CARPATHIAN J. MATH.
Volume **37** (2021), No. 2,
Pages 145 - 160

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2021.02.01

Dedicated to Prof. Ioan A. Rus on the occasion of his 85^{th} *anniversary*

A novel iterative approach for solving common fixed point problems in Geodesic spaces with convergence analysis

THANATPORN BANTAOJAI[1], CHANCHAL GARODIA[2], IZHAR UDDIN[2], NUTTAPOL PAKKARANANG[3] and PANU YIMMUANG [4,†]

ABSTRACT. In this paper, we introduce a new iterative method for nonexpansive mappings in CAT(κ) spaces. First, the rate of convergence of proposed method and comparison with recently existing method is proved. Second, strong and Δ-convergence theorems of the proposed method in such spaces under some mild conditions are also proved. Finally, we provide some non-trivial examples to show efficiency and comparison with many previously existing methods.

1. INTRODUCTION

It was M. Gromov [9] who coined the term CAT(κ) to denote a distinguished class of geodesic metric spaces with curvature bounded above by $\kappa \in \mathbb{R}$. CAT(κ) spaces can be understood as a generalization of Riemannian manifolds with bounded sectional curvature. In recent years, CAT(κ) spaces have attracted the attention of many young researchers owing to their important role in different aspects of geometry. A very thorough discussion on these spaces and the role they play in geometry can be found in the book by M. R. Bridson and A. Haefliger [4].
In 2003-2004, Kirk who noticed the richness of geometry of CAT(κ) spaces and introduced the fixed point theory in CAT(κ) spaces [12, 13]. Following this, different authors produced a series of work mainly focussing on CAT(0) spaces (see e.g., [5–7, 14, 15, 19, 23, 26]). Also, it is worth mentioning that any CAT(κ) space is a CAT(κ') space for every $\kappa' \geq \kappa$ (see in [4]). So, the results of CAT(0) space holds good for any CAT(κ) space with $\kappa \leq 0$. Further, CAT(κ) spaces for $\kappa > 0$, were studied by some authors (see for instance [8, 10, 18, 20–22, 24, 27]) and many authors have introduced various iteration processes for approximating fixed points in CAT(κ) spaces. In 2011, B. Piatek [22] proved that an iterative sequence generated by the Halpern algorithm converges to a fixed point in the complete CAT(κ) spaces. Further, in 2012, He et al. [10] showed that the famous Mann algorithm converges to a fixed point in complete CAT(κ) spaces and B. Panyanak [20] proved the convergence of Ishikawa iteration for multivalued mappings in CAT(κ) spaces.
Very recently, Thounthong et al. [28] introduced the following modified iteration process to approximate common fixed point of two nonexpansive mappings.
Let C be a non-empty closed convex subset of a complete CAT(κ) space X and $T, S : K \to K$ be two nonexpansive mappings. Suppose that a sequence $\{c_n\}$ is generated iteratively

Received: 15.10.2020. In revised form: 16.04.2021. Accepted: 22.04.2021
2010 *Mathematics Subject Classification.* 47H09, 47H10, 65K15.
Key words and phrases. *CAT(κ) spaces; Iterative method; Nonexpansive mappings; Common fixed point.*
Corresponding author: †Panu Yimmuang; panu.y@sut.ac.th

by:

$$\begin{cases} c_1 \in K \\ a_n = (1-\alpha_n)c_n \oplus \alpha_n T c_n \\ b_n = (1-\beta_n)a_n \oplus \beta_n S a_n \\ c_{n+1} = (1-\gamma_n)Ta_n \oplus \gamma_n S b_n \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$.
Motivated by (1.1) we propose the following iteration to locate the common fixed point of T and S.

$$\begin{cases} x_1 \in K \\ z_n = (1-\alpha_n)x_n \oplus \alpha_n T x_n \\ x_{n+1} = S(T((1-\beta_n)z_n \oplus \beta_n S z_n)) \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.
In this paper, we propose a new iteration process for two nonexpansive mappings in complete CAT(κ) spaces. We prove that our proposed iteration process converges faster than iteration process (1.1) for contractive like mappings. We have also constructed a an example to support our claim. Further, we prove strong and Δ-convergence results involving the proposed iteration process under some conditions. We finally provide numerical experiments of two non-trivial examples to demonstrate the speed of convergence of iteration process (1.2) with existing iterations which further supports our main results.

2. Preliminaries

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0)=x$, $c(l)=y$ and $d(c(t),c(t')) = |t–t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $d(x,y)=l$. The image α of c is called a geodesic (or metric) segment joining x and y. We say that X is (i) a geodesic space if any two points of X are joined by a geodesic, and (ii) uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$, which we will denote by $[x,y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that $d(x,z)=(1–\alpha)d(x,y)$ and $d(y,z)=\alpha d(x,y)$. In this case, we write $z=\alpha x \oplus (1–\alpha)y$. The space (X,d) is said to be a geodesic space (D-geodesic space) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be uniquely geodesic (D-uniquely geodesic) if there is exactly one geodesic joining x and y for each $x,y \in X$ (for $x,y \in X$ with $d(x,y) < D$). A subset K of X is said to be convex if K includes every geodesic segment joining any two of its points. The set K is said to be bounded if

$$diam(K) := sup\{d(x,y) : x,y \in K\} < \infty.$$

Denote $F(T) = \{x \in K : Tx = x\}$ is the set of fixed points of mapping T.

Definition 2.1. Given $k \in \mathbb{R}$, we denote by M^n_κ the following metric spaces:

(i) if $\kappa = 0$ then M^n_0 is the Euclidean space $\mathbb{E}^n$;
(ii) if $\kappa > 0$ then M^n_κ is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $\frac{1}{\sqrt{\kappa}}$;
(iii) if $\kappa < 0$ then M^n_κ is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $\frac{1}{\sqrt{-\kappa}}$.

A geodesic triangle $\Delta(x,y,z)$ in the metric space (X,d) consists of three points x,y,z in X (the vertices of Δ) and three geodesic segments between each pair of vertices (the edges of Δ. We write $p \in \Delta(x,y,z)$ when $p \in [x,y] \cup [y,z] \cup [z,x]$. A comparison triangle for a geodesic triangle $\Delta(x,y,z)$ in (X,d) is a triangle $\Delta(\bar{x},\bar{y},\bar{z})$ in M_κ^2 such that

$$d(x,y) = d_{M_\kappa^2}(\bar{x},\bar{y}), d(x,z) = d_{M_\kappa^2}(\bar{x},\bar{z}) \; and \; d(z,x) = d_{M_\kappa^2}(\bar{z},\bar{x}).$$

If $\kappa \leq 0$ then such a comparison triangle always exists in M_κ^2. If $\kappa > 0$ then such a triangle exists whenever $d(x,y)+d(y,z)+d(z,x) < 2D_\kappa$, where $D_\kappa = \dfrac{\pi}{\sqrt{\kappa.}}$ A point $\bar{p} \in [\bar{x},\bar{y}]$ is called a comparison point for $p \in [x,y]$ if $d(x,p) = d(\bar{x},\bar{p})$.
A geodesic triangle $\Delta(x,y,z)$ in X is said to satisfy the $CAT(\kappa)$ inequality if for any $p,q \in \Delta(x,y,z)$ and for their comparison points $\bar{p},\bar{q} \in \Delta(\bar{x},\bar{y},\bar{z})$, one has

$$d(p,q) \leq d_{M_\kappa^2}(\bar{p},\bar{q}).$$

Now, we recall the following important lemmas which will be useful in our subsequent discussion.

