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Dedicated to Prof. Ioan A. Rus on the occasion of his 85^{th} anniversary

Variational approach to nonlinear stochastic differential equations in Hilbert spaces

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ABSTRACT. Here we survey a few functional methods to existence theory for infinite dimensional stochastic differential equations of the form $dX + A(t)X(t) = B(t, X(t))dW(t)$, $X(0) = X_0$, where $A(t)$ is a nonlinear maximal monotone operator in a variational couple (V, V'). The emphasis is put on a new approach of the classical existence result of N. Krylov and B. Rozovski on existence for the infinite dimensional stochastic differential equations which is given here via the theory of nonlinear maximal monotone operators in Banach spaces. A variational approach to this problem is also developed.

1. THE MAIN RESULT

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)_H$ and the norm $\cdot|_H$ and let V be a separable reflexive Banach space with the dual V^* such that

$$V \subset H \subset V^*$$

with dense and continuous injections. In the sequel, the duality ${}_{V'}\langle\cdot,\cdot\rangle_V$ is simply denoted $\langle\cdot,\cdot\rangle$ and the norms of V and V^* are denoted by $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$, respectively.

In the following, we denote by $\mathcal{B}(H)$, $\mathcal{B}(V)$ and $\mathcal{B}(V^*)$ the σ-algebra of Borelian sets in H, V and V^*, respectively.

Let W be the cylindrical Wiener process in the Hilbert space U

$$W(t) = \sum_{j=1}^{\infty} e_j\beta_j(t),\ t \geq 0,$$

where $\{e_j\}$ is an orthonormal basis in U and $\{\beta_j\}_{j=1}^{\infty}$ is a sequence of mutually independent Brownian motions in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the normal filtration $(\mathcal{F}_t)_{t\geq 0}$.

We shall study here the stochastic differential equation (SDE)

$$\begin{aligned} &dX(t) + \mathcal{A}(t, X(t))dt = B(t, X(t))dW(t),\ t \in (0,T), \\ &X(0) = X_0, \end{aligned} \tag{1.1}$$

where $X : [0,T]\times\Omega \to V$. The operators $\mathcal{A} : [0,T]\times\Omega\times V \to V^*$ and $B : [0,T]\times\Omega\times V \to L_2(U,H)$ are assumed to satisfy the following hypotheses (in the sequel, we sometimes write $\mathcal{A}(t)u$, $\mathcal{A}(t)(u)$ or, simply, $\mathcal{A}(u)$ instead of $\mathcal{A}(t,u)$).

(H1) $\mathcal{A} : [0,T]\times\Omega\times V \to V^*$, $B : [0,T]\times\Omega\times V \to L_2(U,H)$ *are progressively measurable, i.e.,* $\forall t \in (0,T)$, *these functions restricted to* $[0,t]\times\Omega\times V$ *are* $\mathcal{B}([0,T])\otimes\mathcal{F}_t\otimes\mathcal{B}(V)$ *measurable.*

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(H2) *For each $t \in [0,T]$ and $\omega \in \Omega$, the operator $u \to \mathcal{A}(t,\omega,u)$ is demicontinuous from V to V^* (that is strongly-weakly continuous). Moreover, there is $\alpha \in \mathbb{R}$ such that, for all $u, v \in V$,*

$$\langle \mathcal{A}(t)u - \mathcal{A}(t)v, u-v\rangle \geq \|B(t)u - B(t)v\|^2_{L_2(U,H)} - \alpha|u-v|^2_H, \quad \forall (t,\omega) \in (0,T)\times\Omega. \tag{1.2}$$

(H3) *There are $1 < p < \infty$, $\alpha_1, \alpha_2 > 0$, $\alpha_q \in \mathbb{R}$ such that, for all $u \in V$ and $\mathbb{P}$-a.s.,*

$$\langle \mathcal{A}(t)u, u\rangle \geq \|B(t)u\|^2_{L_2(U,H)} + \alpha_1\|u\|^p_V - \alpha_2|u|^2_H + g_1(t), \quad \forall (t,u) \in (0,T)\times V, \tag{1.3}$$

$$\|\mathcal{A}(t)u\|_{V^*} \leq \alpha_3\|u\|^{p'}_V + g_2(t), \ \forall (t,u) \in (0,T)\times V, \tag{1.4}$$

where $\frac{1}{p'} = 1 - \frac{1}{p}$ and $g_1, g_2 \in L^1((0,T)\times\Omega)$ are $\mathcal{F}_t$-adapted processes.

Here $\|\cdot\|_{L_2(U,H)}$ is the Hilbert-Schmidt norm $L_2(U,H)$ and the spaces $L^q((0,T)\times\Omega)$ are considered with the measure $dt \otimes \mathbb{P}$. The significance of the term $B(t,X)dW$ is that of a *Gaussian noise*. We refer to the book [1] for the basic results on the theory of nonlinear monotone operators in Banach spaces which will be used in the following.

Definition 1.1. A (pathwise) continuous H-valued $(\mathcal{F}_t)_{t\geq 0}$-adapted process $X : [0,T] \to H$ is called solution to (1.1) if

$$X \in L^p((0,T)\times\Omega; V) \cap L^2((0,T)\times\Omega; H), \tag{1.5}$$

$$X(t) + \int_0^t \mathcal{A}(s,\widetilde{X}(s))ds = X_0 + \int_0^t B(s,\widetilde{X}(s))dW(s), \quad \forall t \in [0,T],\ \mathbb{P}\text{-a.s.}, \tag{1.6}$$

where $\widetilde{X}$ is any V-valued progressively measurable version of X, that is,

$$X = \widetilde{X}, \ \text{ a.e. } dt \otimes \mathbb{P}.$$

The integral from the right-hand side of (1.6) is taken in the sense of Itô (see, e.g., [8]). Theorem 1.1 below is the main result.

Theorem 1.1. *Under hypotheses* (H1)–(H3), *for each $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}, H)$, there is a unique solution $X = X(t, X_0)$ to* (1.1). *Moreover, one has*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X(t,X_0) - X(t,\overline{X}_0)|^2_H\right] \leq C_1\mathbb{E}|X_0 - \overline{X}_0|^2_H, \tag{1.7}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X(t,X_0)|^2_H\right] + \mathbb{E}\int_0^T \|X(t,X_0)\|^p_V dt \leq C_2(\mathbb{E}|X_0|^2_H + 1), \tag{1.8}$$

where C_1, C_2 are independent of X_0.

We note that (1.7) and (1.8) can be equivalently written as

$$\|X(\cdot,X_0) - X(\cdot,\overline{X}_0)\|)^2_{L^2(\Omega;C([0,T];H)} \leq C_1(\mathbb{E}|X_0 - \overline{X}_0|_H)^2, \tag{1.9}$$

and

$$\|X(\cdot,X_0)\|^2_{L^2(\Omega;L^\infty(0,T;H))} + \|X(\cdot,X_0)\|_{L^1(\Omega;L^p(0,T,V))} \leq C_2(\mathbb{E}|X_0|^2_H + 1). \tag{1.10}$$

In many cases, X_0 is taken deterministic, that is, $X_0 \in H$.

Examples

Very often, the stochastic differential equations arise as deterministic differential equations perturbed by a Gaussian (Wiener) noise of the form $\sigma(t,X)dW$. Here are few examples.

