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Dedicated to Prof. Ioan A. Rus on the occasion of his 85^{th} anniversary

Implicit functional differential equations with linear modification of the argument, via weakly Picard operator theory

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ABSTRACT. Let $\mathbf{K} := \mathbf{R}$ or $\mathbf{C}$, $0 < \lambda < 1$ and $f \in C([0, b] \times \mathbf{K}^3, \mathbf{K})$.

In this paper we use the weakly Picard operator theory technique to study the following functional-differential equation

$$y'(x) = f(x, y(x), y'(x), y(\lambda x)), x \in [0, b].$$

1. INTRODUCTION

The theory of functional-differential equations and of functional-integral equations are both active fields in mathematics.

Many problems from physics, chemistry, astronomy, biology, engineering, social sciences lead to mathematical models described by functional-differential and functional-integral equations (see [7], [8], [10], [13], [15], [19], [21]). The theory of this kind of equations has developed very much.

For the monographs in this field we quote here [1]-[4], such as a large number of papers, which contain a lot of techniques, ideas and applications.

Let $\mathbf{K} := \mathbf{R}$ or $\mathbf{C}$, $0 < \lambda < 1$ and $f \in C([0, b] \times \mathbf{K}^3, \mathbf{K})$.

In this paper we use the weakly Picard operator theory technique to study the following functional differential equation

$$y'(x) = f(x, y(x), y'(x), y(\lambda x)), x \in [0, b].$$

We obtain existence, uniqueness and data dependence results for the solution.

2. PRELIMINARIES

2.1. **Notations and terminology.** Let X be a nonempty set and $A : X \longrightarrow X$ an operator.

We denote by $A^0 := 1_X$, $A^1 := A$, ..., $A^{n+1} := A \circ A^n$, $n \in \mathbb{N}$, the iterate operators of A. Also:

$$P(X) \; : \; = \{Y \subset X \mid Y \neq \emptyset\},$$
$$I(A) \; : \; = \{Y \in P(X) \mid A(Y) \subset Y\},$$

the family of all nonempty invariant subsets of A,

$$F_A = \{x \in X \,|A(x) = x\},$$

the fixed point set of the operator A.

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Let $(X, \to)$ be an $L-$ space (see [14],[18]).

Following Rus I. A. ([11], [14], [15]), we have:

Definition 2.1. A is a Picard operator if there exists $x^* \in X$ such that

1) $F_A = \{x^*\}$;

2) the successive approximation sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$

Definition 2.2. A is a weakly Picard operator if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which generally depends on x_0) is a fixed point of A.

Definition 2.3. If $A : X \to X$ is a weakly Picard operator then we define the operator A^∞ as follows:

$$A^\infty : X \to X, \;\; A^\infty(x) := \lim_{n\to\infty} A^n(x), \text{ for all } x \in X.$$

We remark that $A^\infty(X) = F_A$. So A^∞ is an retract of X on F_A.

2.2. **Weakly Picard operators.** We have the following theorems:

Theorem 2.1. (**Contraction mapping principle**). Let (X, d) *be a complete metric space and* $A : X \to X$ *a contraction. Then* A *is a Picard operator.*

Theorem 2.2. (**Characterization theorem**). Let (X, d) *be a metric space and* $A : X \to X$ *an operator. The operator* A *is a weakly Picard operator if and only if there exists a partition of* X, $X = \cup_{\lambda\in\Lambda} X_\lambda$, *such that:*

(i) $X_\lambda \in I(A)$;

(ii) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator, for all* $\lambda \in \Lambda$.

Fibre generalized contraction theorem is a fixed point theorem for operators on cartezian product. This theorem is useful for proving solution of operatorial equations to be differentiable and it is a generalization of a result given by Hirsch and Pugh in [5]. See also [12], [20], [21].

Theorem 2.3. (**Fibre contraction theorem**). Let (X, d) *be a metric space,* (Y, ρ) *be a complete metric space and* $T : X \times Y \to X \times Y$. *We suppose that:*

(i) $T(x, y) = (T_1(x), T_2(x, y))$, *for all* $x \in X, y \in Y$;

(ii) $T_1 : X \to X$ *is a weakly Picard operator;*

(iii) *there exists* $c \in]0, 1[$, *such that*

$$\rho(T_2(x, y_1), T_2(x, y_2)) \leq c\rho(y_1, y_2),$$

for all $x \in X$, $y_1, y_2 \in Y$.

Then the operator T *is a weakly Picard operator. Moreover, if* T_1 *is a Picard operator, then* T *is a Picard operator.*

We will use the previous result to study the differentiability with respect to parameter λ for the solution of our equation.