Lemma 2.1. [4] *Let (X,d) be a CAT(κ) space and let K be a closed and π-convex subset of X. Then for each point $x \in X$ such that $d(x,K) < \dfrac{\pi}{2}$, there exists a unique point $y \in K$ such that $d(x,y) = d(x,K)$.*

Lemma 2.2. [17] *Let (X,d) be a CAT(1) space. Then there is a constant $M > 0$ such that*

$$d^2(x, ty \oplus (1-t)z) \leq td^2(x,y) + (1-t)d^2(x,z) - \frac{M}{2}t(1-t)d(y,z)$$

for any $t \in [0,1]$ and any point $x,y,z \in X$ such that $d(x,y) \leq \dfrac{\pi}{4}$, $d(x,z) \leq \dfrac{\pi}{4}$ and $d(y,z) \leq \frac{\pi}{2}$.

Let $\{x_n\}$ be a bounded sequence in a CAT(κ) space (X,d). For $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,\{x_n\}).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = inf\{r(x,\{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

Definition 2.2. [14, 16] A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.3. [10] *Let (X,d) be a complete CAT(1) space and let K be a non-empty subset of X. Suppose that the sequence $\{x_n\}$ in X is Fejer monotone with respect to K and the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is lees than $\dfrac{\pi}{2}$. If any Δ-cluster point x of $\{x_n\}$ belongs to K, then $\{x_n\}$ Δ-converges to a point in K.*

Lemma 2.4. [10] *Let (X,d) be a complete CAT(1) space and let $p \in X$. Suppose that the sequence $\{x_n\}$ in X Δ-converges to x such that $r(p,\{x_n\}) < \dfrac{D_k}{2}$. then*

$$d(x,p) \leq \liminf_{n\to\infty} d(x_n,p).$$

Definition 2.3. Let (X,d) be a metric space and K be its nonempty subset. Then $T : K \to K$ is called semi-compact if for a sequence x_n in K with $\lim_{n\to\infty} d(x_n,Tx_n) = 0$, there exist a subsequence x_{n_k} of x_n such that $x_{n_k} \to p \in K$.

In 1972, Zamfirescu [29] introduced Zamfirescu mappings which serves as an important generalization for Banach contraction principle [1]. In 2004, Berinde [2] gave a more general class of mappings known as quasi-contractive mappings. Following this, Imoru and Olantiwo [11] gave the following definition:

Definition 2.4. A mapping $T : K \to K$ is known as contractive-like mapping if there exists a strictly increasing and continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and a constant $\delta \in [0, 1)$ such that for all $x, y \in K$, we have

$$\|Tx - Ty\| \leq \delta\|x - y\| + \varphi(\|x - Tx\|).$$

Clearly, the class of contractive-like mappings is wider than the class of quasi-contractive mappings.

Recall that the following definitions about the rate of convergence were given by Berinde [3].

Definition 2.5. Let $\{a_n\}$ and $\{b_n\}$ be two real sequences converging to a and b respectively. Then, $\{a_n\}$ converges faster then $\{b_n\}$ if $\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0$.

Definition 2.6. Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration processes converging to the same fixed point p. If $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero such that $\|u_n - p\| \leq a_n$ and $\|v_n - p\| \leq b_n$ for all $n \geq 1$, then we say that $\{u_n\}$ converges faster than $\{v_n\}$ to p if $\{a_n\}$ converges faster then $\{b_n\}$.

3. Rate of convergence

In this section, we prove that our proposed iteration process (1.2) is having a better rate of convergence than (1.1) for contractive-like mappings.

Theorem 3.1. *Let T and S be two contractive-like mappings defined on a nonempty, closed, convex subset K of a complete CAT(1) space (X, d) such that $F := F(T) \cap F(S) \neq \emptyset$. If $\{x_n\}$ is a sequence defined by* (1.2)*, then $\{x_n\}$ converges faster than the iterative algorithm* (1.1)

Proof. As, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, we can find $\eta, \varsigma \in \mathbb{R}$ such that $0 < \eta \leq \alpha_n, \beta_n, \gamma_n \leq \varsigma < 1$ for all $n \in \mathbb{N}$.
From (1.2), for any $p \in F$, we have

$$\begin{aligned} d(z_n, p) &= d((1 - \alpha_n)x_n \oplus \alpha_n Tx_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n \delta d(x_n, p) \\ &= (1 - (1 - \delta)\alpha_n)d(x_n, p); \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(S(T((1 - \beta_n)z_n \oplus \beta_n Sz_n)), p) \\ &\leq \delta d(T((1 - \beta_n)z_n \oplus \beta_n Sz_n, p)) \\ &\leq \delta^2 d((1 - \beta_n)z_n \oplus \beta_n Sz_n, p) \\ &\leq \delta^2((1 - \beta_n)d(z_n, p) + \beta_n d(Sz_n, p)) \\ &\leq \delta^2(1 - (1 - \delta)\beta_n)d(z_n, p) \\ &\leq \delta^2(1 - (1 - \delta)\beta_n)(1 - (1 - \delta)\alpha_n)d(x_n, p) \\ &\ \ \vdots \\ &\leq \delta^{2n}(1 - (1 - \delta)\varsigma)^n(1 - (1 - \delta)\varsigma)^n d(x_1, p). \end{aligned}$$