1. Stochastic differential equations in $\mathbb{R}^N$. Consider the stochastic differential equation

$$\begin{aligned} &dX + f(t,X)dt = \sigma(t,X)dW \\ &X(0) = X_0 \end{aligned} \tag{1.11}$$

where $f:[0,T]\times\Omega\times\mathbb{R}^N\to\mathbb{R}^N$, $\sigma:[0,T]\times\Omega\times\mathbb{R}^M\to\mathbb{R}^N$ are progressively measurable and W is a Gaussian process of the form

$$W(t) = \left\{\sum_{j=1}^{M} a_{ij}(t)\beta_j(t)\right\}_{i=1}.$$

Moreover, for each $\omega\in\Omega$ and $t\in(0,T)$, the functions $u\to f(t,\omega,u)$ and $u\to\sigma(t,\omega,u)$ are continuous and

$$\begin{aligned} &(f(t,u)-f(t,v))\cdot(u-v) \geq \|\sigma(t,u)-\sigma(t,v)\|^2_{L(\mathbb{R}^N,\mathbb{R}^M)} - \alpha\|u-v\|^2_{\mathbb{R}^N}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall u,v\in\mathbb{R}^N, \\ &f(t,u)\cdot u \geq \|\sigma(t,u)\|^2_{L(\mathbb{R}^M,\mathbb{R}^N)} - \alpha\|u\|^2_{\mathbb{R}^N} + g_1(t),\ \forall u\in\mathbb{R}^N,\ t\in(0,T), \\ &\|f(t,u)\|_{\mathbb{R}^N} \leq \|u\|^p_{\mathbb{R}^N} + g_2(t),\ \forall u\in\mathbb{R}^N,\ t\in(0,T), \end{aligned}$$

where $g_1,g_2\in L^1((0,T)\times\Omega)$ are $\mathcal{F}_t$-adapted processes in $(\Omega,\mathcal{F},\mathbb{P})$. Here, one applies Theorem 1.1 with $H=V=V^*=\mathbb{R}^N$ and $\mathcal{A}(t)\equiv f(t)$, $B(t)=\sigma(t)$.

2. Stochastic reaction-diffusion equations. Let $V=H^1_0(\mathcal{O})$, $V^*=H^{-1}(\mathcal{O})$, $H=L^2(\mathcal{O})$, $\mathcal{O}\subset\mathbb{R}^d$, $U=L^2(\mathcal{O})$, $d>1$ and

$$(Au)(x) = -\Delta u(x) + \gamma(x,u(x)),\ x\in\mathcal{O},$$

defined from V to V^* by the variational formula

$$_{V^*}\langle\mathcal{A}u,v\rangle_V = \int_{\mathcal{O}}\nabla u\cdot\nabla v\,dx + \int_{\mathcal{O}}\gamma(x,u)v\,dx,\ \forall v\in V. \tag{1.12}$$

Here, $\mathcal{O}$ is a bounded and open set of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $\gamma:\mathbb{R}\to\mathbb{R}$ is monotonically increasing and continuous in u, Lebesgue measurable in x, $\gamma(x,0)\equiv 0$ and

$$|\gamma(x,u)| \leq C_1|u|^\alpha + C_2,\ \forall x\in\mathcal{O},\ u\in\mathbb{R},$$

where $0\leq\alpha\leq\frac{d+1}{d-2}$. By (1.12), we see that

$$\langle\mathcal{A}u,u\rangle \geq |\nabla u|^2_H = \|u\|^2_V,\ \forall u\in V,$$

and, by Hölder's inequality,

$$|\langle\mathcal{A}u,v\rangle| \leq |\nabla u|_H|\nabla v|_H + \left(\int_{\mathcal{O}}|u|^{\alpha q}\right)^{\frac{1}{2}}\left(\int_{\mathcal{O}}|v|^{p^*}\right)^{\frac{1}{p^*}} + C_3\|v\|_{L^p(\mathcal{O})},$$

where $\frac{1}{p^*}=\frac{1}{2}-\frac{1}{d}$, $\frac{1}{q}=\frac{1}{2}+\frac{1}{d}$.

Recalling that, by the Sobolev embedding theorem,

$$V=H^1_0(\mathcal{O})\subset L^{p^*}(\mathcal{O}),\ V^*=H^{-1}(\mathcal{O})\supset L^q(\mathcal{O}),$$

it follows that

$$|\langle\mathcal{A}u,v\rangle| \leq C_4(\|u\|_V+1)\|v\|_V,\ \forall u,v\in V,$$

and, therefore,

$$\|Au\|_{V^*} \le C_4(\|u\|_V + 1),\ \forall u \in V.$$

Hence hypothesis (H3) holds. Finally, W is a Wiener cylindrical process in $H = L^2(\mathcal{O})$ and $B : H \to L_2(H,H)$ is assumed to satisfy the condition

$$\|B(u) - B(v)\|^2_{L_2(H,H)} \le \alpha|u-v|^2_H + \|u-v\|^2_V,\ \ \forall u, v \in H^1_0(\mathcal{O}), \tag{1.13}$$

for some $\alpha \in \mathbb{R}$. Then, by Theorem 1.1, the equation

$$\begin{aligned} & dX - \Delta X\, dt + \gamma(x, X)dt = B(X)dW,\ t \in (0,T), \\ & X(0,x) = X_0(x),\ x \in \mathcal{O}, \\ & X = 0 \text{ on } \partial\mathcal{O}, \end{aligned} \tag{1.14}$$

has, for each $X_0 \in L^2(\Omega; \mathcal{F}_0, \mathbb{P}, L^2(\mathcal{O}))$, a unique solution

$$X \in L^2(\Omega; C([0,T]; L^2(\mathcal{O})) \cap L^2(\Omega; L^2(0,T; H^1_0(\mathcal{O})).$$

A few special cases of Gaussian perturbation

1° Let

$$B(X)dW = \sum_{j=1}^{\infty} \mu_j X\, e_j\, d\beta_j,\ W = \sum_{j=1}^{\infty} e_j \beta_j(t). \tag{1.15}$$

In this case, $B : H \to L_2(H,H)$ is defined by

$$(B(u))(h) = u \sum_{j=1}^{\infty} \mu_j\, e_j (h, e_j)_{L^2(\mathcal{O})},\ \forall h \in L^2(\mathcal{O}) = H,$$

where $\{e_j\}$ is an orthonormal basis in $L^2(\mathcal{O})$. Then

$$\|B(u)\|^2_{L_2(H,H)} = \sum_{j=1}^{\infty} \mu_j^2 \|ue_j\|^2_{L^2(\mathcal{O})}$$

and so (1.13) holds if either $\sum\limits_{j=1}^{\infty} \mu_j^2 \|e_j\|^2_{L^\infty(\mathcal{O})} < \alpha < \infty$, or $\sum\limits_{j=1}^{\infty} \mu_j^2 \|e_j\|^2_{H^{-1}(\mathcal{O})} \le 1$.