3. Cauchy problem: existence and uniqueness

Let $\mathbf{K} := \mathbf{R}$ or $\mathbf{C}$, $0 < \lambda < 1$ and $f \in C([0, b] \times \mathbf{K}^3, \mathbf{K})$.

We consider the following Cauchy problem:

$$y'(x) = f(x, y(x), y'(x), y(\lambda x)), x \in [0, b], \tag{3.1}$$

$$y(0) = y_0, \tag{3.2}$$

where $y_0 \in \mathbf{K}$.

The problem (3.1)+(3.2) is equivalent to the following:

$$\left\{\begin{array}{c} y'(x) = v(x) \\ v(x) = f\left(x, y_0 + \int\limits_0^x v(s)ds, v(x), y_0 + \int\limits_0^{\lambda x} v(s)ds\right) \\ y(0) = y_0, \end{array}\right. \tag{3.3}$$

or

$$\left\{\begin{array}{c} y(x) = y_0 + \int\limits_0^x v(s)ds,\ x \in [0,b] \\ v(x) = f\left(x, y_0 + \int\limits_0^x v(s)ds, v(x), y_0 + \int\limits_0^{\lambda x} v(s)ds\right), \\ x \in [0,b]\ . \end{array}\right. \tag{3.4}$$

Let $T : C([0,b],\mathbf{K}) \to C([0,b],\mathbf{K})$ be the operator defined by

$$(T(v))(x) := f\left(x, y_0 + \int\limits_0^x v(s)ds, v(x), y_0 + \int\limits_0^{\lambda x} v(s)ds\right).$$

So we obtain a fixed point problem

$$v(x) = (T(v))(x),\ v \in C([0,b],\mathbf{K}). \tag{3.5}$$

The problem (3.1)+(3.2) has a unique solution if and only if the problem (3.5) has a unique solution, $v^* \in C([0,b],\mathbf{K})$.

Consequently, in our paper, we will study the fixed point problem (3.5).

By using Contraction principle we give an existence and uniqueness theorem.

Theorem 3.4. *We suppose that there exist* $L_1 > 0$, $0 < L_2 < 1$, $L_3 > 0$ *such that*

$$|f(x,u_1,u_2,u_3) - f(x,u_4,u_5,u_6)| \le L_1|u_1 - u_4| + L_2|u_2 - u_5| + L_3|u_3 - u_6|$$

for all $x \in [0,b]$ *and all* $u_k \in K$, $k = \overline{1,6}$.

Then

(a) *the problem* (3.5) *has a unique solution* $v^* \in C([0,b],K)$;

(b) *for all* $v_0 \in C([0,b],K)$ *the sequence* $(v_n)_{n\in\mathbb{N}}$ *defined by*

$$v_{n+1}(x) := f\left(x, y_0 + \int\limits_0^x v_n(s)ds, v_n(x), y_0 + \int\limits_0^{\lambda x} v_n(s)ds\right),$$

converges uniformly to v^* *on* $[0,b]$.

Proof. Let $||\cdot||_B$ be a Bielecki norm on $C([0,b],\mathbf{K})$ defined by

$$||v||_B = \max_{x\in[0,b]} |v(x)|e^{-\tau x},\ \text{where } \tau > 0.$$

Consider the above operator $T : (C([0,b],\mathbf{K}), ||\cdot||_B) \to (C([0,b],\mathbf{K}), ||\cdot||_B)$.

We have

$$|(T(v))(x) - (T(w))(x)| \le L_1 \int\limits_0^x |v(s) - w(s)|ds +$$

$$+ L_2|v(x) - w(x)| + L_3 \int\limits_0^{\lambda x} |v(s) - w(s)|ds \le$$

$$\leq L_1 \int_0^x |v(s) - w(s)| e^{-\tau s} e^{\tau s} ds + L_2 |v(x) - w(x)| e^{-\tau x} e^{\tau x} +$$

$$+ L_3 \int_0^x |v(s) - w(s)||e^{-\tau s} e^{\tau s} ds \leq \frac{L_1 + L_3}{\tau}(e^{\tau x} - 1)||v - w||_B +$$

$$+ L_2 ||v - w||_B e^{\tau x} \leq (\frac{L_1 + L_3}{\tau} + L_2)||v - w||_B e^{\tau x},$$

for all $x \in [0, b]$.

Therefore,

$$|(T(v))(x) - (T(w))(x)| e^{-\tau x} \leq (\frac{L_1 + L_3}{\tau} + L_2)||v - w||_B,$$

for all $x \in [0, b]$.