From (1.1), we get

$$\begin{aligned} d(a_n, p) &= d((1 - \alpha_n)c_n \oplus \alpha_n Tc_n, p) \\ &\leq (1 - \alpha_n)d(c_n, p) + \alpha_n d(Tc_n, p) \\ &= (1 - (1 - \delta)\alpha_n)d(c_n, p); \end{aligned}$$

$$\begin{aligned} d(b_n,p) &= d((1-\beta_n)a_n \oplus \beta_n Sa_n, p) \\ &\leq (1-\beta_n)d(a_n,p) + \beta_n d(Sa_n,p) \\ &\leq (1-(1-\delta)\beta_n)(1-(1-\delta)\alpha_n)d(c_n,p); \end{aligned}$$

and

$$\begin{aligned} d(c_{n+1},p) &= d((1-\gamma_n)Ta_n \oplus \gamma_n Sb_n, p) \\ &\leq (1-\gamma_n)d(Ta_n,p) + \gamma_n d(Sb_n,p) \\ &\leq \delta((1-\gamma_n)d(a_n,p) + \gamma_n d(b_n,p)) \\ &= \delta(1-(1-\delta)\gamma_n)(1-(1-\delta)\beta_n)(1-(1-\delta)\alpha_n)d(c_n,p) \\ &\ . \\ &\ . \\ &\ . \\ &\leq \delta^n(1-(1-\delta)\varsigma)^n(1-(1-\delta)\varsigma)^n(1-(1-\delta)\varsigma)^n d(c_1,p). \end{aligned}$$

Now, since $\delta, \varsigma < 1$, we have $1-(1-\delta)\varsigma) < 1$. So,

$$d(x_{n+1},p) \leq \delta^{2n} d(x_1,p) \ \text{ and } \ d(c_{n+1},p) \leq \delta^n d(c_1,p).$$

Let $b_n = \delta^{2n} d(x_1,p)$ and $a_n = \delta^n d(c_1,p)$, then

$$\begin{aligned} \frac{b_n}{a_n} &= \frac{\delta^{2n} d(x_1,p)}{\delta^n d(c_1,p)} \\ &\to 0 \ as \ n \to \infty. \end{aligned}$$

Hence in view of Definitions 2.5 and 2.6, $\{x_n\}$ converges faster than $\{c_n\}$. □

Now, we present a example of a contractive-like mapping which is not a contraction.

Example 3.1. Let $X = \mathbb{R}$ and $K = [0,6]$. Let $T : K \to K$ be a mapping defined as

$$Tx = \begin{cases} \dfrac{x}{5} & x \in [0,3) \\ \dfrac{x}{10} & x \in [3,6]. \end{cases}$$

Proof. Clearly $x = 0$ is the fixed point of T. First, we prove that T is a contractive-like mapping but not a contraction. Since T is not continuous at $x = 3 \in [0,6]$, so T is not a contraction. We show that T is a contractive-like mapping. For this, define $\varphi : [0,\infty) \to [0,\infty)$ as $\varphi(x) = \dfrac{x}{8}$. Then, φ is a strictly increasing as well as continuous function. Also, $\varphi(0) = 0$.
We need to show that

$$(A) \qquad \|Tx - Ty\| \leq \delta\|x - y\| + \varphi(\|x - Tx\|)$$

for all $x, y \in [0,6]$ and δ is a constant in $[0,1)$.
Before going ahead, let us note the following. When $x \in [0,3)$, then

$$\|x - Tx\| = \left\|x - \frac{x}{5}\right\| = \frac{4x}{5}$$

and

$$\varphi\left(\frac{4x}{5}\right) = \frac{x}{10}. \tag{3.3}$$

Similarly, when $x \in [3,6]$, then

$$\|x - Tx\| = \left\|x - \frac{x}{10}\right\| = \frac{9x}{10}$$

and

$$\varphi\left(\frac{9x}{10}\right) = \frac{9x}{80}. \tag{3.4}$$

Consider the following cases:
Case A: Let $x, y \in [0, 3)$, then using (3.3) we get

$$\begin{aligned}
\|Tx - Ty\| &= \|\frac{x}{5} - \frac{y}{5}\| \\
&\leq \frac{1}{5}\|x - y\| \\
&\leq \frac{1}{5}\|x - y\| + \frac{x}{10} \\
&= \frac{1}{5}\|x - y\| + \varphi\left(\frac{4x}{5}\right) \\
&= \frac{1}{5}\|x - y\| + \varphi(\|x - Tx\|).
\end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{5}$.
Case B: Let $x \in [0, 3)$ and $y \in [3, 6]$ then using (3.3) we get

$$\begin{aligned}
\|Tx - Ty\| &= \|\frac{x}{5} - \frac{y}{10}\| \\
&= \|\frac{x}{10} + \frac{x}{10} - \frac{y}{10}\| \\
&\leq \frac{1}{10}\|x - y\| + \left\|\frac{x}{10}\right\| \\
&\leq \frac{1}{5}\|x - y\| + \varphi\left(\frac{4x}{5}\right) \\
&= \frac{1}{5}\|x - y\| + \varphi(\|x - Tx\|).
\end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{5}$.
Case C: Let $x \in [3, 6]$ and $y \in [0, 3)$ then using (3.4) we get

$$\begin{aligned}
\|Tx - Ty\| &= \|\frac{x}{10} - \frac{y}{5}\| \\
&= \|\frac{x}{5} - \frac{x}{10} - \frac{y}{5}\| \\
&\leq \frac{1}{5}\|x - y\| + \left\|\frac{x}{10}\right\| \\
&\leq \frac{1}{5}\|x - y\| + \left\|\frac{9x}{80}\right\| \\
&= \frac{1}{5}\|x - y\| + \varphi(\|x - Tx\|).
\end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{5}$.
Case D: Let $x, y \in [3, 6]$ then using (3.4) we get

$$\begin{aligned}
\|Tx - Ty\| &= \|\frac{x}{10} - \frac{y}{10}\| \\
&\leq \frac{1}{10}\|x - y\| + \left\|\frac{9x}{80}\right\| \\
&\leq \frac{1}{5}\|x - y\| + \left\|\frac{9x}{80}\right\| \\
&= \frac{1}{5}\|x - y\| + \varphi(\|x - Tx\|).
\end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{5}$.

Consequently, (A) is satisfied for $\delta = \frac{1}{5}$ and $\varphi(x) = \frac{x}{8}$ in all the possible cases. Thus, T is a contractive-like mapping. Similarly, define $S : K \to K$ as

$$Sx = \begin{cases} \frac{x}{6} & x \in [0,4) \\ \frac{x}{12} & x \in [4,8]. \end{cases}$$

We can show that S is a contractive-like mapping and it is not a contraction mapping. Also, zero is the common fixed point of T and S.