2° $B(X)dW = \sum\limits_{j=1}^{\infty} (\eta_j \cdot \nabla X) e_j\, d\beta_j$, where $\{\eta_j\}_{j=1}^{\infty} \subset L^\infty(\mathcal{O})$. Then

$$B(u)(h) = \sum_{j=1}^{\infty} (\eta_j \cdot \nabla u) e_j (h, e_j)_{L^2(\mathcal{O})},\ \ \forall h \in L^2(\mathcal{O}), \tag{1.16}$$

and, therefore,

$$\|B(u)\|^2_{L_2(U,H)} = \sum_{j=1}^{\infty} \|(\eta_j \cdot \nabla u) e_j\|^2_{L^2(\mathcal{O})} \le \|u\|^2_{H^1_0(\mathcal{O})} \sum_{j=1}^{\infty} \|\eta_j e_j\|^2_{L^\infty},\ \ \forall u \in H.$$

Then hypothesis (1.13) holds if

$$\sum_{j=1}^{\infty} \|\eta_j e_j\|^2_{L^\infty(\mathcal{O})} \le 1.$$

3. Parabolic nonlinear SDE of divergence type. Consider here the equation

$$\begin{aligned} & dX - \mathrm{div}_x(a(t, x, \nabla_x X(t,x))) = B(t, X)dW,\ t \in (0,T),\ x \in \mathcal{O}, \\ & X(0,x) = X_0(x),\ x \in \mathcal{O}, \\ & X(t,x) = 0\ \text{ for } x \in \partial\mathcal{O},\ t \in [0,T], \end{aligned} \tag{1.17}$$

where $X_0 \in L^2(\mathcal{O})$, $a : [0,T] \times \mathcal{O} \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following hypotheses

(i) $a = a(t,x,\omega,\eta)$ is measurable on $(0,T) \times \mathcal{O} \times \Omega$ and continuous in η. Moreover, there exists $\alpha_1 > 0$ such that

$$(a(t,x,\omega,\eta) - a(t,x,\omega,\bar{\eta})) \cdot (\eta - \bar{\eta}) \geq \alpha_1 |\eta - \bar{\eta}|^2,\ \forall \eta, \bar{\eta} \in \mathbb{R}^N.$$

(ii) $a(t,x,\eta) \cdot \eta \geq \alpha_2 |\eta|^p,\ \forall \eta \in \mathbb{R}^N$, $|a(t,x,\eta)| \leq \alpha_3 |\eta|^{p-1},\ \forall \eta \in \mathbb{R}^N$, where $1 < p < \infty$ and $\alpha_1 \geq 0,\ \alpha_2 > 0,\ \alpha_3 > 0$.

(iii) $B : L^2(\mathcal{O}) \to L_2(L^2(\mathcal{O}), L^2\mathcal{O})) = L_2$ satisfies the condition

$$\|B(u) - B(v)\|_{L_2}^2 \leq \lambda |u - v|_H^2,\ \forall u, v \in H = L^2(\mathcal{O}).$$

A standard example is

$$\begin{aligned} &dX - \mathrm{div}_x(|\nabla_x X|^{p-2} \nabla_x X)dt = B(t,x)dW \text{ in } (0,T) \times \mathcal{O}, \\ &X(0) = X_0 \text{ in } \mathcal{O}, \\ &X = 0 \text{ on } (0,T) \times \partial\mathcal{O}, \end{aligned}$$

where $2 \leq p < \infty$ and B is of the form (1.15) or (1.16). We apply Theorem 1.1, where $V = W_0^{1,p}(\mathcal{O}) \cap L^2(\mathcal{O})$, $H = L^2(\mathcal{O})$ and $\mathcal{A}(t) : V \to V^*$ defined by

$${}_{V^*}\langle \mathcal{A}(t)u, v \rangle_V = \int_{\mathcal{O}} a(t,x,\nabla u(x)) \cdot \nabla v(x) dx,\ \forall u, v \in V.$$

We get

Corollary 1.1. *For each $X_0 \in L^2(\mathcal{O})$ there is a unique solution*

$$X \in L^2(\Omega; C([0,T]; L^2(\mathcal{O}))) \cap L^1((0,T) \times \Omega; W_0^{1,p}(\mathcal{O}))$$

to equation (1.17).

4. The stochastic porous media equation. We consider here the stochastic version of the differential equation

$$\begin{aligned} &dX - \Delta\gamma(X)dt = B(t,X)dW \text{ in } (0,T) \times \mathcal{O}, \\ &X(0,x) = X_0(x),\ x \in \mathcal{O}, \\ &\gamma(X(t)) \in H_0^1(\mathcal{O}),\ \forall t \in [0,T]. \end{aligned} \tag{1.18}$$

Here $\gamma : \mathbb{R} \to \mathbb{R}$ is continuous, monotonically increasing, W is a Wiener cylindrical process in $H^{-1} = H^{-1}(\mathcal{O})$ and $B : [0,T] \times H^{-1} \to L_2(H^{-1}, H^{-1})$ satisfies

$$\|B(t,u) - B(t,v)\|_{L_2(H^{-1},H^{-1})} \leq \lambda \|u - v\|_{H^{-1}}. \tag{1.19}$$

An example is

$$(B(t)u)(h) = u \sum_{j=1}^{\infty} \mu_j e_j (h, e_j), e_j,\ \forall h \in H^{-1},$$

where $\{e_j\} \subset H^{-1}$ is an orthonormal basis such that

$$\sum_{j=1}^{\infty} \mu_j^2 \|e_j u\|_{H^{-1}}^2 \leq C \|u\|_{H^{-1}}^2,\ \forall u \in H^{-1}.$$

We apply Theorem 1.1, where $H = H^{-1}(\mathcal{O})$, $V = L^q(\mathcal{O}) \subset H^{-1}(\mathcal{O})$, $\mathcal{A}u = -\Delta\gamma(u)$, $\forall u \in L^q(\mathcal{O})$,

$${}_{V^*}(\mathcal{A}u, v)_V = \int_{\mathcal{O}} \gamma(u) v\, dx,\ \forall v \in L^p(\mathcal{O}).$$

Here $q \geq \frac{2d}{d+2}$ and V^* is the dual of $V = L^q(\Omega)$ is the duality with $H = H^{-1}$ as pivot space, that is,

$$L^q(\mathcal{O}) \subset H^{-1} \subset V^*.$$

Then, assuming that

$$\begin{aligned} \gamma(u)u &\geq \alpha_1 |u|^q, && \text{for } u \in \mathbb{R}, \\ |\gamma(u)| &\geq \alpha_2 |u|^{q-1}, && \text{for } u \in \mathbb{R}, \end{aligned}$$

where $\alpha_1, \alpha_2 > 0$, we see that $\mathcal{A}$ satisfies hypotheses (H1), (H2) and, by Theorem 1.1, we get

Corollary 1.2. *Under the above assumptions, there is a unique solution X to SDE* (1.18) *satisfying $X \in L^2(\Omega; C([0,T]; H^{-1}(\mathcal{O}))) \cap L^q((0,T) \times \Omega \times \mathcal{O})$, where $q \geq \frac{2d}{d+2}$.*

2. Proof of Theorem 1.1

For simplicity, we take $g_1, g_2 \equiv 0$. For each $\lambda > 0$, consider the operator $J_\lambda(t)(u) = (I + \lambda \mathcal{A}_H(t))^{-1} u,\ t \in [0,T],\ u \in H$, where $\mathcal{A}_H(t) : H \to H$ is defined by

$$\begin{aligned} \mathcal{A}_H(t)(u) &= \mathcal{A}(t)u, \forall u \in D(A_H(t)),\ \forall t \in (0,T), \\ D(\mathcal{A}_H(t)) &= \{u \in V; \mathcal{A}(t)u \in H\}. \end{aligned}$$

