k_i(z)}-1\right|\right]=$$

$$=\sum_{i=1}^{n}\left[|\alpha_i-1|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+(|\beta_i|\,b_i+|\beta_i|)\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\right]+$$

$$+\sum_{i=1}^{n}\left[(|\gamma_i|+|\delta_i|\,b_i+|\delta_i|)\left(\left|\frac{zk_i'(z)}{k_i(z)}-1\right|+\left|\frac{zh_i'(z)}{h_i(z)}-1\right|\right)+|\beta_i|\,b_i+2\,|\delta_i|\,b_i\right]\leq$$

$$\leq\sum_{i=1}^{n}\left[|\alpha_i-1|+(2b_i+1)\,|\beta_i|+2\,|\gamma_i|+(4b_i+2)\,|\delta_i|\right],\tag{2.6}$$

which readily shows that

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq\frac{1-|z|^{2c}}{c}\left(\sum_{i=1}^{n}\left[|\alpha_i-1|+(2b_i+1)\,|\beta_i|+2\,|\gamma_i|+(4b_i+2)\,|\delta_i|\right]\right)\leq$$

$$\leq\frac{1}{c}\left(\sum_{i=1}^{n}\left[|\alpha_i-1|+(2b_i+1)\,|\beta_i|+2\,|\gamma_i|+(4b_i+2)\,|\delta_i|\right]\right)\leq 1.\tag{2.7}$$

By Theorem 1.1 it results that the integral operator $\mathcal{T}_n$ given by (1.3) is in the class $\mathcal{S}$. □

Theorem 2.5. *Let $\alpha_i,\beta_i,\gamma_i,\delta_i$ be complex numbers, $M_i\geq 1$, $N_i\geq 1$, $P_i\geq 1$, $Q_i\geq 1$ are real numbers, for all $i=\overline{1,n}$ and $\gamma\in\mathbb{C}$ with $c=Re\gamma$*

$$c\geq\sum_{i=1}^{n}\left[|\alpha_i-1|\,(2M_i+1)+(b_i\,|\beta_i|+|\beta_i|)\,(2N_i+1)\right]+$$

$$+\sum_{i=1}^{n}\left[(|\gamma_i|+|\delta_i|\,b_i+|\delta_i|)\,(2P_i+2Q_i+2)+b_i\,|\beta_i|+2b_i\,|\delta_i|\right].\tag{2.8}$$

If for all $i=\overline{1,n}$, $g_i,h_i,k_i\in\mathcal{G}_{b_i}$, $0<b_i\leq 1$ $f_i\in\mathcal{A}$ satisfy

$$\left|\frac{z^2f_i'(z)}{[f_i(z)]^2}-1\right|<1,\ \left|\frac{z^2g_i'(z)}{[g_i(z)]^2}-1\right|<1,\ \left|\frac{z^2h_i'(z)}{[h_i(z)]^2}-1\right|<1,\ \left|\frac{z^2k_i'(z)}{[k_i(z)]^2}-1\right|<1,\tag{2.9}$$

$$|f_i(z)|\leq M_i,\ |g_i(z)|\leq N_i,\ |h_i(z)|\leq P_i,\ |k_i(z)|\leq Q_i,\tag{2.10}$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$, then for any complex number δ with $Re\delta\geq Re\gamma$, the integral operator $\mathcal{T}_n$, given by (1.3) is in the class $\mathcal{S}$.

Proof. From the proof of Theorem 2.4, we have

$$\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \sum_{i=1}^{n}\left[|\alpha_i-1|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+(|\beta_i|\,b_i+|\beta_i|)\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\right]+$$

$$+\sum_{i=1}^{n}\left[(|\gamma_i|+|\delta_i|\,b_i+|\delta_i|)\left(\left|\frac{zk_i'(z)}{k_i(z)}-1\right|+\left|\frac{zh_i'(z)}{h_i(z)}-1\right|\right)+|\beta_i|\,b_i+2\,|\delta_i|\,b_i\right].$$