Now, using T and S, we show that our iterative algorithm (1.2) has a better rate of convergence. Set $\alpha_n = \beta_n = \gamma_n = \frac{n}{n+1}$ for each $n \in \mathbb{N}$. Then, we get the following tables and graphs with the initial value 4.5. □

TABLE 1

No. of Iter.	Thounthong Iter.	Proposed Iter.
1	4.5	4.5
2	0.3678125	0.048125
3	0.0199193930041152	0.000332716049382716
4	0.000771876478909465	$1.66358024691358 \times 10^{-6}$
5	0.0000234650449588477	$6.65432098765432 \times 10^{-9}$
6	$5.92661372160198 \times 10^{-7}$	$2.25918305136412 \times 10^{-11}$
7	$1.29245103899658 \times 10^{-8}$	$6.76218056190622 \times 10^{-14}$
8	$2.50075813014183 \times 10^{-10}$	$1.83142390218293 \times 10^{-16}$
9	$4.38023356677181 \times 10^{-12}$	$4.57227915222625 \times 10^{-19}$
10	$7.05217604250261 \times 10^{-14}$	$1.06686513551946 \times 10^{-21}$

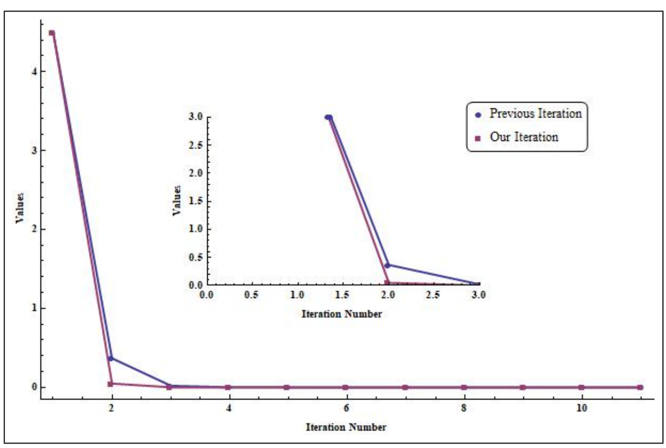


FIGURE 1. Graph corresponding to Table 1.

Clearly, the proposed iterative method converges faster than the previous one.

4. Convergence analysis

Lemma 4.5. *Let (X, d) be a complete CAT(1) space and K be a non-empty, closed and convex subset of X. Let T and S be two nonexpansive mappings on K such that $F := F(T) \cap F(S) \neq \emptyset$, where $F(T)$ and $F(S)$ be two sets of fixed point for mappings T and S respectively. Let $\{x_n\}$ be sequence defined by* (1.2) *for $x_0 \in K$ such that $d(x_0, F) \leq \dfrac{\pi}{4}$. Then there exists a unique point $p \in F$ such that $d(y_n, p) \leq d(z_n, p) \leq d(x_n, p) \leq \dfrac{\pi}{4}$ for all $n \geq 0$.*

Proof. By the paper of Piatek in [22] and Lemma 2.1 there exists a unique point $p \in F$ such that $d(x_0, p) = d(x_0, F)$. By the condition of nonexpansive mapping $d(Tx_0, p) \leq d(x_0, p) \leq \frac{\pi}{4}$ and $B_{\frac{\pi}{4}}(p)$ is convex, we get

$$\begin{aligned} d(y_0, p) &= d(T((1-\beta_0)z_0 \oplus \beta_0 S z_0), p) \\ &\leq d(z_0, p) = d((1-\alpha_0)x_0 \oplus \alpha_0 T x_0, p) \leq d(x_0, p) \leq \tfrac{\pi}{4}. \end{aligned}$$

Suppose that $d(y_k, p) \leq d(z_k, p) \leq d(x_k, p) \leq \frac{\pi}{4}$. Since $B_{\frac{\pi}{4}}(p)$ is convex, we get

$$\begin{aligned} d(x_{k+1}, p) &= d(Sy_k, p) \\ &\leq d(y_k, p) \leq d(x_k, p) \leq \tfrac{\pi}{4} \end{aligned}$$

and

$$\begin{aligned} d(y_{k+1}, p) &= d(T((1-\beta_{k+1})z_{k+1} \oplus \beta_{k+1} S z_{k+1}), p) \\ &\leq d(z_{k+1}, p) = d((1-\alpha_{k+1})x_{k+1} \oplus \alpha_{k+1} T x_{k+1}, p) \leq d(x_{k+1}, p) \leq \tfrac{\pi}{4}. \end{aligned}$$

It follows that $d(y_{k+1}, p) \leq d(z_{k+1}, p) \leq d(x_{k+1}, p) \leq \dfrac{\pi}{4}$. By mathematical induction, we get $d(y_n, p) \leq d(z_n, p) \leq d(x_n, p) \leq \dfrac{\pi}{4}$ for all $n \geq 0$. This completes the proof. □

Lemma 4.6. *Let (X, d) be a complete CAT(1) space and K be a non-empty, closed and convex subset of X. Let T and S be two nonexpansive mappings on K such that $F := F(T) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be sequence defined by* (1.2) *for $x_0 \in K$ such that $d(x_0, F) \leq \dfrac{\pi}{4}$, then*

(i) $\lim_{n\to\infty} d(x_n, p)$ *exist for $p \in F$;*
(ii) $\lim_{n\to\infty} d(Tx_n, x_n) = 0 = \lim_{n\to\infty} d(Sx_n, x_n)$.

Proof. From Lemma 2.2 and Lemma 4.5, there exist $p \in F$ and $M > 0$ such that

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(S(T((1-\beta_n)z_n \oplus \beta_n S z_n)), p) \\ &\leq d^2(T((1-\beta_n)z_n \oplus \beta_n S z_n), p) \\ &\leq (1-\beta_n)d^2(z_n, p) + \beta_n d^2(Sz_n, p) - \tfrac{M}{2}\beta_n(1-\beta_n)d^2(z_n, Sz_n) \\ &\leq d^2(z_n, p) - \tfrac{M}{2}\beta_n(1-\beta_n)d^2(z_n, Sz_n) \leq d^2(z_n, p) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} d^2(z_n, p) &= d^2((1-\alpha_n)x_n \oplus \alpha_n T x_n, p) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(Tx_n, p) - \tfrac{M}{2}\alpha_n(1-\alpha_n)d^2(x_n, Tx_n) \\ &\leq d^2(x_n, p) - \tfrac{M}{2}\alpha_n(1-\alpha_n)d^2(x_n, Tx_n) \leq d^2(x_n, p). \end{aligned} \tag{4.6}$$

By (4.5) and (4.6), we get

$$d^2(x_{n+1}, p) \leq d^2(x_n, p).$$

Therefore, $d(x_{n+1}, p) \leq d(x_n, p)$ which gives that $\lim_{n\to\infty} d(x_n, p)$ exist.
Next, we prove (ii). Let

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{4.7}$$

Then, from (4.5) and (4.6), we have $d(x_{n+1}, p) \leq d(z_n, p) \leq d(x_n, p)$ which on using (4.7) gives

$$\lim_{n\to\infty} d(z_n, p) = c. \tag{4.8}$$

Now, from (4.6) we see that

$$d^2(z_n, p) \leq d^2(x_n, p) - \frac{M}{2}\alpha_n(1-\alpha_n)d^2(x_n, Tx_n)$$

thus,

$$d^2(x_n, Tx_n) \leq \frac{2}{\alpha_n(1-\alpha_n)M}[d^2(x_n, p) - d^2(z_n, p)].$$