By (1.2) it follows that the operator $\mathcal{A}_H$ is quasi-maximal monotone in $H \times H$, that is, $R(I + \lambda \mathcal{A}_H) = H$ for $0 < \lambda < \frac{1}{\alpha}$ and

$$(\mathcal{A}_H(t)u - \mathcal{A}_H(t)v, u - v) \geq -\alpha |u - v|_H^2,\ \forall u, v \in D(\mathcal{A}_H),$$

and so the Yosida approximation (see, e.g., [1])

$$\mathcal{A}_\lambda(t) = \frac{1}{\lambda}\left(I - (I + \lambda \mathcal{A}_H(t))^{-1}\right) = \mathcal{A}_H(t) J_\lambda(t) \tag{2.20}$$

is Lipschitz for $0 < \lambda < \frac{1}{\alpha}$ and $t \in (0,T)$.

Now, consider the approximating equation

$$\begin{aligned} &dX_\lambda(t) + \mathcal{A}_\lambda(t) X_\lambda(t) dt = B_\lambda(t)(X_\lambda(t)) dW(t), \\ &X_\lambda(0) = X_0, \end{aligned} \tag{2.21}$$

where

$$B_\lambda(t) = B(t) J_\lambda(t),\ t \in (0,T). \tag{2.22}$$

By (1.2) and (1.3), we see that

$$\begin{aligned} &(\mathcal{A}_\lambda(t)u - \mathcal{A}_\lambda(t)v, J_\lambda(t)u - J_\lambda(t)v)_H \\ &\qquad \geq \|B_\lambda(t)u - B_\lambda(t)v\|_{L_2(U,H)}^2 - \alpha |J_\lambda(t)u - J_\lambda(t)v|_H^2, \end{aligned} \tag{2.23}$$

$$\begin{aligned} (\mathcal{A}_\lambda(t)u, J_\lambda(t)u)_H \geq \|B_\lambda(t)u\|_{L_2(U,H)}^2 + \alpha_1 \|J_\lambda(t)u\|_V^p - \alpha_2 |J_\lambda(t)u|_H^2, \\ \forall u, v \in H. \end{aligned} \tag{2.24}$$

In particular, it follows that

$$\begin{aligned} &\langle \mathcal{A}_\lambda(t)u - \mathcal{A}_\lambda(t)v, u - v \rangle \geq \|B_\lambda(t)u - B_\lambda(t)v\|_{L_2(U,H)}^2 \\ &\quad + \lambda \langle \mathcal{A}_\lambda u - \mathcal{A}_\lambda v, \mathcal{A}_\lambda u - \mathcal{A}_\lambda v \rangle - \alpha |J_\lambda(t)u - J_\lambda(t)v|_H \\ &\quad \geq \|B_\lambda(t)u - B_\lambda(t)v|_{L_2(U,H)}^2 - C_\lambda^1 |u - v|_H^2, \end{aligned} \tag{2.25}$$

$$\|B_\lambda(t)u - B_\lambda(t)v\|_{L_2(U,H)}^2 \leq C_\lambda |u - v|_H^2,\ \forall u, v \in H. \tag{2.26}$$

It turns out that, for each $0 < \lambda \leq \frac{1}{\alpha}$, equation (2.21) has a unique solution X. More precisely, we have

Lemma 2.1. *Equation* (2.21) *has a unique solution*

$$X_\lambda \in L^2(\Omega; C([0,T]; H)).$$

Moreover, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_\lambda(t) - Y_\lambda(t)|_H^2\right] \leq C\mathbb{E}|X_\lambda(0) - Y_\lambda(0)|_H^2, \tag{2.27}$$

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} |X_\lambda(t)|_H^2\right] &+ \mathbb{E}\int_0^T \|J_\lambda(t, X_\lambda(t))\|_V^p dt \\ &+\mathbb{E}\int_0^T |B_\lambda(s), X_\lambda(s)|^2_{L^2(U,H)} \leq C\mathbb{E}|X_0|_H^2,\end{aligned} \tag{2.28}$$

where C is independent of $\lambda \in \left(0, \frac{1}{\alpha}\right)$.

Proof. The existence of the solution X_λ follows by the standard existence theory. By (2.21), we get

$$\begin{aligned}&\frac{1}{2}|X_\lambda(t)|_H^2 + \int_0^t (\mathcal{A}_\lambda(s)X_\lambda(s), X_\lambda(s))_H ds \\ &\quad = \frac{1}{2}|X_0|_H^2 + \frac{1}{2}\int_0^t \|B_\lambda(s)X_\lambda(s)\|^2_{L_2(U,H)} ds \\ &\quad + \int_0^t (B_\lambda(s)X_\lambda(s)dW(s), X_\lambda(s))_H.\end{aligned} \tag{2.29}$$

On the other hand, we have

$$\begin{aligned}&(\mathcal{A}_\lambda(s)X_\lambda(s), X_\lambda(s))_H \\ &\quad \geq (\mathcal{A}(s)J_\lambda(s)X_\lambda(s), J_\lambda(s)X_\lambda(s)) + \lambda|\mathcal{A}_\lambda(s)X_\lambda(s)|_H^2, \ \forall s \in (0,T),\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}|X_\lambda(t)|_H^2 + \alpha_1\int_0^t \|J_\lambda(s)X_\lambda(s)\|_V^p ds \\ &\quad + \frac{1}{2}\int_0^t \|B_\lambda(s)X_\lambda(s)\|^2_{L_2(U,H)} ds \leq \lambda\int_0^t |\mathcal{A}_\lambda(s)X_\lambda(s)|_H^2 ds \\ &\quad \leq \frac{1}{2}|X_0|_H^2 + \alpha_2\int_0^t |J_\lambda(s)X_\lambda(s)|_H^2 ds \\ &\quad + \int_0^t (B_\lambda(s)X_\lambda(s)dW(s), X_\lambda(s))_H.\end{aligned} \tag{2.30}$$

By (H3), this yields

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]} |X_\lambda(t)|_H^2 + \mathbb{E}\int_0^t \|J_\lambda(s)X_\lambda(s)\|_V^p ds + \mathbb{E}\int_0^t \|B_\lambda(s)X_\lambda(s)\|^2_{L_2(U,H)} ds \\ &\quad + \lambda\mathbb{E}\int_0^t |A_\lambda(s)X_\lambda(s)|_H^2 ds \leq C\mathbb{E}|X_0|_H^2 + \mathbb{E}\int_0^t |X_\lambda(s)|_H^2 ds \\ &\quad + \mathbb{E}\Big[\sup_{\tau\in[0,t]} |X_\lambda(\tau)|_H^2 \int_0^t (1 + |X_\lambda(\tau)|_H^2) d\tau\Big]^{\frac{1}{2}},\end{aligned}$$

and so, by Gronwall's lemma, we get (2.28), as desired.

