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Dedicated to Prof. Ioan A. Rus on the occasion of his 85^{th} *anniversary*

Functional differential equations with maxima, via step by step contraction principle

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ABSTRACT. T. A. Burton presented in some examples of integral equations a notion of progressive contractions on $C([a,\infty[)$. In 2019, I. A. Rus formalized this notion (I. A. Rus, *Some variants of contraction principle in the case of operators with Volterra property: step by step contraction principle*, Advances in the Theory of Nonlinear Analysis and its Applications, **3** (2019) No. 3, 111–120), put "step by step" instead of "progressive" in this notion, and give some variant of step by step contraction principle in the case of operators with Volterra property on $C([a,b],\mathbb{B})$ and $C([a,\infty[,\mathbb{B})$ where $\mathbb{B}$ is a Banach space. In this paper we use the abstract result given by I. A. Rus, to study some classes of functional differential equations with maxima.

1. INTRODUCTION

In 1990, Corduneanu investigated functional differential equations involving abstract Volterra operators. In this sense, around the year 2000 Corduneanu [7] presented a general study on functional differential equations with abstract or causal Volterra operators.

On the other hand, T. A. Burton ([3]-[6]) presented in some examples of integral equations a notion of progressive contractions on $C([a,\infty[)$. In 2019, following the idea of T. A. Burton and the forward step method ([19]), I. A. Rus formalized this notion ([21]), with "step by step" instead of "progressive", and give some variant of step by step contraction principle in the case of operators with Volterra property on $C([a,b],\mathbb{B})$ and $C([a,\infty[,\mathbb{B})$ where $\mathbb{B}$ is a Banach space.

In this paper we consider the following functional differential equation with maxima

$$x'(t)=f(t,x(t),\max_{a\le\xi\le t}x(\xi)),\ t\in[a,b] \tag{1.1}$$

with the condition

$$x(a)=\alpha, \tag{1.2}$$

where $\alpha\in\mathbb{R}$ and $f\in C([a,b]\times\mathbb{R}^2)$ are given. To prove our results, we shall use the abstract result given by I. A. Rus [21].

2. PRELIMINARIES

2.1. **Weakly Picard operators.** In the sequel, the following results are useful for some of the proofs in the paper (see [16, 17]).

Let (X,d) be a metric space. An operator $A:X\to X$ is called weakly Picard operator (WPO) if the sequence of successive approximations, $\{A^n(x)\}_{n\in\mathbb{N}}$, converges for all $x\in X$ and its limit (which generally depend on x) is a fixed point of A. If an operator A is WPO

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with a unique fixed point, that is, $F_A = \{x^*\}$, then, by definition, A is called a Picard operator (PO).

If $A : X \to X$ is a WPO, we can define the operator $A^\infty : X \to F_A$, by $A^\infty(x) := \lim_{n\to\infty} A^n(x)$.

2.2. **G-contractions.** Let (X, d) be a metric space and $G \subset X \times X$ be a nonempty binary relation. An operator $A : X \to X$ is a G-contraction if there exists $l \in (0, 1)$ such that,

$$d(A(x), A(y)) \leq ld(x, y),\ \forall (x, y) \in G.$$

Let us give an example of G-contraction. For other examples see [2], [18], [21] and [22].

Let $a < c < b$ and $X := C[a, b]$, with $d(x, y) := \max_{a\leq t\leq b} |x(t) - y(t)|$. For $H \in C([a, b] \times [a, b] \times \mathbb{R})$ we consider the operator, $A : C[a, b] \to C[a, b]$ defined by

$$A(x)(t) := \int_a^t H(t, s, \max_{a\leq\xi\leq s} x(\xi))ds.$$

We suppose that there exists $L > 0$ such that

$$|H(t, s, u) - H(t, s, v)| \leq L |u - v|,\ t, s \in [a, b],\ u, v \in \mathbb{R}.$$

Let $G := \{(x, y)|\ x, y \in C([a, b], \mathbb{R}),\ x|_{[a,c]} = y|_{[a,c]}\}$. If $L(b - c) < 1$, then A is a G-contraction.

