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On a class of nonlinear nonlocal fractional differential equations

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ABSTRACT. We investigate the existence of extremal solutions for a class of fractional differential equations in the area of fluid dynamics. By establishing a new comparison theorem and applying the classical monotone iterative approach, we establish sufficient conditions to ensure the existence of the extremal solutions and construct twin convergent monotone explicit iterative schemes. Generalized nonlinear nonlocal Bagley-Torvik equation and generalized Basset equation with nonlinear source functions are some special cases of our discussed problem.

1. INTRODUCTION

We consider the existence of extremal solutions of the following fractional order differential equation

$$u^{(m)}(t) + MD^{\alpha}u(t) = f\left(t, u(t)\right), \quad m-1 < \alpha < m, \ t \in [0,T], \tag{1.1}$$

subject to the nonlinear nonlocal conditions

$$g_k\left(u^{(k)}(t_0), u^{(k)}(t_1), \cdots, u^{(k)}(t_r)\right) = 0, \tag{1.2}$$

where $0 = t_0 < t_1 < \cdots < t_r = T$, $k = 0, 1, \cdots, m-1$, $m \in \mathbb{N}$ and D^{α} is the Caputo fractional derivative of order $\alpha > 0$. The nonlinear functions f and g_k are assumed to satisfy certain conditions, which will be specified later.

There are two types of primary incentives for studying problem (1.1)-(1.2). The first one is the fact that, equation (1.1) serves as a prototype for a large class of fractional differential equations involving more than one differential operator and appears in mathematical models of physical phenomena. For example, when $m = 2$ we obtain a general case of Bagley-Torvik equation which arises in the modeling of the motion of a rigid plate immersed in a Newtonian fluid [22, 24, 11]. Another example for an application of our discussed problem is the Basset equation which describes the forces that occur when a spherical object sinks in a incompressible viscous fluid [5, 6]. And the second one is using nonlinear boundary conditions (1.2) as a generalization of the classical initial and boundary conditions. These conditions are of significance because they have applications in the problems of physics and other areas of applied mathematics. Conditions of this type can be applied in the theory of elasticity with better effect than the initial or boundary conditions. For the importance of nonlinear boundary conditions in different fields, we refer to [7, 8, 9, 1, 2, 3, 23] and the references cited therein.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. For example,, we can find numerous applications in a number of fields such as physics, geophysics,

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polymer rheology, viscoelasticity, capacitor theory, electrical circuits, electron-analytical chemistry, biology, etc. For more details and applications, we refer the reader to the books [16, 10] and references therein. The fractional operators are nonlocal, therefore they are suitable for constructing models possessing memory and hereditary properties of various materials and processes. The presence of memory term in such models not only takes into account the history of the process involved but also carries its impact to present and future development of the dynamic process [18].

Analysis of fractional differential equations has been carried out by various researchers. Recently, there are many papers dealing with the existence, uniqueness, approximation and regularity of solutions of fractional differential equations using various methods. We refer the interested readers to the valuable monographs of Kilbas et al. [16] as well to [10] and references [18, 12, 13, 14, 15, 21, 4, 25, 17, 19]. Among the most important of all fractional differential equations are undoubtedly fractional differential equations involving nonlinear boundary conditions. The nonlinear boundary condition was motivated by mathematical physical problems such as flow fluid through fissured rocks, diffusion of gas in a transparent tube, heat conduction and so on. The pioneering work in this direction is due to Byszewski [7, 8, 9]. Subsequently, many authors contributed to the study of nonlocal problems, see [1, 2, 3, 23] and references therein.

In the present article, we wish to investigate the existence of extremal solutions for the problem (1.1) subject to the nonlocal nonlinear conditions (1.2) using monotone iterative approach. Some auxiliary facts and results are stated in Section 2 which help us to give regularity properties of the solution under some conditions on the source function f. Our method not only establishes excellent conditions to ensure the existence of the extremal solutions, but also constructs twin convergent monotone explicit iterative schemes to seek them. This is done in Section 3. Finally in the last section, we provide two examples useful in the area of fluid dynamics, illustrating the main result. The first example is nonlinear Bagley-Torvik equation involving nonlinear boundary conditions and the other one is nonlinear nonlocal Basset equation.