As regards (2.27), it follows similarly by the equation

$$\begin{aligned}&\frac{1}{2}\,|X_\lambda(t)-Y_\lambda(t)|_H^2+\int_0^t(\mathcal{A}_\lambda(s)X_\lambda(s)-\mathcal{A}_\lambda(s)Y_\lambda(s),X_\lambda(s)-Y_\lambda(s))ds\\ &\quad=\frac{1}{2}\,|X_\lambda(0)-Y_\lambda(0)|_H^2+\frac{1}{2}\int_0^t\|B_\lambda(s)(X_\lambda(s))-B_\lambda(s)(Y_\lambda(s))\|_{L_2(U,H)}^2ds\\ &\quad+\int_0^t((B_\lambda(s)(X_\lambda(s)))-B_\lambda(Y_\lambda(s))))dW(s),X_\lambda(s)-Y_\lambda(s))_H\end{aligned}$$

taking into account that, by (2.20), we have

$$\begin{aligned}&(\mathcal{A}_\lambda(s)X_\lambda(s)-\mathcal{A}_\lambda(s)Y_\lambda(s),X_\lambda(s)-Y_\lambda(s))_H\\ &\quad\geq\|B_\lambda(s)X_\lambda(s)-B_\lambda(s)Y_\lambda(s)\|^2-\alpha|J_\lambda(s)(Y_\lambda(s)))\\ &\quad-J_\lambda(s)(Y_\lambda(s)))|_H^2+\lambda|\mathcal{A}_\lambda(s)X_\lambda(s)-\mathcal{A}_\lambda(s)Y_\lambda(s)|_H^2.\end{aligned}$$

The details are omitted. □

Proof of Theorem 1.1. (continued) By estimates (2.27)–(2.28), it follows that on a subsequence $\{\lambda\}\to 0$

$$\begin{array}{lcll}X_\lambda & \to & \widetilde{X} & \text{weak star in } L^2(\Omega;L^\infty(0,T;H)),\\ J_\lambda(X_\lambda) & \to & \widetilde{Y} & \text{weakly in } L^p((0,T)\times\Omega;V),\\ B_\lambda(t,X_\lambda) & \to & \widetilde{Z} & \text{weakly in } L^2((0,T)\times\Omega;L_2(U,H)),\\ J_\lambda(X_\lambda)X_\lambda & \to & 0 & \text{strongly in } L^2((0,T)\times\Omega;H),\end{array}\tag{2.31}$$

This yields $\widetilde{X}=\widetilde{Y}\;dt\otimes\mathbb{P}$, a.e. Moreover, we define the process

$$X(t)=X_0-\int_0^t\widetilde{Y}(s)ds+\int_0^t\widetilde{Z}(s)dW(s),\ t\in(0,T),\tag{2.32}$$

and note that it is a version of $\overline{X}$, that is,

$$\widetilde{X}=X\;dt\otimes\mathbb{P},\ \text{a.e. in }(0,T)\times\Omega.\tag{2.33}$$

Here is the argument. By (2.21), we have, for all $\Psi\in L^p((0,T)\times\Omega;V)$,

$$\begin{aligned}&\mathbb{E}\int_0^T(X_\lambda(t),\Psi(t))dt+\mathbb{E}\int_0^T\left(\int_0^t\mathcal{A}_\lambda(s)X_\lambda(s)ds,\Psi(t)\right)dt\\ &\quad=\mathbb{E}\int_0^T(X_0,\Psi(t))dt+\mathbb{E}\int_0^T\left(\Psi(t),\int_0^tB_\lambda(s)X_\lambda(s)dW(s)\right).\end{aligned}$$

This yields, via Fubini's theorem,

$$\begin{aligned}&\mathbb{E}\int_0^T(X_\lambda(t),\Psi(t))dt+\mathbb{E}\int_0^T\left(\mathcal{A}_\lambda(t)X_\lambda(t),\int_0^T\Psi(s)ds\right)dt\\ &\quad=\mathbb{E}\int_0^T(X_0,\Psi(t))dt+\mathbb{E}\int_0^Tdt\left(\int_0^tB_\lambda(s)X_\lambda(s)dW(s),\Psi(t)\right).\end{aligned}$$

Letting $\lambda\to 0$, we get by (2.31) that

$$\mathbb{E}\int_0^T(\widetilde{X}(t),\Psi(t))dt+\mathbb{E}\int_0^T\left(\widetilde{Y}(t),\int_t^T\Psi(s)ds\right)dt$$

and so (2.33) follows. We also note that $X:[0,T]\to H$ is continuous for all $\omega\in\Omega$ and

$$\mathbb{E}\sup_{t\in[0,T]}|X(t)|_H^2\le C(1+|X_0|_H^2). \tag{2.34}$$

We note first that X is a V^*-valued and by

$$X(t)=\lim_{\lambda\to 0}X_\lambda(t)=X_0+\lim_{\lambda\to 0}\int_0^t\mathcal{A}_\lambda(s)X_\lambda(s)ds+\lim_{\lambda\to 0}\int_0^t B_\lambda(s)X_\lambda(s)dW(s),\quad \forall t\in[0,T],\ \mathbb{P}\text{-a.s.}$$

On the other hand, by (2.31) we see that $X(t)\in H$, $\mathbb{P}$-a.s. for all $t\in[0,T]$ and (2.33) holds. Moreover, since $t\to X(t)$ is V^*-continuous, it follows that $t\to X(t)$ is weakly H-continuous on $[0,T]$. To prove that $t\to X(t)$ is H-valued continued, it suffices to show that $t\to|X(t)|_H^2$ is continuous. To this end, we note that, by (2.32), we have

$$\begin{aligned}\frac12|X(t)|_H^2&=\frac12|X(s)|_H^2-\int_s^t(\widetilde{Y}(\tau),X(\tau))d\tau\\&+\frac12\int_s^t\|\widetilde{Z}\|^2_{L_2(U,H)}d\tau+\int_s^t(\widetilde{Z}dW,X)d\tau.\end{aligned}$$

Hence,

$$\lim_{s\to t}(|X(t)|_H^2-|X(s)|_H^2)=0,\ \mathbb{P}\text{-a.s.}$$

To prove that X is solution to (1.1), it remains to be shown that

$$\widetilde{Y}=\mathcal{A}(t,\widetilde{X}),dt\otimes\mathbb{P},\ \text{a.e. in }(0,T)\times\Omega, \tag{2.35}$$

$$\widetilde{Z}=B(t,\widetilde{X}),dt\otimes\mathbb{P},\ \text{a.e. in }(0,T)\times\Omega. \tag{2.36}$$

To this end, we note that, by (2.21) and (2.32), we have

$$\begin{aligned}&\limsup_{\lambda\to0}\Big\{\mathbb{E}\Big|\int_0^t(\mathcal{A}_\lambda(X_\lambda(s))),J_\lambda(X_\lambda(s))ds\Big|-\frac12\int_0^t\|B_\lambda(X_\lambda(s))\|^2_{L_2(U,H)}ds\Big\}\\&\le\frac12\,\mathbb{E}(|X_0|_H^2-|X(t)|_H^2)\\&=\mathbb{E}\Big(\int_0^t(\widetilde{Y}(s),\widetilde{X}(s))ds+\frac12\int_0^t\|\widetilde{Z}(s)\|^2_{L_2(U,H)}ds\Big).\end{aligned} \tag{2.37}$$