Indeed for $t \in [a, c]$ if $x|_{[a,c]} = y|_{[a,c]}$, then $A(x)(t) = A(y)(t)$.

If $t \in [c, b]$, then

$$A(x)(t) = \int_a^c H(t, s, \max_{a\leq\xi\leq s} x(\xi))ds + \int_c^t H(t, s, \max_{a\leq\xi\leq s} x(\xi))ds,$$

$$x, y \in G \Rightarrow \|A(x) - A(y)\| \leq L(b - c) \|x - y\|.$$

2.3. **Step by step contraction.** Let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space and $C([a, c], \mathbb{B})$ the Banach space with max-norm, $\|\cdot\|$. In what follows, in all spaces of functions we consider max-norm. For $m \in \mathbb{N},\ m \geq 2$, let $t_0 := a,\ t_k := t_0 + k\frac{b-a}{m},\ k = \overline{1, m}$. Let $V : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$ be an operator with Volterra property. Let $V_k : C([t_0, t_k], \mathbb{B}) \to C([t_0, t_k], \mathbb{B}), k = \overline{1, m-1}$ the operator induced by V on $C([t_0, t_k], \mathbb{B})$. We also consider the following sets,

$$G_k := \{(x, y)|\ x, y \in C([t_0, t_{k+1}], \mathbb{B}),\ x|_{[t_0,t_k]} = y|_{[t_0,t_k]}\},\ k = \overline{1, m-1}.$$

For $x_k \in C([t_0, t_k], \mathbb{B}),\ k = \overline{1, m-1}$, we denote

$$X_{x_k} := \{y \in C([t_0, t_{k+1}], \mathbb{B}),\ y|_{[t_0,t_k]} = x_k\}.$$

The following result is given in [21].

Theorem 2.1. (Theorem of step by step contraction). *We suppose that:*

(1) $V : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$ has the Volterra property;
(2) V_1 is a contraction;
(3) V_k is a G_{k-1}-contraction, for $k = \overline{2, m}$.

Then

(i) $F_V = \{x^*\}$;

(ii)

$$x^*|_{[t_0,t_1]} = V_1^\infty(x),\ \forall x \in C([t_0,t_1],\mathbb{R}),$$
$$x^*|_{[t_0,t_2]} = V_2^\infty(x),\ \forall x \in X_{x^*|_{[t_0,t_1]}},$$
$$\vdots$$
$$x^*|_{[t_0,t_{m-1}]} = V_{m-1}^\infty(x),\ \forall x \in X_{x^*|_{[t_0,t_{m-2}]}};$$

(iii) $x^* = V^\infty(x),\ \ \forall x \in X_{x^*|_{[t_0,t_{m-1}]}}$.

For other details and results concerning the theory of G-contraction, step by step contraction, Picard operator, weakly Picard Operator and equations with maxima, see: [1], [8]-[21].

3. Main result

In this section, we shall establish a new result of existence and uniqueness of the solution of the functional differential equation with maxima (1.1).

The problem (1.1)–(1.2), $x \in C^1([a,b],\mathbb{R})$ is equivalent with the fixed point equation

$$x(t) = \alpha + \int_a^t f(s, x(s), \max_{a\le\xi\le s} x(\xi))ds,\ t \in [a,b]. \tag{3.3}$$

It is clear that equation (3.3) is equivalent with $x = V(x)$, where the operator $V : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$, defined by

$$V(x)(t) := \alpha + \int_a^t f(s, x(s), \max_{a\le\xi\le s} x(\xi))ds,\ t \in [a,b]. \tag{3.4}$$

The operator V has the Volterra property, i.e.,

$$t \in (a,b),\ x,y \in C[a,b],\ x|_{[a,t]} = y|_{[a,t]} \Rightarrow V(x)|_{[a,t]} = V(y)|_{[a,t]}.$$

This implies that the operator V induced, for each c with $a < c < b$ and, the operator $V_c : C[a,c] \to C[a,c]$, defined by, $V_c(x)(t) := V(\widetilde{x})$, where $\widetilde{x} \in C[a,b]$ is such that, $\widetilde{x}|_{[a,c]} = x$.