Thus, we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+(|\beta_i|\,b_i+|\beta_i|)\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\right]+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[(|\gamma_i|+|\delta_i|\,b_i+|\delta_i|)\left(\left|\frac{zk_i'(z)}{k_i(z)}-1\right|+\left|\frac{zh_i'(z)}{h_i(z)}-1\right|\right)+|\beta_i|\,b_i+2\,|\delta_i|\,b_i\right]\le$$

$$\le \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\left(\left|\frac{z^2f_i'(z)}{[f_i(z)]^2}\right|\left|\frac{f_i(z)}{z}\right|+1\right)\right]+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[(b_i\,|\beta_i|+|\beta_i|)\left(\left|\frac{z^2g_i'(z)}{[g_i(z)]^2}\right|\left|\frac{g_i(z)}{z}\right|+1\right)\right]+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[(|\gamma_i|+|\delta_i|\,b_i+|\delta_i|)\left(\left|\frac{z^2h_i'(z)}{[h_i(z)]^2}\right|\left|\frac{h_i(z)}{z}\right|+1\right)\right]+$$

$$(2.11)\qquad +\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[(|\gamma_i|+|\delta_i|\,b_i+|\delta_i|)\left(\left|\frac{z^2k_i'(z)}{[k_i(z)]^2}\right|\left|\frac{k_i(z)}{z}\right|+1\right)+|\beta_i|\,b_i+2\,|\delta_i|\,b_i\right].$$

Since $|f_i(z)|\le M_i$, $|g_i(z)|\le N_i$, $|h_i(z)|\le P_i$, $|k_i(z)|\le Q_i$, $z\in\mathbb{U}$, $i=\overline{1,n}$ and for each f_i, g_i, h_i, k_i satisfy conditions (2.9) and (2.10) , then applying General Schwarz Lemma, we obtain

$$|f_i(z)|\le M_i\,|z|,\quad |g_i(z)|\le N_i\,|z|,\quad |h_i(z)|\le P_i\,|z|,\quad |k_i(z)|\le Q_i\,|z|,$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$. Using these inequalities from (2.11) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}|\alpha_i-1|\left(\left|\frac{z^2f_i'(z)}{[f_i(z)]^2}-1\right|M_i+M_i+1\right)+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}(b_i\,|\beta_i|+|\beta_i|)\left(\left|\frac{z^2g_i'(z)}{[g_i(z)]^2}-1\right|N_i+N_i+1\right)+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}(|\gamma_i|+|\delta_i|\,b_i+|\delta_i|)\left(\left|\frac{z^2h_i'(z)}{[h_i(z)]^2}-1\right|P_i+P_i+1\right)+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[(|\gamma_i|+|\delta_i|\,b_i+|\delta_i|)\left(\left|\frac{z^2k_i'(z)}{[k_i(z)]^2}-1\right|Q_i+Q_i+1\right)+|\beta_i|\,b_i+2\,|\delta_i|\,b_i\right]\le$$

$$\le \frac{1}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\,(2M_i+1)+(b_i\,|\beta_i|+|\beta_i|)\,(2N_i+1)\right]+$$

$$+\frac{1}{c}\sum_{i=1}^{n}\left[(|\gamma_i|+|\delta_i|\,b_i+|\delta_i|)\,(2P_i+2Q_i+2)+b_i\,|\beta_i|+2b_i\,|\delta_i|\right],$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and from the hypothesis, we get

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq 1, \qquad z \in \mathbb{U}.$$

Applying Theorem 1.2 for the function H_n, we prove that $\mathcal{T}_n$ defined by (1.3) is in the class $\mathcal{S}$. □

Theorem 2.6. *Let $\alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $\delta \in \mathbb{C}$ with*

$$Re\delta \geq \sum_{i=1}^{n}\left[|\alpha_i - 1| + (2b_i+1)|\beta_i| + 2|\gamma_i| + (4b_i+2)|\delta_i|\right] \tag{2.12}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\delta}\sum_{i=1}^{n}\left[|\alpha_i - 1| + (2b_i+1)|\beta_i| + 2|\gamma_i| + (4b_i+2)|\delta_i|\right]. \tag{2.13}$$

If for all $i = \overline{1,n}$, $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$ $f_i \in \mathcal{A}$ and

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < 1, \quad \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| < 1, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| < 1, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| < 1, \tag{2.14}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{T}_n$, given by (1.3) is in the class $\mathcal{S}$.