On the other hand, we have by (H2) that, for all $\varphi\in L^p((0,T)\times\Omega;V)$,

$$\begin{aligned}&(\mathcal{A}_\lambda(X_\lambda)-\mathcal{A}\varphi,J_\lambda(X_\lambda)-\varphi)+\alpha|J_\lambda(X_\lambda)-\varphi|_H^2\\&\qquad\ge\frac12\|B_\lambda(X_\lambda)-B\varphi\|^2_{L_2(U,H)},\end{aligned} \tag{2.38}$$

while, by (2.37) and (2.31), we have

$$\begin{aligned}&\limsup_{\lambda\to0}\Big\{\mathbb{E}\int_0^t((\mathcal{A}_\lambda(X_\lambda)-\mathcal{A}\varphi,J_\lambda(X_\lambda)-\varphi)-\frac12\|B_\lambda(X_\lambda)-B_\lambda(\varphi)\|^2_{L_2(U,H)})ds\Big\}\\&\le\mathbb{E}\int_0^t((\mathcal{A}\varphi,\varphi)-(\mathcal{A}\varphi,\widetilde{X})-(\widetilde{Y},\varphi)ds\\&+\frac12\,\mathbb{E}\int_0^t\|\widetilde{Z}\|^2_{L_2(U,H)}-\|B_\lambda\varphi\|^2_{L_2(U,H)}+2\,\langle B_\lambda(X_\lambda),B_\lambda\varphi\rangle)ds.\end{aligned} \tag{2.39}$$

By (2.38)–(2.39), we obtain that, for $\varphi=\widetilde{X}$,

$$\mathbb{E}\int_0^t\|\widetilde{Z}-B\widetilde{X}\|^2_{L_2(U,H)}\le\alpha\lim_{\lambda\to0}\mathbb{E}\int_0^t|J_\lambda(\widetilde{X})-\widetilde{X}|_H^2ds,$$

because, for $\lambda \to 0$, $B_\lambda(\varphi) \to B\varphi$ in $L^2((0,T)\times\Omega; L_2(U,H))$. Hence

$$\widetilde{Z} = B\widetilde{X}, \ \text{ a.e. } dt \otimes \mathbb{P},$$

Now, coming back to (2.37), we get

$$\limsup_{\lambda\to 0} \mathbb{E}\int_0^t (\mathcal{A}_\lambda(X_\lambda(s)), J_\lambda(X_\lambda(s)))ds \le \mathbb{E}\int_0^t (\widetilde{Y}(s), \widetilde{X}(s))ds, \tag{2.40}$$

because

$$\liminf_{\lambda\to 0} \mathbb{E}\int_0^t \|B_\lambda(X_\lambda)\|^2_{L_2(U,H)} ds \ge \mathbb{E}\int_0^t \|\widetilde{Z}(s)\|^2_{L_2(U,H)}.$$

Taking into account that the operator $u \to A(t,u)+\alpha u$ is maximal monotone in $L^p((0,T)\times\Omega; V)\times L^{p'}((0,T)\times\Omega; V^*)$, we conclude by (2.40) that $\widetilde{Y} = \mathcal{A}(\widetilde{X})\,dt\times\mathbb{P}$, a.e. in $(0,T)\times\Omega$. Finally, (1.7) follows by (2.27). This concludes the proof. □

Remark 2.1. As seen from the above proof, Theorem 1.1 extends to maximal monotone multivalued operators $\mathcal{A}(t): V \to V'$ satisfying (H1)-(H3). Applied to stochastic PDEs of the form (1.17)-(1.18), such a general case allows the treatment of equations with discontinuous nonlinearities, a and γ.

3. The potential case $\mathcal{A}(t) = \partial\varphi(t)$

Assume now in Theorem 1.1 that $\mathcal{A}(t): V \to V^*$ is of the form

$$\mathcal{A}(t)(u) = \partial\varphi(t,u), \ \forall t\in[0,T], \ u \in V,$$

where $\varphi = \varphi(t,\omega,u): [0,T]\times\Omega\times V \to \mathbb{R} = (-\infty,+\infty)$ is a convex and lower semicontinuous function on V and $\partial\varphi(t,\cdot)$ is the subdifferential of $\varphi(t,\cdot)$, that is,

$$_{V^*}\langle \mathcal{A}(t)u, u-v\rangle_V \ge \varphi(t,u)-\varphi(t,v), \ \forall u,v\in V, \ \omega\in\Omega.$$

If we denote by $\varphi^*(t): V^* \to \mathbb{R}$ the conjugate of φ^*, that is,

$$\varphi^*(t,u) = \sup\{_{V^*}\langle u,v\rangle_V - \varphi(v); \forall v\in V\},$$

and recall that (see, e.g., [1])

$$\begin{aligned} \varphi(v)+\varphi^*(u) &= {}_{V^*}(v,u)_V \quad \text{ if } v\in\partial\varphi(u),\\ \varphi(v)+\varphi^*(u) &\ge {}_{V^*}(v,u)_V, \quad \forall u\in V, \ v\in V^*, \end{aligned}$$

we may rewrite equation (1.1) as

$$\begin{aligned} dX + u\,dt &= B(t)X\,dW, \ t\in(0,T),\\ _{V^*}\langle u(t), X(t)\rangle_V &= \varphi(t,X(t)) + \varphi^*(t,u(t)), \ \forall t\in[0,T],\\ _{V^*}\langle v, X(t)\rangle_V &\ge \varphi(t,X(t)) + \varphi^*(t,v), \ \forall v\in V^*. \end{aligned} \tag{3.41}$$

This means that, if X^* is the solution to equation (1.1), then $(X^*, u^* = \mathcal{A}(t)(X^*))$ is the solution to the minimization problem

$$\begin{aligned} \text{Min}\Big\{\mathbb{E}\int_0^T (\varphi(t,X(t)) + \varphi^*(t,v(t)) - {}_{V^*}\langle u(t), X(t)\rangle_V)dt;\\ dX + u(t)dt = B(t)X\,dW, \ X(0) = X_0\Big\}, \end{aligned} \tag{3.42}$$

and this minimum equals zero. Taking into account that, by Itô's formula,

$$\mathbb{E}\int_0^T \langle u, X\rangle\,dt\,dt = -\frac{1}{2}\left(\mathbb{E}|X(T)|^2_H - \mathbb{E}|X_0|^2_H\right) + \frac{1}{2}\mathbb{E}\int_0^T \|B(t)(X(t))\|^2_{L_2(U,H)}dt,$$

we can rewrite (3.42) as

$$\tag{3.43} \begin{aligned} \operatorname*{Min}_u \Big\{ \mathbb{E}\int_0^T \Big(\varphi(t, X(t)) + \varphi^*(t, u) - \frac{1}{2}\, \|B(t)(X(t))\|^2_{L_2(U,H)} \Big) dt \\ + \frac{1}{2}\, \mathbb{E}(|X(T)|^2_H - |X_0|^2_H);\ u \in L^{p'}((0,T)\times\Omega; V^*),\ \mathcal{F}_t\text{-adapted} \Big\}. \end{aligned}$$

By hypothesis (1.4), we know that (take $g_1 = 0$)

$$\varphi(t,x) \ge (\mathcal{A}(t)x, x) \ge \|B(t)x\|^2_{L_2(U,H)} + \alpha_1\|x\|^p_V - \alpha_2|x|^2_H,\ \forall x \in V.$$

(We note that, in this case, $\mathcal{A}(t) : V \to V'$ might be multivalued.)