2. Preliminaries and auxiliary results

Here, we recall several known definitions and properties from fractional calculus theory. For details, see [22, 10, 16]. Throughout the paper $AC^n[0,T], n \in \mathbb{N}$, denotes the set of functions having absolutely continuous n-th derivative on $[0,T]$, and $AC[0,T]$ is the set of absolutely continuous functions on $[0,T]$. It is known that $u \in AC[0,T]$ if and only if there exists a pair $(c,\varphi) \in \mathbb{R} \times L^1[0,T]$ such that $u(t) = c + \int_0^t \varphi(\tau)\, d\tau$.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ for the function $u : [0,T] \to \mathbb{R}$, $u \in L^1[0,T]$ is defined as

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds.$$

Definition 2.2. Let $n-1 < \alpha \leq n$ and $n \in \mathbb{N}$. The Caputo fractional derivative of order $\alpha > 0$ of a function $u : [0,T] \to \mathbb{R}$ is defined as

$$D^\alpha u(t) \quad = \quad D^n I^{n-\alpha} \left(u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!}\, t^k \right),$$

provided that the right-hand side integral exists and is finite. Note that if $u \in AC^{n-1}[0,T]$, then

$$D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds = I^{n-\alpha} u^{(n)}(t).$$

Lemma 2.1. *Let $\alpha, \beta \geq 0$. The fractional integral and derivative operators satisfy the following conditions:*

(i) $I^\alpha I^\beta u = I^{\alpha+\beta} u$, $D^\alpha I^\alpha u = u$ *when* $u \in C[0,T]$.

(ii) $I^\alpha : C[0,T] \to C[0,T]$ *and* $I^\alpha : AC[0,T] \to AC[0,T]$ *are linear and continuous. Precisely* $\|I^\alpha u\|_{C[0,T]} \leq \frac{T^\alpha}{\Gamma(1+\alpha)} \|u\|_{C[0,T]}$.

Now, we state Weissinger's Fixed Point Theorem [10, Theorem D.7] as a generalization of Banach's fixed point theorem which is needed to prove the existence of at least one solution of (1.1).

Theorem 2.1. *Assume (X,d) to be a nonempty complete metric space, and let $\theta_n \geq 0$ for every $n \in \mathbb{N} \cup \{0\}$ and such that $\sum_{n=0}^{\infty} \theta_n$ converges. Furthermore, let the mapping $A : X \to X$ satisfy the inequality $d(A^n x, A^n y) \leq \theta_n d(x,y)$ for every $n \in \mathbb{N}$ and every $x, y \in X$. Then, A has a uniquely determined fixed point x^*. Moreover, for any $x_0 \in X$, the sequence $\{A^n x_0\}_{n=0}^{\infty}$ converges to this fixed point x^*.*

In relation to (1.1)-(1.2), we introduce the following linear problem

$$\left\{ \begin{array}{l} u^{(m)}(t) + M D^\alpha u(t) = h(t), \;\; m-1 < \alpha < m, \; t \in [0,T], \\ u^{(k)}(0) = u_k, \;\; k = 0, 1, \cdots, m-1. \end{array} \right. \tag{2.3}$$

The following lemma plays a crucial role in what follows. Particularly, this lemma is very important in discussion of regularity properties of the solution of (1.1)-(1.2).

Lemma 2.2. *Let $h \in L^1[0,T]$. A function $u \in AC^{m-1}[0,T]$ is a solution of the initial value problem* (2.3) *if and only if u is a solution of the integral equation*

$$\left(u(t) - \sum_{k=0}^{m-1} u_k \frac{t^k}{k!} \right) + M I^{m-\alpha} \left(u(t) - \sum_{k=0}^{m-1} u_k \frac{t^k}{k!} \right) = I^m h(t), \tag{2.4}$$

in the set $C[0,T]$.