This implies that

$$||T(v) - T(w)||_B \leq (\frac{L_1 + L_3}{\tau} + L_2)||v - w||_B,$$

for all $v, w \in C([0, b], \mathbf{K})$.

We can choose τ large enough such that $\frac{L_1 + L_3}{\tau} + L_2 < 1$.

By applying Contraction principle we obtain (a) and (b). □

Remark 3.1. Let us consider the operator

$$A : C([0, b], \mathbf{K}) \times C([0, b], \mathbf{K}) \to C([0, b], \mathbf{K}) \times C([0, b], \mathbf{K})$$

defined by

$$A(y, v)(x) := \left(y(0) + \int_0^x v(s) ds,\ f\left(x, y(0) + \int_0^x v(s) ds), v(x), y(0) + \int_0^{\lambda x} v(s) ds \right) \right).$$

From the Theorem 3.4. it is clear that the operator A, in the conditions of Theorem 3.4., is a weakly Picard operator. Indeed, let for $y_0 \in \mathbf{K}$

$$X_{y_0} := \{ y \in C([0, b], \mathbf{K}) \mid y(0) = y_0 \}.$$

Then

$$C([0, b], \mathbf{K}) \times C([0, b], \mathbf{K}) = \bigcup_{y_0 \in \mathbf{K}} (X_{y_0} \times C([0, b], \mathbf{K}))$$

is an invariant partition of $C([0, b], \mathbf{K}) \times C([0, b], \mathbf{K})$, i.e.,

$$A(X_{y_0} \times C([0, b], \mathbf{K})) \subset X_{y_0} \times C([0, b], \mathbf{K}), \text{ for all } y_0 \in \mathbf{K}.$$

From a similar proof as in Theorem 3.4. we have that $A_{|X_{y_0} \times C([0,b], \mathbf{K})}$ is a Picard operator for each $y_0 \in \mathbf{K}$. So, from Theorem 2.2., the operator A is a weakly Picard operator.

From the definition of operator A we have that:

- if y is a solution of the equation 3.1., then $(y, y') \in F_A$;
- if $(y, v) \in F_A$, then y is a solution of (3.1).

4. Data dependence

By using Fibre contraction theorem we give a data dependence theorem for the solution of the following equation:

$$v(x,\lambda) = (T(v))(x,\lambda),\ \ v \in C([0,b]\times]0,1[,\mathbf{K}). \tag{4.6}$$

We have

Theorem 4.5. *We suppose that:*

(i) $f(x,\cdot,\cdot,\cdot) \in C^1(\mathbf{K}^3)$, *for all* $x \in [0,b]$;

(ii) there exist $M_k > 0$, $k = \overline{1,3}$, *such that*

$$\left|\frac{\partial f}{\partial u_k}(x,u_1,u_2,u_3)\right| \leq M_k,\ k = \overline{1,3},$$

for all $x \in [0,b]$ *and all* $u_k \in \mathbf{K}$, $k = \overline{1,3}$;

(iii) $0 < M_2 < 1$.

Then

(a) the equation (4.6) *has a unique solution* $v^* \in C([0,b]\times]0,1[,\mathbf{K})$;

(b) for all $v_0 \in C([0,b]\times]0,1[,\mathbf{K})$ *the sequence* $(v_n)_{n\in\mathbb{N}}$ *defined by*

$$v_{n+1}(x,\lambda) := f\left(x, y_0 + \int_0^x v_n(s,\lambda)ds, v_n(x,\lambda), y_0 + \int_0^{\lambda x} v_n(s,\lambda)ds\right)$$

converges uniformly to v^* *on each compact of* $[0,b]\times]0,1[$ *;*

(c) the sequence $(w_n)_{n\in\mathbb{N}}$ *defined by*

$$\begin{aligned} w_{n+1}(x,\lambda) :=& \\ =& \frac{\partial f}{\partial u_1}\left(x, y_0 + \int_0^x v_n(s,\lambda)ds, v_n(x,\lambda), y_0 + \int_0^{\lambda x} v_n(s,\lambda)ds\right)\int_0^x w_n(s,\lambda)ds + \\ &+ \frac{\partial f}{\partial u_2}\left(x, y_0 + \int_0^x v_n(s,\lambda)ds, v_n(x,\lambda), y_0 + \int_0^{\lambda x} v_n(s,\lambda)ds\right) w_n(x,\lambda) + \\ &+ \frac{\partial f}{\partial u_3}\left(x, y_0 + \int_0^x v_n(s,\lambda)ds, v_n(x,\lambda), y_0 + \int_0^{\lambda x} v_n(s,\lambda)ds\right)\cdot \\ &\cdot\left(\int_0^{\lambda x} w_n(s,\lambda)ds + x v_n(\lambda x,\lambda)\right), \end{aligned}$$

where $w_0 = \dfrac{\partial v_0}{\partial \lambda}$, *converges uniformly to* $\dfrac{\partial v^*}{\partial \lambda}$ *on each compact of* $[0,b]\times]0,1[$.