In what follows we consider the notations from Section 2.3, where $\mathbb{B} = \mathbb{R}$.

We have

Theorem 3.2. *We suppose that:*

(1) There exists $L > 0$, such that

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le L\max(|u_1 - v_1|, |u_2 - v_2|),$$

for all $t \in [a,b], u_i, v_i \in \mathbb{R}, i = 1,2$.

(2) $m \in \mathbb{N}^*$ is such that

$$\frac{L(b-a)}{m} < 1.$$

Then, we have

(i) $F_V = \{x^*\}$, i.e., the problem (1.1)-(1.2) has a unique solution.

(ii)

$$x^*|_{[t_0,t_1]} = V_1^\infty(x),\ \forall x \in C[t_0,t_1],$$
$$x^*|_{[t_0,t_2]} = V_2^\infty(x),\ \forall x \in X_{x^*}$$
$$\vdots$$
$$x^*|_{[t_0,t_{m-1}]} = V_{m-1}^\infty(x),\ \forall x \in X_{x^*}|_{[t_0,t_{m-1}]}.$$

(iii) $x^* = V^\infty(x),\ \forall x \in X_{x^*}|_{[t_0,t_{m-1}]}$.

Proof. We shall prove that in the conditions (1) and choosing m as in (2), we are in the conditions of Theorem of step by step contractions.

Let us prove that V_1 is an contraction. We have:

$$\begin{aligned}|V_1(x)(t) - V_1(y)(t)| &\leq \left|\int_a^t f(s, x(s), \max_{a\leq\xi\leq s} x(\xi))ds - \int_a^t f(s, y(s), \max_{a\leq\xi\leq s} y(\xi))ds\right| \\ &\leq L\int_a^t \max\left(|x(s)-y(s)|, \left|\max_{a\leq\xi\leq s} x(\xi) - \max_{a\leq\xi\leq s} y(\xi)\right|\right) ds \\ &\leq \frac{L(b-a)}{m}\max_{t_0\leq t\leq t_1}|x(t)-y(t)|.\end{aligned}$$

It follows that

$$\max_{t_0\leq t\leq t_1}|V_1(x)(t) - V_1(y)(t)| \leq \frac{L(b-a)}{m}\max_{t_0\leq t\leq t_1}|x(t)-y(t)|.$$

So, V_1 is a contraction.

Let us prove that V_2 is a G_1-contraction. First we remark that, for $t \in [t_0,t_1]$

$$V_2(x)(t) = V_2(y)(t),\ \text{for } x, y \in G_1.$$

$$\begin{aligned}|V_2(x)(t) - V_2(y)(t)| &= \left|\int_a^{t_1}\left[f(s, x(s), \max_{a\leq\xi\leq s} x(\xi)) - f(s, y(s), \max_{a\leq\xi\leq s} y(\xi))\right] ds\right. \\ &\quad \left.+ \int_{t_1}^t \left[f(s, x(s), \max_{a\leq\xi\leq s} x(\xi)) - f(s, y(s), \max_{a\leq\xi\leq s} y(\xi))\right] ds\right| \\ &= \left|\int_{t_1}^t \left[f(s, x(s), \max_{a\leq\xi\leq s} x(\xi)) - f(s, y(s), \max_{a\leq\xi\leq s} y(\xi))\right] ds\right| \\ &\leq \frac{L(b-a)}{m}\max_{t_0\leq t\leq t_2}|x(t)-y(t)|.\end{aligned}$$

In a similar way we prove that $V_3,\ldots,V_m$ are $G_2,\ldots,G_{m-1}$ contractions.

Now the prove follows from the Theorem of step by step contractions. □

Remark 3.1. In the conditions of the Theorem 3.2 let us denote $x^*|_{[t_0,t_k]} = x_k^*,\ 1 \leq k \leq m-1$. Then we have that:

The sequence of successive approximations

$$x_{1,n+1}(t) = \int_a^t f(s, x_{1,n}(s), \max_{a\leq\xi\leq s} x_{1,n}(\xi))ds,\ t \in [t_0,t_1]$$

converges uniformly on $[t_0,t_1]$ to $x_1^* = x^*|_{[t_0,t_1]}$.