Proof. Let $u \in AC^{m-1}[0,T]$ be a solution of (2.3). Then, we have

$$u^{(m)}(t) + M I^{m-\alpha} u^{(m)}(t) = h(t), \tag{2.5}$$

holds almost everywhere on $[0,T]$. Applying the integral operator I^m to both sides of (2.5), we deduce the integral equation (2.4). Conversely, suppose that $u \in C[0,T]$ is a solution of the integral equation (2.4). Applying the operator $MI^{m-\alpha}$ to both sides of (2.4), we have

$$M I^{m-\alpha} \left(u(t) - \sum_{k=0}^{m-1} u_k \frac{t^k}{k!} \right) + M^2 I^{2(m-\alpha)} \left(u(t) - \sum_{k=0}^{m-1} u_k \frac{t^k}{k!} \right) = M I^{2m-\alpha} h(t). \tag{2.6}$$

Combining now (2.4) and (2.6), we deduce

$$\left(u(t) - \sum_{k=0}^{m-1} u_k \frac{t^k}{k!} \right) - M^2 I^{2(m-\alpha)} \left(u(t) - \sum_{k=0}^{m-1} u_k \frac{t^k}{k!} \right) = I^m h(t) - M I^{2m-\alpha} h(t). \tag{2.7}$$

Applying again the operator $MI^{m-\alpha}$ to both sides of (2.7) and combining the resulting equality with (2.4), we have

$$\begin{aligned} \left(u(t) - \sum_{k=0}^{m-1} u_k \frac{t^k}{k!} \right) + M^3 I^{3(m-\alpha)} \left(u(t) - \sum_{k=0}^{m-1} u_k \frac{t^k}{k!} \right) &= I^m h(t) - M I^{2m-\alpha} h(t) \\ &\quad + M^2 I^{3m-2\alpha} h(t). \end{aligned}$$

Repeating this process (n_0-1) times with $n_0(m-\alpha)\geq m$, we get

$$u(t)=\sum_{k=0}^{m-1}u_k\frac{t^k}{k!}+(-M)^{n_0}I^{n_0(m-\alpha)}\left(u(t)-\sum_{k=0}^{m-1}u_k\frac{t^k}{k!}\right)+\sum_{k=0}^{n_0-1}(-M)^kI^{m+k(m-\alpha)}h(t).$$

Then $u\in AC^{m-1}[0,T]$. Now, differentiating (2.4) m times leads to the relation $u^{(m)}(t)+MD^\alpha u(t)=h(t)$ holds almost everywhere on $[0,T]$. On the other hand, a simple calculation shows that the initial conditions are satisfied. □

Lemma 2.3. *Let $h\in L^1[0,T]$. Then the integral equation* (2.4) *has a unique solution in $C[0,T]$.*

Proof. Let $A:C[0,T]\to C[0,T]$ as

$$Aw(t):=-MI^{m-\alpha}w(t)+I^mh(t).$$

A is well defined because of Lemma 2.1. We apply the Weissinger's fixed point theorem to prove that A has a unique fixed point. Let $n\in\mathbb{N}$ and $w_1,w_2\in C[0,T]$. Then, using the semigroup property of the fractional integral operator I^α, we have

$$\begin{aligned}\|A^nw_2-A^nw_1\|_{C[0,T]} &= \left\|-MI^{m-\alpha}\left(A^{n-1}w_2-A^{n-1}w_1\right)\right\|_{C[0,T]}\\ &= \left\|-MI^{m-\alpha}\left(-MI^{m-\alpha}\left(A^{n-2}w_2-A^{n-2}w_1\right)\right)\right\|_{C[0,T]}\\ &\vdots\\ &= \left\|(-M)^nI^{n(m-\alpha)}\left(w_2-w_1\right)\right\|_{C[0,T]}\\ &\leq \frac{(-M)^nT^{n(m-\alpha)}}{\Gamma\left(n(m-\alpha)+1\right)}\|w_2-w_1\|_{C[0,T]},\end{aligned}$$

where the last equality following from Lemma 2.1. In order to apply Weissinger's fixed point theorem, we only need to show that the series $\sum_{n=0}^\infty\frac{(-MT^{m-\alpha})^n}{\Gamma(n(m-\alpha)+1)}$ is convergent. This, however, is trivial in view of the fact that $\sum_{n=0}^\infty\frac{(-MT^{m-\alpha})^n}{\Gamma(n(m-\alpha)+1)}=E_{m-\alpha}(-MT^{m-\alpha})$ where $E_{m-\alpha}(z)$ is the Mittag-Leffler function. This completes the proof. □

3. Main result

In this section, we present an important comparison result about fractional differential equations and use it to construct two monotone iterative sequences which converge to extremal solutions for the problem (1.1)-(1.2).