If the function $x \to B(t)x$ which happens in the case of the Wiener linear multiplicative noise is linear, it follows that the function

$$x \to \varphi(t,x) - \frac{1}{2}\, \|B(t)x\|^2_{L_2(U,H)}$$

is convex if

$$\alpha_1\|x\|^p_V - \alpha|x|^2 + \frac{1}{2}\, \|B(t)x\|^2_{L_2(U,H)} \ge 0,\ \forall x \in V,$$

and so, in this case, (3.43) reduces to a convex minimization problem which has a solution (X, u) because the function $u \to \varphi^*(t,u)$ is convex and coercive in $L^p((0,T)\times\Omega; V^*)$. This is an alternative constructive proof of existence in this special case.

4. The case of additive Gaussian noise

Consider here the stochastic differential equation (1.1) in a separable Hilbert space H,

$$\tag{4.44} \begin{cases} dX(t) + AX(t)dt = B\,dW(t), & t \ge 0, \\ X(0) = x. \end{cases}$$

Here $A : D(A) \subset H \to H$ is a nonlinear quasi-m-accretive operator in H, that is, $A + \lambda_0 I$ is m-accretive for some $\lambda_0 > 0$, $B \in L(U,H)$, where U is another Hilbert space and $W(t)$ is a cylindrical Wiener process in U defined on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$. This means that

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k,$$

where $\{e_k\}_k$ is an orthonormal basis in U and $\{\beta_k\}_k$ is a sequence of mutually independent Brownian motions on $\{\Omega, \mathcal{F}, \mathbb{P}\}$. Denote by $\mathcal{F}_t$ the σ-algebra generated by $\beta_k(s)$ for $s \le t$, $k \in \mathbb{N}$.

By solution to (4.44) we mean, as in Definition 1.1, a stochastic process $X = X(t)$ on $\{\Omega, \mathcal{F}, \mathbb{P}\}$ adapted to $\mathcal{F}_t$; that is, $X(t)$ is measurable with respect to the σ-algebra $\mathcal{F}_t$, and satisfies the equation

$$\tag{4.45} X(t) = x - \int_0^t AX(s)ds + \int_0^t B\,dW(s)ds, \qquad \forall t \ge 0,\ \mathbb{P}\text{-a.s.},$$

where the integral $\int_0^t B\,dW(s)$ is considered, as above, in the sense of Itô.

A standard way to study the existence for equation (4.44) is to reduce it via substitution

$$y(t) = X(t) - BW(t)$$

to the random differential equation

$$\tag{4.46} \begin{cases} \dfrac{d}{dt}\, y(t,\omega) + A(y(t,\omega) + BW(t,\omega)) = 0,\ t \ge 0,\ \mathbb{P}\text{-a.s.},\ \omega \in \Omega, \\ y(0,\omega) = x. \end{cases}$$

For almost all $\omega \in \Omega$ (i.e., P-a.s.), (4.46) is a deterministic time-dependent equation in H of the form

$$\begin{cases} \dfrac{dy}{dt}(t) + A(t)y(t) = 0, & t \geq 0, \\ y(0) = x, \end{cases} \tag{4.47}$$

where $A(t)y = A(y + BW(t))$. This fact explains why one cannot expect a complete theory of existence similar to that from the deterministic case. In fact, because the Wiener process $t \to W(t)$ does not have bounded variation and this precludes the well posedness of the Cauchy problem (4.47) because $t \to (I + \lambda A(t))^{-1}$ is not with bounded variation (see [1]). However, we can invoke in this case an abstract existence result for (4.47) which does not require regularity of $A(t)$. Namely, we assume that V is a reflexive Banach space continuously embedded in H and so we have

$$V \subset H \subset V'$$

algebraically and topologically, where V' is the dual space of V.

Let $A : V \to V'$ satisfy the assumption

(l) A is a demicontinuous monotone operator and

$$\begin{aligned} (Au, u) &\geq \gamma \|u\|_V^p + C_1, & \forall u \in V, \\ \|Au\|_{V'} &\leq C_2(1 + \|u\|_V^{p-1}), & \forall u \in V, \end{aligned}$$

where $\gamma > 0$ and $p > 1$.

Then, we have the following theorem.

Theorem 4.2. *Assume that A satisfies hypothesis* (l) *and that*

$$BW \in L^p(0, T; V), \ \mathbb{P}\text{-a.s.} \tag{4.48}$$

Then, for each $x \in H$, equation (4.46) *has a unique $\mathcal{F}$-adapted solution $X = X(t, \omega) \in L^p(0, T; V) \cap C([0, T]; H)$, a.e. $\omega \in \Omega$.*

Proof. Here, one simply applies Theorem 4.17 in [1] to the operator $A(t)y = A(y + BW(t))$ and check that conditions (i)–(iii) are satisfied under hypotheses (l) and (4.48).

Thus, one finds a solution $X = X(t, \omega)$ to (4.46) that satisfies the equation for $\mathbb{P}$-almost all $\omega \in \Omega$. Taking into account that, as seen earlier, such a solution can be obtained as the limit of solutions y_λ to the approximating equations

$$\begin{cases} \dfrac{d}{dt} y_\lambda + A_\lambda(y_\lambda + BW) = 0, & t \in (0, T), \\ y_\lambda(0) = x, \end{cases}$$

for $\lambda \to 0$, where A_λ is the Yosida approximation of $A\big|_H$ (the restriction of the operator A to H), we may conclude that X is adapted with respect to the filtration $\{\mathcal{F}_t\}$. □ □

Theorem 4.2 remains true for time-dependent operators $A = A(t) : (0, T) \times V \to V'$, which are measurable in t and satisfy conditions (i), (ii), a.e. $t \in (0, T)$.

5. The operatorial approach for linear multiplicative noise

Under hypotheses of Theorem 1.1, consider here (1.1) being of the form

$$\begin{aligned} &dX + \mathcal{A}(t)X\,dt = X\,dW, \\ &X(0) = x, \end{aligned} \tag{5.49}$$

where $W = \sum_{j=1}^{\infty} \mu_j e_j \beta_j$.

By the rescaling transformation $X = e^W y$, we reduces (5.49) to

$$\frac{dy}{dt} + e^{-W}\mathcal{A}(t)(e^W y) + \mu y = 0 \quad (5.50)$$
$$y(0) = x \in H,$$

because by Itô's formula $dX = e^W dy + e^W y\, dW + \mu e^W y$, where $\mu = \frac{1}{2}\sum_{j=1}^{\infty} \mu_j^2 e_j$.

We note that the operator $y \mapsto e^{-W(t)}A(t)(e^{W(t)}y)$ is not monotone in $V \times V'$ and so Theorem 3.19 in [1] is not applicable here. In order to rewrite (5.49) as a nonlinear infinite dimensional equation of monotone (accretive) type, we define new spaces $\mathcal{H}, \mathcal{V}$ and $\mathcal{V}'$, as follows.