Proof. For $0 < \lambda_1 < \lambda_2 < 1$, we denote $X = (C([0,b]\times[\lambda_1,\lambda_2],\mathbf{K}), ||\cdot||_\tau)$, where

$$||v||_\tau = \max_{\substack{x\in[0,b]\\ \lambda\in[\lambda_1,\lambda_2]}} |v(x,\lambda)|e^{-\tau x},$$

and $\tau > 0$.

Consider the operator $S_1 : X \to X$, defined by

$$(S_1(v))(x,\lambda) := f\left(x, y_0 + \int_0^x v(s,\lambda)ds, v(x,\lambda), y_0 + \int_0^{\lambda x} v(s,\lambda)ds\right).$$

The operator S_1 is a Lipchitz operator with the constant

$$L_{S_1} = \frac{M_1 + M_3}{\tau} + M_2.$$

Because of condition (iii), and by choosing τ large enough we have that $L_{S_1} < 1$.

By applying Contraction principle to the operator S_1 we obtain (a) and (b).

Let us prove that there exists $\dfrac{\partial v^*}{\partial \lambda}$ and

$$\frac{\partial v^*}{\partial \lambda}(x, \cdot) \in C([\lambda_1, \lambda_2], \mathbf{K}), \text{ for all } x \in [0, b].$$

We have

$$v^*(x,\lambda) = f\left(x, y_0 + \int_0^x v^*(s,\lambda)ds, v^*(x,\lambda), y_0 + \int_0^{\lambda x} v^*(s,\lambda)ds\right). \tag{4.7}$$

If we suppose that there exists $\dfrac{\partial v^*}{\partial \lambda}$, then from (4.7) we obtain

$$\begin{aligned}\frac{\partial v^*}{\partial \lambda}(x,\lambda) = {} & \frac{\partial f}{\partial u_1}\left(x, y_0 + \int_0^x v^*(s,\lambda)ds, v^*(x,\lambda), y_0 + \int_0^{\lambda x} v^*(s,\lambda)ds\right)\int_0^x \frac{\partial v^*}{\partial \lambda}(s,\lambda)ds + \\ & + \frac{\partial f}{\partial u_2}\left(x, y_0 + \int_0^x v^*(s,\lambda)ds, v^*(x,\lambda), y_0 + \int_0^{\lambda x} v^*(s,\lambda)ds\right)\frac{\partial v^*}{\partial \lambda}(x,\lambda) + \\ & + \frac{\partial f}{\partial u_3}\left(x, y_0 + \int_0^x v^*(s,\lambda)ds, v^*(x,\lambda), y_0 + \int_0^{\lambda x} v^*(s,\lambda)ds\right)\cdot \\ & \cdot\left(\int_0^{\lambda x} \frac{\partial v^*}{\partial \lambda}(s,\lambda)ds + xv^*(\lambda x,\lambda)\right).\end{aligned}$$

The previous relationship suggests us to consider the operator $S_2 : X \times X \to X$, defined by

$$\begin{aligned}(S_2(v,y))(x,\lambda) := {} & \\ = {} & \frac{\partial f}{\partial u_1}\left(x, y_0 + \int_0^x v(s,\lambda)ds, v(x,\lambda), y_0 + \int_0^{\lambda x} v(s,\lambda)ds\right)\int_0^x y(s,\lambda)ds + \\ & + \frac{\partial f}{\partial u_2}\left(x, y_0 + \int_0^x v(s,\lambda)ds, v(x,\lambda), y_0 + \int_0^{\lambda x} v(s,\lambda)ds\right)y(x,\lambda) + \\ & + \frac{\partial f}{\partial u_3}\left(x, y_0 + \int_0^x v(s,\lambda)ds, v(x,\lambda), y_0 + \int_0^{\lambda x} v(s,\lambda)ds\right)\cdot \\ & \cdot\left(\int_0^{\lambda x} y(s,\lambda)ds + xv(\lambda x,\lambda)\right).\end{aligned}$$

By using (i) and(ii) we obtain that

$$||S_2(v,y_1) - S_2(v,y_2)||_\tau \le \left(\frac{M_1 + M_3}{\tau} + M_2\right)||y_1 - y_2||_\tau,$$

for all $y_1, y_2 \in X$.