Lemma 3.4. (Comparison result). *Suppose that $u\in AC^{m-1}[0,T]$ is a function with*

$$\begin{cases}u^{(m)}(t)+MD^\alpha u(t)\geq0, & m-1<\alpha<m,\ t\in[0,T],\\ u^{(k)}(0)\geq0, & k=0,1,\cdots,m-1,\end{cases}\tag{3.8}$$

where $M\in\mathbb{R}$ is a constant, then $u^{(k)}(t)\geq u^{(k)}(0)$ for $k=0,1,\cdots,m-1$.

Proof. Since $u\in AC^{m-1}[0,T]$ satisfies (3.8), we have

$$\begin{cases}u^{(m)}(t)+MI^{m-\alpha}u^{(m)}(t)\geq0,\\ u^{(k)}(0)\geq0,\ \ k=0,1,\cdots,m-1.\end{cases}\tag{3.9}$$

Now substitute $v(t)=u^{(m-1)}(t)$ into (3.9), we have

$$\begin{cases}v'(t)+MI^{m-\alpha}v'(t)\geq0,\\ v(0)\geq0,\end{cases}\tag{3.10}$$

Applying the operator $I^{1-(m-\alpha)}$ to both sides of (3.10) and using the monotonicity properties of Riemann-Liouville integral operator, we deduce

$$\begin{cases} D^{m-\alpha}v(t) + M(v(t) - v(0)) \geq 0, \\ v(0) \geq 0. \end{cases}$$

With the substitution $w(t) = v(t) - v(0)$ and using the fact that the Caputo derivative of a constant is zero, we obtain

$$\begin{cases} D^{m-\alpha}w(t) + Mw(t) \geq 0, \\ w(0) = 0. \end{cases}$$

A similar argument to the proof of Maximum Principle Theorem [20, Theorem 3.1] implies $w(t) \geq 0$ on $[0, T]$. Therefore, $u^{(m-1)}(t) \geq u^{(m-1)}(0)$ on $[0, T]$. Now, applying the integral operator to the inequality $u^{(m-1)}(t) \geq u^{(m-1)}(0)$ together with $u^{(m-2)}(0) \geq 0$, we find that $u^{(m-2)}(t) \geq u^{(m-2)}(0)$ on $[0, T]$. Finally, repeating this process $m-2$ times yields the desired result. $\square$

Definition 3.3. We say that $u \in AC^{m-1}[0, T]$ is called a lower solution of (1.1)-(1.2) if

$$\begin{cases} u^{(m)}(t) + MD^{\alpha}u(t) \leq f\left(t, u(t)\right), \quad m-1 < \alpha < m, \ t \in [0, T], \\ g_k\left(u^{(k)}(t_0), u^{(k)}(t_1), \cdots, u^{(k)}(t_r)\right) \leq 0, \quad 0 = t_0 < t_1 < \cdots < t_r = T, \quad k = 0, 1, \cdots, m-1, \end{cases}$$

and it is an upper solution of (1.1)-(1.2) if the above inequalities are reverted.

We list the following assumptions for the convenience.

(H1) Assume that $\underline{u}, \overline{u} \in AC^{m-1}[0, T]$ are lower and upper solutions of the problem (1.1)-(1.2), respectively, and $\underline{u} \preceq \overline{u}$, i.e., $\underline{u}^{(k)}(t) \leq \overline{u}^{(k)}(t), \ k = 0, 1, \cdots, m-1, \ t \in [0, T]$.

(H2) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function with respect to the second variable.