The sequence of successive approximations

$$x_{2,n+1}(t)=\begin{cases} x_1^*(t),\ t\in[t_0,t_1] \\ x_1^*(t_1)+\int_{t_1}^t f(s,x_{2,n}(s),\max\limits_{a\le\xi\le s} x_{2,n}(\xi))ds,\ t\in[t_1,t_2]\end{cases}$$

converges uniformly on $[t_0,t_2]$ to $x_2^*=x^*|_{[t_0,t_2]}$.

...

The sequence of successive approximations

$$x_{m-1,n+1}(t)=\begin{cases} x_{m-2}^*(t),\ t\in[t_0,t_{m-2}] \\ x_{m-2}^*(t_{m-2})+\int_{t_{m-2}}^t f(s,x_{m-1,n}(s),\max\limits_{a\le\xi\le s} x_{m-1,n}(\xi))ds,\ t\in[t_{m-2},t_{m-1}]\end{cases}$$

converges uniformly on $[t_0,t_{m-1}]$ to $x_{m-1}^*=x^*|_{[t_0,t_{m-1}]}$.

The above considerations give rise to the following problem: In which conditions the operator V is Picard operator?

From the Fibre contraction principle (see [21]) the answer is the following: In the conditions of the Theorem 3.2, the operator V is a Picard operator with respect to the uniform convergence on $[t_0,t_m]$·

In order to prove this we consider the following operators induced by the operator V. First of all from (3.4) we have that:

(4.1) $V(x)(t):=\alpha+\int_{t_0}^t f(s,x(s),\max\limits_{t_0\le\xi\le s} x(\xi))ds,\ t\in[t_0,t_1]$,

(4.2) $V(x)(t):=\alpha+\int_{t_0}^{t_1} f(s,x(s),\max\limits_{t_0\le\xi\le s} x(\xi))ds+\int_{t_1}^t f(s,x(s),\max\limits_{t_0\le\xi\le s} x(\xi))ds,\ t\in[t_1,t_2]$,

⋮

(4.k) $V(x)(t):=\alpha+\int_{t_0}^{t_1} f(s,x(s),\max\limits_{t_0\le\xi\le s} x(\xi))ds+...+\int_{t_{k-1}}^t f(s,x(s),\max\limits_{t_0\le\xi\le s} x(\xi))ds,\ t\in[t_{k-1},t_k],k=\overline{1,m}$.

Let $R:C[t_0,t_m]\to C[t_0,t_1]\times C[t_1,t_2]\times...\times C[t_{m-1},t_m]$ be defined by,

$$x\to\left(x|_{[t_0,t_1]},x|_{[t_1,t_2]},...,x|_{[t_{m-1},t_m]}\right).$$

We also consider the following subset:

$$U\subset\prod_{k=1}^m C[t_{k-1},t_k],U:=\left\{(x_1,...,x_m)\mid x_k(t_k)=x_{k+1}(t_k),k=\overline{1,m-1}\right\}\cdot$$

It is clear that $R:C[t_0,t_m]\to U$ is a bijection.

Let us consider the following operators induced by the operator V :

$T_1:C[t_0,t_1]\to C[t_0,t_1]$,

$$T_1(x_1)(t):=\alpha+\int_{t_0}^t f(s,x_1(s),\max_{t_0\le\xi\le s} x_1(\xi))ds,\ t\in[t_0,t_1],$$

$T_2:C[t_0,t_1]\times C[t_1,t_2]\to C[t_1,t_2]$,

$$T_2(x_1,x_2)(t):=\alpha+\int_{t_0}^{t_1} f(s,x_1(s),\max_{t_0\le\xi\le s} x_1(\xi))ds+\int_{t_1}^t f(s,x_2(s),\max_{t_0\le\xi\le s} x_2(\xi))ds,\ t\in[t_1,t_2],$$