Proof. From (2.6), we deduce that

$$\left|c|z|^{2\delta} + \left(1-|z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right| \leq |c| + \left|\frac{1-|z|^{2\delta}}{\delta}\right|\frac{zH_n''(z)}{\delta H_n'(z)} \leq$$

$$\leq |c| + \left|\frac{1-|z|^{2\delta}}{\delta}\right|\sum_{i=1}^{n}\left[|\alpha_i - 1| + (2b_i+1)|\beta_i| + 2|\gamma_i| + (4b_i+2)|\delta_i|\right] \leq$$

$$\leq |c| + \frac{1}{|\delta|}\sum_{i=1}^{n}\left[|\alpha_i - 1| + (2b_i+1)|\beta_i| + 2|\gamma_i| + (4b_i+2)|\delta_i|\right] \leq$$

$$\leq |c| + \frac{1}{Re\delta}\sum_{i=1}^{n}\left[|\alpha_i - 1| + (2b_i+1)|\beta_i| + 2|\gamma_i| + (4b_i+2)|\delta_i|\right] \leq 1.$$

Finally by applying Theorem 1.3, we conclude that $\mathcal{T}_n$ is in the class $\mathcal{S}$. □

Theorem 2.7. *Let $\alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $M_i \geq 1$, $N_i \geq 1$, $P_i \geq 1$, $Q_i \geq 1$ are real numbers, for all $i = \overline{1,n}$ and $\delta \in \mathbb{C}$ with*

$$Re\delta \geq \sum_{i=1}^{n}\left[|\alpha_i - 1|(2M_i+1) + (b_i|\beta_i| + |\beta_i|)(2N_i+1)\right] +$$

$$+\sum_{i=1}^{n}\left[(|\gamma_i| + |\delta_i|b_i + |\delta_i|)(2P_i + 2Q_i + 2) + b_i|\beta_i| + 2b_i|\delta_i|\right] \tag{2.15}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\delta}\sum_{i=1}^{n}\left[|\alpha_i - 1|(2M_i+1) + (b_i|\beta_i| + |\beta_i|)(2N_i+1)\right] +$$

$$+\sum_{i=1}^{n}\left[(|\gamma_i| + |\delta_i|b_i + |\delta_i|)(2P_i + 2Q_i + 2) + b_i|\beta_i| + 2b_i|\delta_i|\right]. \tag{2.16}$$

If for all $i = \overline{1,n}$, $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$ $f_i \in \mathcal{A}$ *satisfy*

$$\left|\frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1\right| < 1, \left|\frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1\right| < 1, \left|\frac{z^2 h_i'(z)}{[h_i(z)]^2} - 1\right| < 1, \left|\frac{z^2 k_i'(z)}{[k_i(z)]^2} - 1\right| < 1, \tag{2.17}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then the integral operator* $\mathcal{T}_n$ *given by (1.3) is in the class* $\mathcal{S}$.

Proof. From the proof of Theorem 2.5, we have

$$\left|c|z|^{2\delta} + \left(1 - |z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right| \leq |c| + \frac{1}{Re\delta}\sum_{i=1}^{n}|\alpha_i - 1|(2M_i + 1) +$$

$$+\sum_{i=1}^{n}\left[\left(\left(b_i|\beta_i| + |\beta_i|\right)(2N_i + 1) + |\gamma_i| + |\delta_i| b_i + |\delta_i|\right)(2P_i + 2Q_i + 2) + b_i|\beta_i| + 2b_i|\delta_i|\right]$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and from the hypothesis, we get

$$\left|c|z|^{2\delta} + \left(1 - |z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right| \leq 1.$$

Applying Theorem 1.3 for the function H_n, we prove that $\mathcal{T}_n$ is in the class $\mathcal{S}$. □

3. Corollaries and consequences

First of all, upon setting $\delta = 1$ in Theorem 2.4, we immediately arrive at the following corollary:

Corollary 3.1. *Let* $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ *be complex numbers,* $0 < Re\gamma \leq 1$, $c = Re\gamma$, *with*

$$c \geq \sum_{i=1}^{n}\left[|\alpha_i - 1| + (2b_i + 1)|\beta_i| + 2|\gamma_i| + (4b_i + 2)|\delta_i|\right].$$

If for all $i = \overline{1,n}$, $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ *and*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < 1, \ \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| < 1, \ \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| < 1, \ \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then the integral operator* $\mathcal{F}_n$, *defined by*

$$\mathcal{F}_n(z) = \int_0^z \prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1}(g_i'(t))^{\beta_i}\left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right]dt, \tag{3.18}$$

is in the class $\mathcal{S}$.