(H3) For every $i = 0, 1, \cdots, r, \ k = 0, 1, \cdots, m-1, \ g_k \in C(\mathbb{R}^{r+1}, \mathbb{R})$ and there exist constants $\lambda_k > 0$, $\mu_k^i \geq 0$ such that for $\underline{u}^{(k)}(t_i) \leq x_i \leq \bar{x}_i \leq \overline{u}^{(k)}(t_i)$,

$$g_k(\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_r) - g_k(x_0, x_1, \cdots, x_r) \leq \lambda_k(\bar{x}_0 - x_0) - \sum_{i=1}^{r} \mu_k^i(\bar{x}_i - x_i).$$

Theorem 3.2. *Suppose that conditions (H1)-(H3) hold. Then there exist sequences $\{u_j\}, \{v_j\} \subseteq AC^{m-1}[0, T]$ such that $\lim_{j\to\infty} u_j = u$ and $\lim_{j\to\infty} v_j = v$ and u, v are minimal and maximal solutions of the problem* (1.1)-(1.2)*, respectively, in $[\underline{u}, \overline{u}]$ where $[\underline{u}, \overline{u}] = \{u \in C[0, T] \mid \ \underline{u}(t) \leq u(t) \leq \overline{u}(t), \ t \in [0, T]\}$.*

Proof. The proof is divided into four steps:

Step 1. Set $u_0 = \underline{u}$ and $v_0 = \overline{u}$ and then given $\{u_j\}_{j=0}^{\infty}$ and $\{v_j\}_{j=0}^{\infty}$ inductively define $u_{j+1} \in AC^{m-1}[0, T]$ and $v_{j+1} \in AC^{m-1}[0, T]$ to be the unique solutions of the linear problems

$$\begin{cases} u_{j+1}^{(m)}(t) + MD^{\alpha}u_{j+1}(t) = f\left(t, u_j(t)\right), \ j \geq 0, \\ u_{j+1}^{(k)}(0) = \eta_j^k, \quad k = 0, 1, \cdots, m-1, \end{cases} \tag{3.11}$$

where $\eta_j^k = u_j^{(k)}(0) - \frac{1}{\lambda_k} g_k\left(u_j^{(k)}(t_0), u_j^{(k)}(t_1), \cdots, u_j^{(k)}(t_r)\right)$ and

$$\begin{cases} v_{j+1}^{(m)}(t) + MD^{\alpha}v_{j+1}(t) = f\left(t, v_j(t)\right), \ j \geq 0, \\ v_{j+1}^{(k)}(0) = \overline{\eta}_j^k, \quad k = 0, 1, \cdots, m-1, \end{cases} \tag{3.12}$$

where $\overline{\eta}_j^k = v_j^{(k)}(0) - \frac{1}{\lambda_k} g_k\left(v_j^{(k)}(t_0), v_j^{(k)}(t_1), \cdots, v_j^{(k)}(t_r)\right)$. From Lemma 2.2 and Lemma 2.3, we know that (3.11) and (3.12) have a unique solutions in $AC^{m-1}[0, 1]$.

Step 2. We claim

$$\underline{u} = u_0 \preceq u_1 \preceq \cdots \preceq u_j \preceq u_{j+1} \preceq \cdots \preceq v_{j+1} \preceq v_j \preceq \cdots \preceq v_1 \preceq v_0 = \overline{u}. \tag{3.13}$$

To confirm this, first note from (3.11) for $j = 0$ that

$$\begin{cases} u_1^{(m)}(t) + MD^\alpha u_1(t) = f\left(t, u_0(t)\right), \\ u_1^{(k)}(0) = u_0^{(k)}(0) - \frac{1}{\lambda_k} g_k \left(u_0^{(k)}(t_0), \cdots, u_0^{(k)}(t_r)\right), \quad k = 0, 1, \cdots, m-1. \end{cases} \tag{3.14}$$

Recalling the definition of lower solution $u_0 = \underline{u}$ and setting $w = u_1 - u_0$, we find

$$\begin{cases} w^{(m)}(t) + MD^\alpha w(t) \geq 0, \\ w^{(k)}(0) = -\frac{1}{\lambda_k} g_k \left(u_0^{(k)}(t_0), u_0^{(k)}(t_1), \cdots, u_0^{(k)}(t_r)\right) \geq 0, \quad k = 0, 1, \cdots, m-1. \end{cases}$$