By (4.7) and (4.8), we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{4.9}$$

Also, from (4.5) we get

$$d^2(x_{n+1}, p) \leq d^2(z_n, p) - \frac{M}{2}\beta_n(1-\beta_n)d^2(z_n, Sz_n)$$

which gives

$$d^2(z_n, Sz_n) \leq \frac{2}{\beta_n(1-\beta_n)M}[d^2(z_n, p) - d^2(x_{n+1}, p)].$$

Thus, on using (4.7) and (4.8) we get

$$\lim_{n\to\infty} d(z_n, Sz_n) = 0. \tag{4.10}$$

Now,

$$d(z_n, x_n) = d((1-\alpha_n)x_n \oplus \alpha_n Tx_n, x_n) = \alpha_n d(Tx_n, x_n).$$

So, on using (4.9) we have

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{4.11}$$

Consider

$$\begin{aligned} d(x_n, Sx_n) &\leq d(x_n, z_n) + d(z_n, Sz_n) + d(Sz_n, Sx_n) \\ &\leq d(x_n, z_n) + d(z_n, Sz_n) + d(z_n, x_n) \end{aligned}$$

which on using (4.10) and (4.11) yields

$$\lim_{n\to\infty} d(Sx_n, x_n) = 0.$$

This completes the proof. □

Theorem 4.2. *Let (X, d) be a complete CAT(κ) space and let K be a non-empty, closed and convex subset of X. Let T and S be two nonexpansive mappings on K such that $F := F(T) \cap F(S) \neq \emptyset$.. Let $\{x_n\}$ be sequence defined by* (1.2) *for $x_0 \in K$ such that $d(x_0, F) \leq \frac{\pi}{4}$, then $\{x_n\}$ $\Delta-$converges to a point in F.*

Proof. Without loss of generality, we assume that $\kappa = 1$. Set $F_0 = F \cap B_{\frac{\pi}{2}}(x_0)$. Let $q \in F_0$. Since the open ball $B_{\frac{\pi}{2}}(q) \in K$ with radius $r < \dfrac{\pi}{2}$ is convex, we get

$$d(z_0, q) = d((1-\alpha_0)x_0 + \alpha_0 Tx_0, q) \le d(x_0, q).$$

Also,

$$d(x_1, q) = d(S(T((1-\beta_0)z_0 + \beta_0 Sz_0)), q) \le d(z_0, q) \le d(x_0, q)$$

By mathematical induction, we can prove that

$$d(x_{n+1}, q) \le d(x_n, q) \le d(x_0, q).$$

for all $n \ge 0$. Thus, a sequence $\{x_n\}$ is a Fejer monotone sequence with respect to F_0. Let $p \in F$ such that $d(x_0, p) \le \dfrac{\pi}{4}$. Then, $p \in F_0$. Also, we have

$$d(x_{n+1}, p) \le d(x_n, p) \le d(x_0, p) \le \frac{\pi}{4} \tag{4.12}$$

for all $n \ge 0$. This proves that $r(\{x_n\}) < \dfrac{\pi}{4}$. From Lemma 2.3, let $x \in k$ be a $\Delta-$ cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which $\Delta-$converges to x. By (4.8), we obtain

$$r(p, \{x_{n_k}\}) \le d(x_0, p) \le \frac{\pi}{4}.$$

From Lemma 2.4, we get

$$d(x, x_0) \le d(x, p) + d(x_0, p) \le \liminf_{k\to\infty} d(x_{n_k}, p) + d(x_0, p) < \frac{\pi}{2}.$$

This implies that $x \in B_{\frac{\pi}{2}}(x_0)$. From Lemma 4.6, we get

$$\begin{aligned} \limsup_{k\to\infty} d(Tx, x_{n_k}) &\le \limsup_{k\to\infty} d(Tx, Tx_{n_k}) + \limsup_{k\to\infty} d(Tx_{n_k}, x_{n_k}) \\ &= \limsup_{k\to\infty} d(Tx, Tx_{n_k}) \end{aligned}$$

and

$$\begin{aligned} \limsup_{k\to\infty} d(Sx, x_{n_k}) &\le \limsup_{k\to\infty} d(Sx, Sx_{n_k}) + \limsup_{k\to\infty} d(Sx, x_{n_k}) \\ &= \limsup_{k\to\infty} d(Sx, Sx_{n_k}) \end{aligned}$$

Thus, $Tx, Sx \in A(\{x_{n_k}\})$ and $Tx = x = Sx$. Hence $x \in F_0$. By Lemma 3, we thus complete the proof. □

If $\kappa = 0$, we obtain the following result in CAT(0) spaces.

Corollary 4.1. *Let (X, d) be a complete CAT(0) space and let K be a non-empty, closed and convex subset of X. Let T and S be two nonexpansive mappings of K such that $F := F(T) \cap F(S) \ne \emptyset$.. Let $\{x_n\}$ be sequence defined by* (1.2) *for $x_0 \in K$, then $\{x_n\}$ $\Delta-$ converges to a point in F.*

Theorem 4.3. *Let (X, d) be a complete CAT(κ) space and let K be a non-empty, closed and convex subset of X. Let T and S be two nonexpansive mappings of K such that $F := F(T) \cap F(S) \ne \emptyset$.. Suppose that T, S are semi-compact for some $m \in \mathbb{N}$. If $\{x_n\}$ is defined by* (1.2) *for $x_0 \in K$ such that $d(x_0, F) \le \dfrac{\pi}{4}$, then $\{x_n\}$ converges strongly to a point in F.*

Proof. By Lemma 4.6, we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0$$

and

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

Thus, $\liminf_{n\to\infty} d(x_n, F) = 0$. Since $d(x_{n+1}, p) \le d(x_n, p)\ \forall\, p \in F$, it follows that

$$d(x_{n+1}, F) \le d(x_n, F).$$

Hence, $\lim_{n\to\infty} d(x_n, F)$ exists and $\lim_{n\to\infty} d(x_n, F) = 0$. That is, $\{x_n\}$ is an approximate common fixed point sequence for T and S. By Definition 2, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in K$ such that $\lim_{j\to\infty} x_{n_j} = p$. Next, we get

$$\begin{aligned} d(Tp, Sp) &\le d(Tp, p) + d(Sp, p) \\ &\le d(Tp, Tx_{n_j}) + d(Tx_{n_j}, x_{n_j}) + d(x_{n_j}, p) \\ &\quad + d(Sp, Sx_{n_j}) + d(Sx_{n_j}, x_{n_j}) + d(x_{n_j}, p) \\ &\to 0\ as\ j \to \infty. \end{aligned}$$

That is, $p \in F$. From Lemma 4.6, we have $\lim_{n\to\infty} d(x_n, p)$ exists, thus p is the strong limit of the sequence $\{x_n\}$ itself. This completes the proof. □

5. Numerical examples

Some non trivial examples are presented in this section to demonstrate the efficiency of the proposed iteration process. All the codes are written in Mathlab2020a running on a new surface pro, Core(TM)i5-7300U CPU, Intel(R) with 2.7GHz and memory 8 GB RAM.