Letting $\delta = 1$ and $\delta_i = 0$ in Theorem 2.4, we obtain the following corollary:

Corollary 3.2. *Let* $\gamma, \alpha_i, \beta_i, \gamma_i$ *be complex numbers,* $0 < Re\gamma \leq 1$, $c = Re\gamma$, *with*

$$c \geq \sum_{i=1}^{n}\left[|\alpha_i - 1| + (2b_i + 1)|\beta_i| + 2|\gamma_i|\right].$$

If for all $i = \overline{1,n}$, $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i, h_i, k_i \in \mathcal{A}$ *and*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < 1, \ \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| < 1, \ \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| < 1, \ \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then the integral operator* $\mathcal{S}_n$, *defined by*

$$\mathcal{S}_n(z) = \int_0^z \prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1}(g_i'(t))^{\beta_i}\left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\right]dt, \tag{3.19}$$

is in the class $\mathcal{S}$.

Remark 3.2. On the integral operator from Corollary 3.2, given by (3.19) if we take $\gamma_i = 0$, we obtain a known result proven in [6].

Letting $\delta = 1$ and $\beta_i = 0$ in Theorem 2.4, we have the following corollary:

Corollary 3.3. *Let $\gamma, \alpha_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, with*

$$c \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| + 2\,|\gamma_i| + (4b_i + 2)\,|\delta_i| \right].$$

If for all $i = \overline{1,n}$, $h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| < 1, \quad \left| \frac{z k_i'(z)}{k_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{X}_n$, defined by

$$\mathcal{X}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \left(\frac{h_i{}'(t))}{k_i{}'(t)} \right)^{\delta_i} \right] dt, \tag{3.20}$$

is in the class $\mathcal{S}$.

Remark 3.3. On the integral operator from Corollary 3.3, given by (3.20) if we take $\alpha_i - 1 = 0$, we obtain another known result proven in [15].

Letting $\delta = 1$ and $\alpha_i - 1 = 0$ in Theorem 2.4, we have the next corollary:

Corollary 3.4. *Let $\gamma, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, with*

$$c \geq \sum_{i=1}^{n} \left[(2b_i + 1)\,|\beta_i| + 2\,|\gamma_i| + (4b_i + 2)\,|\delta_i| \right].$$

If for all $i = \overline{1,n}$, $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, and

$$\left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| < 1, \quad \left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| < 1, \quad \left| \frac{z k_i'(z)}{k_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{D}_n$, defined by

$$\mathcal{D}_n(z) = \int_0^z \prod_{i=1}^{n} \left[(g_i'(t))^{\beta_i} \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \left(\frac{h_i{}'(t))}{k_i{}'(t)} \right)^{\delta_i} \right] dt, \tag{3.21}$$

is in the class $\mathcal{S}$.

Remark 3.4. If in (3.21) from Corollary 3.4, take $\beta_i = 0$, we obtain a result that was defined in [15].

Letting $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.4, we have the following corollary:

Corollary 3.5. *Let $\gamma, \alpha_i, \beta_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, with*

$$c \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1)\,|\beta_i| + (4b_i + 2)\,|\delta_i| \right].$$

If for all $i = \overline{1,n}$, $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| < 1, \quad \left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| < 1, \quad \left| \frac{z k_i'(z)}{k_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{Y}_n$, defined by

$$\mathcal{Y}_n(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} (g_i'(t))^{\beta_i} \left(\frac{h_i{}'(t))}{k_i{}'(t)} \right)^{\delta_i} \right] dt, \tag{3.22}$$

is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_1 - 1 = \beta_1 = \gamma_1$ in Theorem 2.4, we obtain:

Corollary 3.6. *Let α be complex number, $Re\alpha > 0$, with*

$$Re\alpha \geq |\alpha - 1| + (6b + 5)\,|\alpha|\,.$$

If $g, h, k \in \mathcal{G}_b$, $0 < b \leq 1$ $f \in \mathcal{A}$ and

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| < 1, \ \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| < 1, \ \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| < 1, \ \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| < 1,$$

then the integral operator $\mathcal{T}$, defined by

$$\mathcal{T}(z) = \left[\alpha \int_0^z \left(f(t) g'(t) \frac{h(t)}{k(t)} \frac{h'(t))}{k'(t)} \right)^{\alpha - 1} dt \right]^{\frac{1}{\alpha}}, \tag{3.23}$$

is in the class $\mathcal{S}$.