Because of condition (iii), and by choosing τ large enough, we have that S_2 is a contraction with respect to the second argument.

If we take the operator $S : X \times X \to X \times X$, $S = (S_1, S_2)$ then we are in the conditions of Fibre contraction theorem. Let (v^*, w^*) the unique fixed point of the operator S.

If we take $v_0 = 0$, $w_0 = 0$ then $w_1 = \dfrac{\partial v_1}{\partial \lambda}$.

By mathematical induction method we can prove that $w_n = \dfrac{\partial v_n}{\partial \lambda}$.

Thus $(v_n)_{n\in\mathbb{N}}$ converges uniformly to v^* and $\left(\dfrac{\partial v_n}{\partial \lambda}\right)_{n\in\mathbb{N}^*}$ converges uniformly to w^*. By using a Weiestrass argument we obtain that $\dfrac{\partial v^*}{\partial \lambda}$ exists and $\dfrac{\partial v^*}{\partial \lambda} = w^*$, and w^* is a continuous function.

So, we have (c). □

References

[1] Azbelev, N. V., Maksimov, V. P. and Rahmatulina, L. F., *Introduction to Functional - Differential Equations Theory*, MIR, Moscow, 1991 (In Russian)

[2] Burton, T. A., *Stability by Fixed Point Theory for Functional Differential Equations*, Dover Publ., Mineola, 2006

[3] Guo, D., Lakshmikantham, V. and Liu, X., *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Acad. Publ., Dordrecht, 1996

[4] Hale, J. K. and Sjoerd, M. Verduyn Lunel, *Introduction to Functional - Differential Equations*, Springer Verlag, New York, 1993

[5] Hirsch, M. W. and Pugh, C. C., *Stable manifolds and hyperbolic sets*, Proc. Symp. in Pure Math., **14** (1970), 133–163

[6] Ionescu, D. V., *L'application de la méthode des approximation successive à l'integration numérique des l'equations differentielles*, Bull. Math. Soc. Sci. Math. Roumanie, **51** (1959), No. 3, 423 -431

[7] Iserles, A., *On the generalized pantograph functional - differential equation*, European J. Appl. Math., **4** (1992), 1–38

[8] Mureşan, V., *Differential Equation with Affine Modification of the Argument*, Transilvania Press, Cluj-Napoca, 1997 (In Romanian)

[9] Mureşan, V., *Functional - Integral Equations*, Ed. Mediamira, Cluj - Napoca, 2003

[10] Ockendon, J. R., *Differential equations and industry*, The Math. Scientist, **5** (1980), No. 1, 1–12

[11] Rus, I. A., *Weakly Picard mappings*, Comment. Math. Univ. Carolin., **34** (1993), No. 4, 769–773

[12] Rus, I. A., *A fiber generalized contraction theorem and applications*, Mathematica, **41(64)** (1999), No. 1, 85–90

[13] Rus, I. A., *A class of nonlinear functional-integral equations, via weakly Picard operators* , Ann. Univ. Craiova, Ser. Mat.-Inform., **28** (2001), 10–15

[14] Rus, I. A., *Picard operators and applications*, Sci. Math. Jpn., **58** (2003), No. 1, 191–219

[15] Rus, I. A., *Some nonlinear functional differential and integral equations, via weakly Picard operators theory: a survey*, Carpathian J. Math., **26** (2010), No. 2, 230–258

[16] Rus, I. A., *Some variants of contraction principles in the case of operators with Volterra property: step by step contraction principle*, Advances in the Theory of Nonlinear Anal. and its Appl., **3** (2019), No. 3, 111–120

[17] Rus, I. A., Petruşel A. and Şerban M. A., *Fibre Picard operators on gauge spaces and applications*, Z. Anal. Anwend., **27** (2008), No. 4, 407–423

[18] Rus, I. A. and Şerban M. A., *Basic problem of the metric fixed point theory and the relevance of a metric fixed point theorem*, Carpathian J. Math., **29**(2013), No. 2, 239–258

[19] Szep, G., *On an integral equation with deviating argument* , Seminar on Fixed Point Theory Cluj-Napoca, **1** (2000), 103–108

[20] Şerban, M. A., *Fixed Point Theory for Operators defined on Cartezian Product*, Presa Univ. Clujeană, Cluj-Napoca, 2002 (In Romanian)

[21] Şerban, M. A., Rus, I. A. and Petruşel, A., *A class of abstract Volterra equations, via weakly Picard operators' technique* , Math. Inequal. Appl., **13** (2010), No. 2, 255–269

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