Consequently Lemma 3.4 implies $0 \preceq w$, so that $u_0 \preceq u_1$. Now, from (3.14) and using assumptions (H2) and (H3), we infer $u_1^{(m)}(t) + MD^\alpha u_1(t) \leq f\left(t, u_1(t)\right)$ and

$$\begin{aligned} g_k\left(u_1^{(k)}(t_0), \cdots, u_1^{(k)}(t_r)\right) &\leq g_k\left(u_0^{(k)}(t_0), \cdots, u_0^{(k)}(t_r)\right) + \lambda_k\left(u_1^{(k)}(0) - u_0^{(k)}(0)\right) \\ &\quad - \textstyle\sum_{i=1}^r \mu_k^i \left(u_1^{(k)}(t_i) - u_0^{(k)}(t_i)\right) \\ &= -\textstyle\sum_{i=1}^r \mu_k^i \left(u_1^{(k)}(t_i) - u_0^{(k)}(t_i)\right) \\ &\leq 0. \end{aligned}$$

Therefore, u_1 is lower solution of problem (1.1)-(1.2). We can now repeat the argument above to deduce $u_1 \preceq u_2$ and then an induction verifies that $u_j \preceq u_{j+1}$ for $j \geq 2$. Assertion $v_j \preceq v_{j-1}$ for $j \in \mathbb{N}$ follows similarly. Now, we put $w = v_1 - u_1$. From (H2) and (H3), we have

$$\begin{cases} w^{(m)}(t) + MD^\alpha w(t) \geq 0, \\ w^{(k)}(0) \geq \sum_{i=1}^r \frac{\mu_k^i}{\lambda_k}\left(u_1^{(k)}(t_i) - u_0^{(k)}(t_i)\right) \geq 0, \quad k = 0, 1, \cdots, m-1. \end{cases}$$

Consequently, $0 \preceq w$, so that $u_1 \preceq v_1$. Using mathematical induction, we see that $u_j \preceq v_j$ for $j \geq 2$.

Step 3. In light of (3.13), it is easy to show $\{u_j\}$ and $\{v_j\}$ are uniformly bounded and equicontinuous in $[\underline{u}, \overline{u}]$. By the Arzela-Ascoli Theorem, we have

$$\lim_{j\to\infty} u_j = u^*, \quad \lim_{j\to\infty} v_j = v^*,$$

uniformly on $[0, T]$, and the limit functions u^*, v^* satisfy (1.1)-(1.2). Moreover, $u^*, v^* \in [\underline{u}, \overline{u}]$.

Step 4. Finally, we prove u^* and v^* are the extremal solutions of (1.1)-(1.2) in $[\underline{u}, \overline{u}]$. Let $u \in [\underline{u}, \overline{u}]$ be any solution of (1.1)-(1.2). We suppose that $u_j \preceq u \preceq v_j$ for some $j \in \mathbb{N}$. Then, by assumption (H2), we see that $f\left(t, u_j(t)\right) \leq f\left(t, u(t)\right) \leq f\left(t, v_j(t)\right)$ and

$$\begin{aligned} u_{j+1}^{(k)}(0) &= u_j^{(k)}(0) - \frac{1}{\lambda_k} g_k\left(u_j^{(k)}(t_0), u_j^{(k)}(t_1), \cdots, u_j^{(k)}(t_r)\right) \\ &= u_j^{(k)}(0) + \frac{1}{\lambda_k} g_k\left(u^{(k)}(t_0), u^{(k)}(t_1), \cdots, u^{(k)}(t_r)\right) \\ &\quad - \frac{1}{\lambda_k} g_k\left(u_j^{(k)}(t_0), u_j^{(k)}(t_1), \cdots, u_j^{(k)}(t_r)\right) \\ &\leq u^{(k)}(0) - \textstyle\sum_{i=1}^r \frac{\mu_k^i}{\lambda_k}\left(u^{(k)}(t_i) - u_j^{(k)}(t_i)\right) \\ &\leq u^{(k)}(0). \end{aligned}$$

Similarly, we have $u^{(k)}(0) \leq v_{j+1}^{(k)}(0)$. Hence

$$\begin{cases} u_{j+1}^{(m)}(t) + MD^\alpha u_{j+1}(t) \leq u^{(m)}(t) + MD^\alpha u(t) \leq v_{j+1}^{(m)}(t) + MD^\alpha v_{j+1}(t), \\ u_{j+1}^{(k)}(0) \leq u^{(k)}(0) \leq v_{j+1}^{(k)}(0), \; k = 0, 1, \cdots .m-1. \end{cases}$$