5.1. **m-sphere $\mathbb{S}^m$.** The m-sphere $\mathbb{S}^m$ is defined by

$$\{x = (x_1, ..., x_{m+1}) \in \mathbb{R}^{m+1} : \langle x, x\rangle = 1\}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product.

The normalized geodesic $c : \mathbb{R} \to \mathbb{S}^m$ beginning from $x \in \mathbb{S}^m$ is denoted by

$$c(l) = (\cos l)x + (\sin l)v, \quad \forall l \in \mathbb{R},$$

where $v \in T_x\mathbb{S}^m$ is a unit vector; which respect distance d on $\mathbb{S}^m$ such that

$$d(x, y) = \arccos(\langle x, y\rangle),$$

for all $x, y \in \mathbb{S}^m$.

Example 5.2. [28, Example 1] Let $K = \mathbb{S}^3$ and $T, S : K \to K$ be two nonexpansive mappings which are defined by

$$Tx = Sx = (x_1, -x_2, -x_3, -x_4), \quad \forall x = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3.$$

Then $F(T) \cap F(S) = \{(1, 0, 0, 0)\}$.

Now, Thounthong iteration (1.1) can be written in the form

$$\begin{aligned} w_n &= (\cos((1-c_n)r(x_n, x_n)))x_n + (\sin((1-c_n)r(x_n, x_n)))U(x_n, x_n), \\ y_n &= (\cos((1-b_n)\overline{r}(w_n, w_n)))w_n + (\sin((1-b_n)\overline{r}(w_n, w_n)))U(w_n, w_n), \\ x_{n+1} &= (\cos((1-a_n)\overline{r}(Tw_n, y_n)))Tw_n + (\sin((1-a_n)\overline{r}(Tw_n, y_n)))\overline{U}(Tw_n, y_n), \end{aligned}$$

and proposed iteration (1.2) can be written in the form

$$\begin{aligned} z_n &= (\cos((1-b_n)r(x_n, x_n)))x_n + (\sin((1-b_n)r(x_n, x_n)))U(x_n, x_n), \\ x_{n+1} &= T(S((\cos((1-a_n)\overline{r}(z_n, z_n)))z_n + (\sin((1-a_n)\overline{r}(z_n, z_n)))U(z_n, z_n))), \end{aligned}$$

for all $n \ge 1$, where

$$r(x,y) = \arccos(\langle x, Ty\rangle), \quad \overline{r}(x,y) = \arccos(\langle x, Sy\rangle),$$

$$U(x,y) = \frac{Ty - \langle x, Ty\rangle x}{\sqrt{1-\langle x, Ty\rangle^2}}, \quad \overline{U}(x,y) = \frac{Sy - \langle x, Sy\rangle x}{\sqrt{1-\langle x, Sy\rangle^2}}, \quad \text{for all } x, y \in \mathbb{R}^{m+1}.$$

Setting control parameters $a_n = \dfrac{n}{20n+1}$, $b_n = \dfrac{n}{10n+1}$, $c_n = \dfrac{n}{30n+1}$ and stop criterion as $\|x_n - F(T)\cap F(S)\| \leq 10^{-7}$.

We test different initial values as

Choice 1 : x_1 = (0.9, 0.3, 0.3, 0.1);

Choice 2 : x_1 = (0.8, 0.4, 0.4, 0.2);

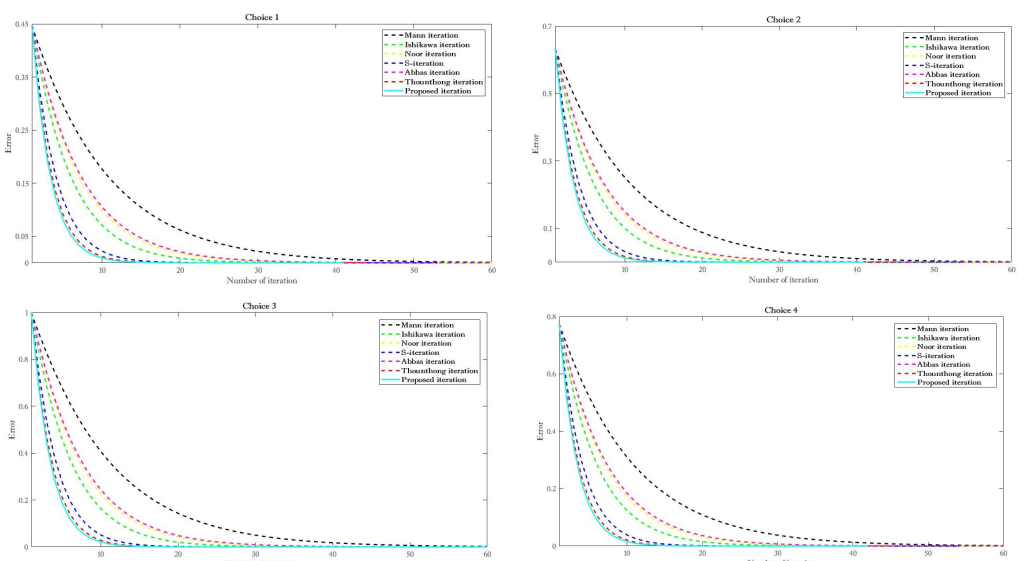
Choice 3 : x_1 = (0.5, 0.5, 0.5, 0.5);

Choice 4 : x_1 = (0.7, 0.5, 0.5, 0.1).