$\mathcal{H}$ is the Hilbert space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $y : [0,T] \to H$ such that

$$|y|_{\mathcal{H}} = \left(\mathbb{E}\int_0^T |e^{W(t)}y(t)|_H^2 dt\right)^{\frac{1}{2}} < \infty, \quad (5.51)$$

where $\mathbb{E}$ denotes the expectation in the above probability space. The space $\mathcal{H}$ is endowed with the norm $|\cdot|_{\mathcal{H}}$ coming from the scalar product

$$\langle y, z\rangle_{\mathcal{H}} = \mathbb{E}\int_0^T \left\langle e^{W(t)}y(t), e^{W(t)}y(t)\right\rangle dt. \quad (5.52)$$

$\mathcal{V}$ is the space of $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $y : [0,T] \to V$ such that

$$|y|_{\mathcal{V}} = \left(\mathbb{E}\int_0^T |e^{W(t)}(t)|_V^p dt\right)^{\frac{1}{p}} < \infty. \quad (5.53)$$

Clearly, the space $\mathcal{V}$ is reflexive. The space $\mathcal{V}'$ (the dual of $\mathcal{V}$) is the space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $y : [0,T] \to V'$ such that

$$|y|_{\mathcal{V}'} = \left(\mathbb{E}\int_0^T |e^{W(t)}y(t)|_{\mathcal{V}'}^{p'} dt\right)^{\frac{1}{p'}} < \infty, \quad (5.54)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. If $\leq p < \infty$, we have

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}' \quad (5.55)$$

with continuous defined embeddings. We note that

$${}_{\mathcal{V}'}\langle u, v\rangle_{\mathcal{V}} = \mathbb{E}\int_0^T \left\langle e^{W(t)}u(t), e^{W(t)}v(t)\right\rangle dt,\ v \in \mathcal{V},\ u \in \mathcal{V}'. \quad (5.56)$$

is just the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$. We also have, for $p \geq 2$,

$${}_{\mathcal{V}'}\langle u, v\rangle_{\mathcal{V}} = \langle u, v\rangle_{\mathcal{H}},\ \forall u \in \mathcal{H},\ v \in V. \quad (5.57)$$

In the case where $1 < p < 2$, we repalce $\mathcal{V}$ by $\mathcal{V} \cap \mathcal{H}$ and still have (5.55).

We also note that we have the continuous embeddings

$$L^{p_2}((0,T) \times \Omega; V) \subset \mathcal{V} \subset L^{p_1}((0,T) \times \Omega; V), \quad (5.58)$$
$$\forall 1 \leq p_1 < p,\ \max(p,2) < p_2.$$

Now, we fix $x \in H$ and define the operators $\widetilde{A} : \mathcal{V} \to \mathcal{V}'$ and $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{V} \to \mathcal{V}'$ as follows:

$$(5.59)\qquad \begin{aligned}&(\mathcal{A}y)(t) = e^{-W(t)}\mathcal{A}(t)(e^{W(t)}y(t)) - \nu y(t),\ \text{a.e. } t \in (0,T),\ y \in \mathcal{V},\\ &(\mathcal{B}y)(t) = \frac{dy}{dt}(t) + (\mu+\nu)y(t),\ \text{a.e. } t \in (0,T), y \in D(\mathcal{B}),\end{aligned}$$

$$(5.60)\qquad D(\mathcal{B}) = \Big\{ y \in \mathcal{V} : y \in AC([0,T];V') \cap C([0,T];H),\ \mathbb{P}\text{-a.s.},\ \frac{dy}{dt} \in \mathcal{V}',\ y(0) = x \Big\}.$$

Here, $AC([0,T];V')$ is the space of all absolutely continuous V'-valued functions on $[0,T]$. The fact that indeed $\mathcal{A}(\mathcal{V}) \subset \mathcal{V}'$ follows from (1.4) since $p \geq p'$ if $p \geq 2$, and $\mathcal{V}$ is replaced by $\mathcal{V} \cap \mathcal{H}$ for $1 < p < 2$.

We note that, if $p \geq 2$, $y \in L^p(0,T;V)$ and $\frac{dy}{dt} \in L^{p'}(0,T;V')$, that is, if $y \in W^{1,p'}([0,T];V') \cap L^p(0,T;V)$, then $y \in C([0,T];H)$ and $\frac{dy}{dt}$ is just the derivative of y in the sense of V'-valued distributions on $(0,T)$, and so the condition $y \in C([0,T];H)$ in the definition of $D(\mathcal{B})$ is redundant.

We also note that we have

$$\frac{dy}{dt} \in L^{p_1}((0,T) \times \Omega; V'),\ \forall y \in D(\mathcal{B}),$$

for any $1 \leq p_1 < p'$ and so $y : [0,T] \to L^{p_1}(\Omega; V')$ is absolutely continuous $\mathbb{P}$-a.s.

The idea is to represent the Cauchy problem as a stationary equation in the pair of spaces $(\mathcal{V}, \mathcal{V}')$. Namely, the Cauchy problem (5.49) can be written as the operatorial equation

$$\mathcal{B}y + \widetilde{\mathcal{A}}y = 0.$$

We have

Lemma 5.2. *Assume that hypotheses* (H1)–(H3) *with $B(t) \equiv 0$ and $\alpha = 0$, $\alpha_2 = 0$. Then, the operators $\widetilde{\mathcal{A}}$ and $\mathcal{B}$ are maximal monotone from $\mathcal{V}$ to $\mathcal{V}'$. Moreover, the equation $\mathcal{B}y + \widetilde{\mathcal{A}}y = 0$ has a solution.*

Proof. It is easily seen that $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ is monotone, that is,

$${}_{\mathcal{V}'}\langle \mathcal{A}y - \mathcal{A}\bar{y}, y - \bar{y}\rangle_{\mathcal{V}} \geq 0,\ \ \forall y, \bar{y} \in \mathcal{V}.$$

Moreover, since $\mathcal{A}$ is also demicontinuous from $\mathcal{V}$ to $\mathcal{V}'$, it follows that it is maximal monotone (see [1]). By (H3), it also follows that $\widetilde{\mathcal{A}}$ is coercive.

The maximal monotonicity of $\mathcal{B} : \mathcal{V} \to \mathcal{V}'$ is more delicate but follows as in [1]. Then, it follows that $\mathcal{B} + \widetilde{\mathcal{A}}$ is maximal monotone. Since $\mathcal{B} + \widetilde{\mathcal{A}}$ is also coercive, it follows that $R(\mathcal{B} + \widetilde{\mathcal{A}}) = X^*$, as claimed. □ □

By Lemma 5.2 we get, therefore,

Theorem 5.3. *For each $x \in H$ there is a unique solution X to* (5.49). *Moreover, X is of the form*

$$X(t,\omega) = y(t,\omega),$$

where $y \in W^{1,p}([0,T];V') \cap L^p(0,T;V) \cap C([0,T];H)$ $\mathbb{P}$-a.s. $\omega \in \Omega$.

Theorem 1.1 was previously established by E. Pardoux [10] and later on by N. Krylov and B. Rozovski [9] in the general setting given here. However, the proof given here is conceptually different and simpler. The variational approach briefly described in Section 3 was firstly developed in the works [2], [3], [6]. Some recent results on these lines are given in [7]. Theorem 5.3 along with the operatorial approach presented above was given in the work [6]. For other results on the variational of stochastic differential equations we refer to the works [3, 4, 5].

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