⋮

$T_k : C[t_0, t_1] \times C[t_1, t_2] \times ... \times C[t_{k-1}, t_k] \to C[t_{k-1}, t_k],$

$$T_k (x_1, x_2, ..., x_k)(t) := \alpha + \int_{t_0}^{t_1} f(s, x_1(s), \max_{t_0 \le \xi \le s} x_1(\xi))ds + ...$$
$$+ \int_{t_{k-1}}^{t} f(s, x_k(s), \max_{t_0 \le \xi \le s} x_k(\xi))ds,\ t \in [t_{k-1}, t_k], k = \overline{1, m}$$

and

$$T : \prod_{k=1}^{m} C[t_{k-1}, t_k] \to \prod_{k=1}^{m} C[t_{k-1}, t_k], T := (T_1, T_2, ..., T_m).$$

In the conditions of Theorem 3.2, the operators, $T_1, T_2(x_1, \cdot), ..., T_m(x_1, ..., x_{m-1}, \cdot)$ are contractions. From the Fibre Contraction Principle, T is a Picard operator.

Now, we observe that: $V = R^{-1}TR$ and $V^n = R^{-1}T^nR$. These imply that the operator V is a Picard operator.

4. Differential inequalities

In this section we will emphasize the importance of the above result by applying for the operator V the Gronwall type inequalities and the comparison theorem.

In this section we suppose that

(H) there exists $L > 0$ such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le L \max(|u_1 - v_1|, |u_2 - v_2|)$$

for all $t \in [a, b]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$.

We consider on $C([a, b], \mathbb{R})$ the max norm and in condition (H), the operator V defined by (3.4) is a Picard operator. So, in the condition (H), the problem (1.1)-(1.2) has in $C([a, b], \mathbb{R})$ a unique solution x^*. Moreover, for $t \in [a, b],\ x^*(t) = \lim_{n\to\infty} x_n(t)$, for each $x_0 \in C([a, b], \mathbb{R})$, where $(x_n)_{n\in\mathbb{N}}$ is defined by

$$x_{n+1} = \alpha + \int_a^t f(s, x_n(s), \max_{a \le \xi \le s} x_n(\xi))ds,\ t \in [a, b].$$

Now we can apply Abstract Gronwall Lemma (see [21]).

Theorem 4.3. *Let us consider the problem (1.1)-(1.2) in the condition (H) and $f(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is increasing, i.e., $u_1 \le v_1, u_2 \le v_2 \Rightarrow f(t, u_1, u_2) \le f(t, v_1, v_2)$, for all $t \in [a, b]$. Let us denote by x^* the unique solution of (1.1)-(1.2). Then the following implications holds:*

(i) $x \in C([a, b], \mathbb{R}),\ x(a) = \alpha, x'(t) \le f(t, x(t), \max_{a \le \xi \le t} x(\xi)),\ t \in [a, b] \Rightarrow x \le x^*$;

(ii) $x \in C([a, b], \mathbb{R}),\ x(a) = \alpha, x'(t) \ge f(t, x(t), \max_{a \le \xi \le t} x(\xi)),\ t \in [a, b] \Rightarrow x \ge x^*$.

In a similar way, a comparison theorem for equation (1.1) can be obtained, using the Abstract Comparison Lemma.

We consider now the following functional differential equations with maxima

$$x'(t) = f_i(t, x(t), \max_{a \le \xi \le t} x(\xi)),\ t \in [a, b] \tag{4.5}$$

with the condition

$$x(a) = \alpha_i, \tag{4.6}$$

where $\alpha_i \in \mathbb{R}$ and $f_i \in C([a, b] \times \mathbb{R}^2),\ i = 1, 2, 3$ are given. We suppose that

(H') there exists $L_i > 0$ such that

$$|f_i(t, u_1, u_2) - f_i(t, v_1, v_2)| \leq L_i \max(|u_1 - v_1|, |u_2 - v_2|),$$

for all $t \in [a, b]$ and $u_1, v_1, u_2, v_2 \in \mathbb{R}, i = 1, 2, 3$.