TABLE 2. Numerical results for different initial values of Example 5.2

		Mann iter.	Ishikawa iter.	Noor iter.	S iter.	Abbas iter.	Thounthong iter.	Proposed iter.
Choice 1	No. of Iter.	148	75	105	47	96	44	41
	Time	3.236	0.875	2.438	0.641	1.016	0.302	0.114
Choice 2	No. of Iter.	151	76	101	48	98	40	37
	Time	3.911	0.866	2.104	0.525	1.875	0.214	0.159
Choice 3	No. of Iter.	155	79	106	49	101	41	38
	Time	2.841	0.618	2.016	0.321	1.844	0.202	0.116
Choice 4	No. of Iter.	153	77	99	48	99	40	37
	Time	2.996	1.645	2.016	0.611	2.025	0.209	0.102

FIGURE 2. Behavior convergence of algorithms corresponding to Table 2



5.2. **Hyperbolic m-space $\mathbb{H}^m$.** The hyperbolic m-space $\mathbb{H}^m$ is defined by

$$\{x := (x_1, x_2, x_3, ..., x_{m+1}) \in \mathbb{R}^{m+1} : \langle x, x\rangle = -1 \text{ and } x_{m+1} \geq 1\},$$

where

$$\langle x, y\rangle = \sum_{i=1}^{m} x_i y_i - x_{m+1}y_{m+1}, \quad \forall x = (x_i), y = (y_i) \in \mathbb{R}^{m+1}.$$

The normalized geodesic $c : \mathbb{R} \to \mathbb{H}^m$ beginning from $x \in \mathbb{H}^m$ is denoted by

$$c(l) = (\cosh l)x + (\sinh l)v, \quad \forall l \in \mathbb{R},$$

where $v \in T_x\mathbb{H}^m$ be a unit vector; which respect distance d on $\mathbb{H}^m$ such that

$$d(x, y) = \text{arccosh}(-\langle x, y\rangle),$$

for all $x, y \in \mathbb{H}^m$.

Example 5.3. [28, Example 2] Let $K = \mathbb{H}^3$ and $T, S : K \to K$ be two nonexpansive mappings which are defined by

$$Tx = Sx = (-x_1, -x_2, -x_3, x_4), \quad \forall x = (x_1, x_2, x_3, x_4) \in \mathbb{H}^3$$

Then $F(T) \cap F(S) = \{(0, 0, 0, 1)\}$.

Now, Thounthong iteration (1.1) can be written in the form

$$\begin{aligned}
w_n &= (\cosh((1-c_n)r(x_n, x_n)))x_n + (\sinh((1-c_n)r(x_n, x_n)))U(x_n, x_n),\\
y_n &= (\cosh((1-b_n)\overline{r}(w_n, w_n)))w_n + (\sinh((1-b_n)\overline{r}(w_n, w_n)))U(w_n, w_n),\\
x_{n+1} &= (\cosh((1-a_n)\overline{r}(Tw_n, y_n)))Tw_n + (\sinh((1-a_n)\overline{r}(Tw_n, y_n)))\overline{U}(Tw_n, y_n),
\end{aligned}$$

and the proposed iteration process (1.2) can be written in the form

$$\begin{aligned}
w_n &= (\cosh((1-b_n)r(x_n, x_n)))x_n + (\sinh((1-b_n)r(x_n, x_n)))U(x_n, x_n),\\
x_{n+1} &= T(S((\cosh((1-a_n)\overline{r}(w_n, w_n)))w_n + (\sinh((1-a_n)\overline{r}(w_n, w_n)))U(w_n, w_n))), \ \forall n \geq 1,
\end{aligned}$$

for all $n \geq 1$, where

$$r(x, y) = \text{arccosh}(-\langle x, Ty\rangle), \quad \overline{r}(x, y) = \text{arccosh}(-\langle x, Sy\rangle),$$
$$U(x, y) = \frac{Ty - \langle x, Ty\rangle x}{\sqrt{1 - \langle x, Ty\rangle^2}}, \qquad \overline{U}(x, y) = \frac{Sy - \langle x, Sy\rangle x}{\sqrt{1 - \langle x, Sy\rangle^2}}, \qquad \text{for all } x, y \in \mathbb{R}^{m+1}.$$

Setting control parameters $a_n = \dfrac{1}{10n+1} + 0.9$, $b_n = \dfrac{1}{20n+1} + 0.8$, $c_n = \dfrac{1}{90n+1} + 0.2$ and stop criterion as $\|x_n - F(T) \cap F(S)\| \leq 10^{-7}$.

We test different initial values as

Choice 1 : x_1 = (3, 3, 9, 10);

Choice 2 : x_1 = (2, 2, 4, 5);

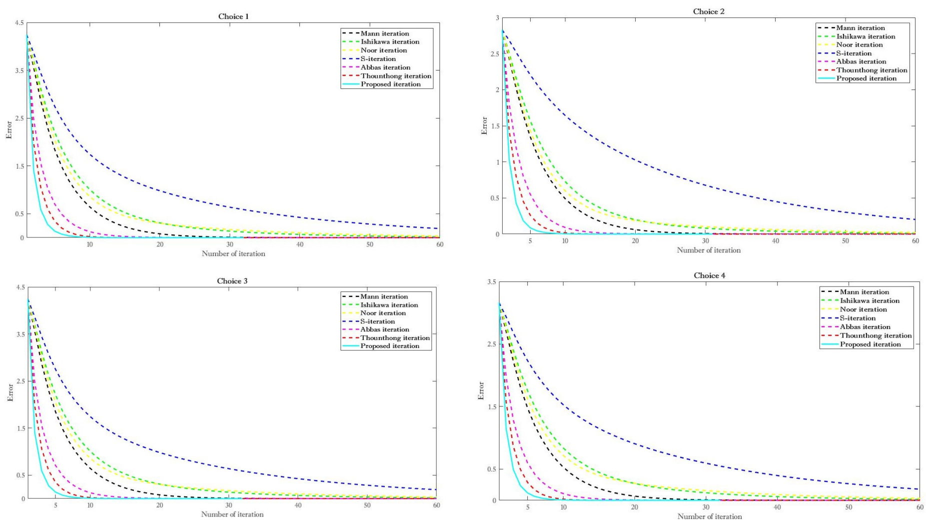
Choice 3 : x_1 = (1, 7, 7, 10);

Choice 4 : x_1 = (1, 3, 5, 6).

TABLE 3. Numerical results for different initial values of Example 5.3

		Mann iter.	Ishikawa iter.	Noor iter.	S iter.	Abbas iter.	Thounthong iter.	Proposed iter.
Choice 1	No. of Iter.	83	231	300	417	58	33	29
	Time	0.994	2.816	3.022	4.178	0.777	0.412	0.109
Choice 2	No. of Iter.	82	224	291	419	56	32	28
	Time	0.771	2.754	3.116	4.011	0.516	0.308	0.096
Choice 3	No. of Iter.	83	231	300	417	58	33	29
	Time	0.987	0.2.711	3.225	4.106	0.764	0.401	0.098
Choice 4	No. of Iter.	82	230	299	415	58	32	29
	Time	0.896	2.566	3.011	4.238	0.711	0.398	0.086

FIGURE 4. Behavior convergence of algorithms corresponding to Table 3



6. CONCLUSIONS

In this paper, we have obtained a new modified two step iteration process in the setting of CAT(κ) spaces. With the help of to guarantee performance of our iteration process, we have shown that the proposed process (1.2) is having a better rate of convergence than number of existing iteration processes in the literature. Moreover, we have provided numerical of two non-trivial examples to show the efficiency of the proposed process (1.2) converges the fastest for a different set of initial values and number of iterations as well as CPU time at least in Tables 2 and 3. Also, Figures 2 and 4 of examples 5.2 and 5.3 are guarantee that the behavior convergence of the proposed algorithm better than existing iterations, respectively.