Letting $n = 1$, $\alpha_1 - 1 = \beta_1 = \gamma_1 = \alpha$, $b_1 = b$ in Theorem 2.5, we have:

Corollary 3.7. *Let α be complex number, $M \geq 1$, $N \geq 1$, $P \geq 1$, $Q \geq 1$, $Re\alpha > 0$ are real numbers, with*

$$Re\alpha \geq [|\alpha - 1|\,(2M + 1) + |\alpha|\, b\, (2N + 2P + 2Q + 6) + |\alpha|\,(2N + 4P + 4Q + 5)]\,.$$

If $g, h, k \in \mathcal{G}_b$, $0 < b \leq 1$ $f \in \mathcal{A}$ satisfy

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| < 1, \left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| < 1, \left| \frac{z^2 h'(z)}{[h(z)]^2} - 1 \right| < 1, \quad \left| \frac{z^2 k'(z)}{[k(z)]^2} - 1 \right| < 1$$

$$|f(z)| \leq M, \quad |g(z)| \leq N, \quad |h(z)| \leq P, \quad |k(z)| \leq Q,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}$, given by (3.23) is in the class $\mathcal{S}$.

Letting $n = 1$, $\alpha_1 - 1 = \beta_1 = \gamma_1$, $b_1 = b$ in Theorem 2.6, we obtain:

Corollary 3.8. *Let $\alpha \in \mathbb{C}^*$ with*

$$Re\alpha \geq [|\alpha - 1| + (6b + 5)\,|\alpha|]$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\alpha}\,[|\alpha - 1| + (6b + 5)\,|\alpha|]\,.$$

If for $g, h, k \in \mathcal{G}_b$, $0 < b \leq 1$ $f \in \mathcal{A}$ and

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \ \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| < 1, \ \left| \frac{zh'(z)}{h(z)} - 1 \right| < 1, \ \left| \frac{zk'(z)}{k(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}$, given by (3.23) is in the class $\mathcal{S}$.

Letting $n = 1$, $\alpha_1 - 1 = \beta_1 = \gamma_1 = \alpha$, $b_1 = b$ in Theorem 2.7, we obtain:

Corollary 3.9. *Let $\alpha \in \mathbb{C}^*$, $M \geq 1$, $N \geq 1$, $P \geq 1$, $Q \geq 1$ are real numbers, with*

$$Re\alpha \geq [|\alpha - 1|(2M+1) + |\alpha|\, b\,(2N+2P+2Q+6) + |\alpha|(2N+4P+4Q+5)]$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\alpha}\left[|\alpha - 1|(2M+1) + |\alpha|\, b\,(2N+2P+2Q+6) + |\alpha|(2N+4P+4Q+5)\right].$$

If $g, h, k \in \mathcal{G}_b$, $0 < b \leq 1$ $f \in \mathcal{A}$ satisfy

$$\left|\frac{z^2 f'(z)}{[f(z)]^2} - 1\right| < 1,\ \left|\frac{z^2 g'(z)}{[g(z)]^2} - 1\right| < 1,\ \left|\frac{z^2 h'(z)}{[h(z)]^2} - 1\right| < 1,\ \left|\frac{z^2 k'(z)}{[k(z)]^2} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}$, given by (3.23) is in the class $\mathcal{S}$.

4. Concluding remarks and observations

Our present investigation was motivated essentially by several recent works dealing with the interesting problem of finding sufficient conditions for univalence of normalized analytic functions which are defined in terms of various families integral operators (see, for example, [2-5, 9, 15-17]). In our study, we derive the univalence conditions for the integral operator $\mathcal{T}_n$, when $g_i, h_i, k_i \in G_{b_i}$ and $f_i \in \mathcal{A}$ for all $i = \overline{1,n}$, by using three univalence criteria, which were derived by Pascu [11], [12], and Pescar [14], respectively. Our main results are shown to yield several corollaries and consequences. Some of these applications of our main results are started here as Corollaries 3.1 - 3.9. Derivations of further corollaries and consequences of the results presented in this paper, including also their connections with known results given in several earlier works, are being left here as exercises for the interested reader.

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