Theorem 4.4. *Let us consider the problems (4.5)-(4.6) in the condition:* (H'), $f_2(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ *is increasing, for all* $t \in [a, b]$ *and* $\alpha_1 \leq \alpha_2 \leq \alpha_3,\ f_1 \leq f_2 \leq f_3$. *Let us denote by* $x_i^*, i = 1, 2, 3$ *the unique solutions of (4.5)-(4.6). Then the following implication holds:*

$$x_1(a) \leq x_2(a) \leq x_3(a) \Rightarrow x_1^* \leq x_2^* \leq x_3^*.$$

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References

[1] Bainov, D. D. and Hristova, S., *Differential equations with maxima*, Chapman & Hall/CRC Pure and Applied Mathematics, 2011

[2] Berzig, M., *Coincidence and common fixed point results on metric spaces endowed with an arbitrary binary relation and applications*, J. Fixed Point Theory Appl., **12** (2012), No. 1-2, 221–238

[3] Burton, T. A., *Integral equations, transformations, and a Krasnoselskii–Schaefer type fixed point theorem*, Electron. J. Qual. Theory Differ. Equ., (2016), No. 66, 1–13; doi: 10.14232/ejqtde.2016.1.66

[4] Burton, T. A., *Existence and uniqueness results by progressive contractions for integro- differential equations*, Nonlinear Dyn. Syst. Theory, **16** (2016), No. 4, 366–371

[5] Burton, T. A., *An existence theorem for a fractional differential equation using progressive contractions*, J. Fract. Calc. Appl, **8** (2017), No. 1, 168–172

[6] Burton, T. A., *A note on existence and uniqueness for integral equations with sum of two operators: progressive contractions*, Fixed Point Theory, **20** (2019), No. 1, 107–112

[7] Corduneanu, C., Abstract Volterra equations: a survey, Math. and Computer Model., **32** (2000), No. (11–13) 1503–1528

[8] Halanay, A., *Differential Equations: Stability, Oscillations*, Time Lags, Acad. Press, New York, 1966

[9] Ilea, V. and Otrocol, D., *On the Burton method of progressive contractions for Volterra integral equations*, Fixed Point Theory, **21** (2020), No. 2, 585–594

[10] Marian, D. and Lungu, N., *Ulam-Hyers-Rassias stability of some quasilinear partial differential equations of first order*, Carpatian J. Math., **35** (2019), No. 2, 165–170

[11] Marian, D., Ciplea, S. A. and Lungu, N., *On the Ulam-Hyers stability of biharmonic equation*, U. P. B. Sci. Bull., Series A, **82** (2020), No. 2, 141–148

[12] Marian, D., Ciplea, S. A. and Lungu, N., *Optimal and nonoptimal Gronwall lemmas*, Symmetry, **12** (2020), No. 10, 1728, 1–10

[13] Otrocol, D., *Ulam stabilities of differential equation with abstract Volterra operator in a Banach space*, Nonlinear Funct. Anal. Appl., **15** (2010), No. 4, 613–619

[14] Otrocol, D. and Rus, I. A., *Functional-differential equations with "maxima" via weakly Picard operators theory*, Bull. Math. Soc. Sci. Math. Roumanie (N. S) , **51 (99)** (2008), No. 3, 253–261

[15] Otrocol, D. and Rus, I. A., *Functional-differential equations with maxima of mixed type argument*, Fixed Point Theory, **9** (2008), No. 1, 207–220

[16] Rus, I. A., *Generalized contractions and applications*, Cluj University Press, 2001

[17] Rus, I. A., *Picard operators and applications*, Scientiae Mathematicae Japonicae, **58** (2003), No. 1, 191–219

[18] Rus, I. A., *Cyclic representations and fixed points*, Ann. T. Popoviciu Seminar of Functional Eq. Approx. Convexity, **3** (2005), 171–178

[19] Rus, I. A., *Abstract models of step method which imply the convergence of successive approximations*, Fixed Point Theory, **9** (2008), No. 1, 293–307

[20] Rus, I. A., *Some nonlinear functional differential and integral equations, via weakly Picard operator theory: a survey*, Carpathian J. Math., **26** (2010), No. 2, 230–258

[21] Rus, I. A., *Some variants of contraction principle in the case of operators with Volterra property: step by step contraction principle*, Advances in the Theory of Nonlinear Analysis and its Applications, **3** (2019), No. 3, 111–120

[22] Samet, B. and Turinici, M., *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications*, Commun. Math. Anal., **13** (2012), No. 2, 82–97

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