Acknowledgments. The project is under the support of SUT Research and Development Fund.

REFERENCES

[1] Banach, S., *Sur les opérations dans les ensembles abstraits et leurs application aux équations intégrales*, Fundam. Math., **3** (1922), 133–181

[2] Berinde, V., *On the convergence of the Ishikawa iteration in the class of quasi contractive operators*, Acta Math. Univ. Comenian., **73** (2004), 119–126

[3] Berinde, V. *Picard iteration converges faster than Mann iteration for a class of quasi contractive operators*, Fixed Point Theory Appl., **2004** (2004), 97–105, doi:10.1155/S1687182004311058

[4] Bridson, M. R. and Haefliger, A., *Metric Spaces of Non-positive Curvature*, Springer-Verlag, Berlin, 1999

[5] Chaoha, P. and Phon-on, A., *A note on fixed point sets in CAT(0) spaces*, J. Math. Anal. Appl., **320** (2006), 983–987

[6] Dhompongsa, S., Kaewkhao, A. and Panyanak, B., *Lim's theorems for multivalued mappings in CAT(0) spaces*, J. Math. Anal. Appl., **312** (2005), 478–487

[7] Dhompongsa, S., Kirk, W. A. and Sims, B., *Fixed points of uniformly lipschitzian mappings*, Nonlinear Anal., **65** (2006), 762–772

[8] Espínola, R. and Fernández-León, A., *CAT(κ)-spaces, weak convergence and fixed points*, J. Math. Anal. Appl. **353** (2009), 410—427

[9] Gromov, M., *Hyperbolic groups*, In: Essays in group theory, Springer, New York, **8** (1987), 75–263

[10] He, J. S., Fang, D. H., Lopez, G. and Li, C., *Mann's algorithm for nonexpansive mappings in CAT(κ) spaces*, Nonlinear Anal., **75** (2012), 445–452

[11] Imoru, C. O. and Olatinwo, M. O., *On the stability of Picard and Mann iteration processes*, Carpathian J. Math., **19** (2003), 155–160

[12] Kirk, W. A., *Geodesic geometry and fixed point theory*, In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). Colecc. Abierta. Universidad de Sevilla Secretariado de Publicaciones, Sevilla, **64**, (2003), 195–225

[13] Kirk, W. A., *Geodesic geometry and fixed point theory II*, In: International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama (2004), 113–142

[14] Kirk, W. A. and Panyanak, B., *A concept of convergence in geodesic spaces*, Nonlinear Anal., **68** (2008), 3689–3696

[15] Leustean, L., *A quadratic rate of asymptotic regularity for CAT(0) spaces*, J. Math. Anal. Appl., **325** (2007), 386–399

[16] Lim, T. C., *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc., **60** (1976), 179–182

[17] Ohta, S., *Convexities of metric spaces*, Geom. Dedic. **125** (2007), 225–250

[18] Panyanak, B., *On total asymptotically nonexpansive mappings in* $\mathrm{CAT}(\kappa)$ *spaces*, J. Inequal. Appl., **336** (2014)

[19] Panyanak, B. and Laokul, T., *On the Ishikawa iteration process in CAT(0) spaces*, Bull. Iranian Math. Soc., **37** (2011), 185–197

[20] Panyanak, B., *On the Ishikawa iteration process for multivalued mappings in some CAT(κ) spaces*, Fixed Point Theory Appl. 2014, 2014:1

[21] Pakkaranang, N., Kumam, P., Cholamjiak, P., Suparatulatorn, R. and Chaipunya, P., *Proximal point algorithms involving fixed point iteration for nonexpansive mappings in CAT (κ) spaces*, Carpathian J. Math., **34** (2018), 229–237

[22] Piatek, B., *Halpern iteration in CAT(κ) spaces*, Acta Math. Sin., **27** (2011), 635–646

[23] Saejung, S., *Halpern's iteration in CAT(0) spaces*, Fixed Point Theory Appl., **2010** (2010), ArticleID 471781, 13 pp.

[24] Saluja, G. S., Postolache, M. and Kurdi, A., *Convergence of three-step iterations for nearly asymptotically nonexpansive mappings in CAT(κ) spaces*, J. Inequal. Appl., **2015** (2015), ArticleID 156, 18 pp.

[25] Schu, J., *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc., **43** (1991), 153–159

[26] Shahzad, N. and Markin, J., *Invariant approximations for commuting mappings in* $\mathrm{CAT}(0)$ *and hyperconvex spaces*, J. Math. Anal. Appl., **337** (2008), 1457–1464

[27] Suparatulatorn, R. and Cholumjiak, P., *The modified S-iteration process for nonexpansive mappings in* $\mathrm{CAT}(\kappa)$ *spaces*,Fixed Point Theory Appl., **2016** (2016), ArticleID 25, 12 pages, doi:10.1186/s13663-016-0515-6

[28] Thounthong, P., Pakkaranang, N., Saipara, P., Phairatchatniyom, P. and Kumam, P., *Convergence analysis of modified iterative approaches in geodesic spaces with curvature bounded above*, Math. Methods Appl. Sci., **42** (2019), 5929–5943

[29] Zamfirescu, T., *Fix point theorems in metric spaces*, Arch. Math., **23** (1972), 292–298, doi:10.1007/BF01304884

[1]MATHEMATICS ENGLISH PROGRAM
FACULTY OF EDUCATION
VALAYA ALONGKORN RAJABHAT UNIVERSITY UNDER THE ROYAL PATRONAGE
1 MOO 20 PHAHOLYOTHIN ROAD, KLONG NEUNG, KLONG LUANG, PATHUMTHANI, 13180, THAILAND
Email address: thanatpron.ban@vru.ac.th

[2]DEPARTMENT OF MATHEMATICS
JAMIA MILLIA ISLAMIA, NEW DELHI-110025, INDIA
Email address: c.garodia85@gmail.com (C. Garodia)
Email address: izharuddin1@jmi.ac.in (I. Uddin)

[3]DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KING MONGKUT'S UNIVERSITY OF TECHNOLOGY THONBURI (KMUTT)
126 PRACHA-UTHIT ROAD, BANG MOD, THRUNG KHRU, BANGKOK 10140, THAILAND
Email address: nuttapol.pakka@gmail.com

[4]SCHOOL OF MATHEMATICS
INSTITUTE OF SCIENCE
SURANAREE UNIVERSITY OF TECHNOLOGY, NAKHON RATCHASIMA 30000, THAILAND
Email address: panu.y@sut.ac.th