Consequently, $u_{j+1} \preceq u \preceq v_{j+1}$. Therefore, we have

$$u_j \preceq u \preceq v_j, \quad j = 0, 1, 2, \cdots . \tag{3.15}$$

Thus, taking limit in (3.15) as $j \to \infty$, we have $u^* \preceq u \preceq v^*$. That is, u^* and v^* are the extremal solutions of (1.1)-(1.2) in $[\underline{u}, \overline{u}]$. □

4. Applications

In this section we discuss some particular cases of (1.1)-(1.2) that are of importance in the area of fluid dynamics.

Example 4.1. Consider the following nonlinear Bagley-Torvik equation involving nonlinear boundary conditions

$$\begin{cases} u''(t) + 5D^{\frac{3}{2}}u(t) = t + \frac{1}{5}u^2(t), & 0 \le t \le 1, \\ 4u(0) - u(1) = 0, \\ u'^2(0) + 4u'(0) - u'(1) = 1, \end{cases} \tag{4.16}$$

where $f(t,u) = t + \frac{1}{5}u^2$, $g_0(x,y) = 4x - y$ and $g_1(x,y) = x^2 + 4x - y - 1$. A relatively simple calculations, with the help of Maple, shows that $\underline{u}(t) = 0$ and $\overline{u}(t) = 1 + t + t^{\frac{3}{2}}$ be lower and upper solutions of (4.16), respectively, and $\underline{u} \preceq \overline{u}$. In addition, it is easy to verify that the assumptions (H2) and (H3) hold and $g_i \in C(\mathbb{R}^2, \mathbb{R}),\ i = 0, 1$ and we have

$$\begin{aligned} g_0(\bar{x}, \bar{y}) - g_0(x,y) &\le 4(\bar{x} - x) - (\bar{y} - y), \quad 0 \le x \le \bar{x} \le 1, \quad 0 \le y \le \bar{y} \le 3, \\ g_1(\bar{x}, \bar{y}) - g_1(x,y) &\le 6(\bar{x} - x) - (\bar{y} - y), \quad 0 \le x \le \bar{x} \le 1, \quad 0 \le y \le \bar{y} \le \frac{5}{2}. \end{aligned}$$

Therefore, all the assumptions of Theorem 3.2 hold and consequently, there exist monotone iterative sequences $\{u_j\}, \{v_j\}$, which converge uniformly on $[0,1]$ to the extremal solutions (u,v) of (4.16).

Example 4.2. Consider the following nonlinear nonlocal Basset equation

$$\begin{cases} u'(t) + 2D^{\frac{1}{2}}u(t) = t\left(1 + \frac{1}{2}u^2(t)\right), & 0 \le t \le 1, \\ 10u(0) - u(\frac{1}{4}) - u(\frac{1}{2}) - u(\frac{3}{4}) - u(1) = 0, \end{cases} \tag{4.17}$$

where $f(t,u) = t\left(1 + \frac{1}{2}u^2(t)\right)$ and $g_0(x_0, x_1, x_2, x_3, x_4) = 10x_0 - x_1 - x_2 - x_3 - x_4$. A relatively simple calculations, with the help of Maple, shows that $\underline{u}(t) = 0$ and $\overline{u}(t) = 1 + t$ be lower and upper solutions of (4.17), respectively, and $\underline{u} \preceq \overline{u}$. In addition, it is easy to verify that the assumptions (H2) and (H3) hold and $g_0 \in C(\mathbb{R}^5, \mathbb{R})$ and we have

$$g_0(\bar{x}_0, \cdots, \bar{x}_4) - g_0(x_0, \cdots, x_4) \le 10(\bar{x}_0 - x_0) - \sum_{i=1}^{4}(\bar{x}_i - x_i),\ 0 \le x_i \le \bar{x}_i \le 3,\ i = 0, \cdots, 4.$$

Therefore, all the assumptions of Theorem 3.2 hold and consequently, there exist monotone iterative sequences $\{u_j\}, \{v_j\}$, which converge uniformly on $[0,1]$ to the extremal solutions (u,v) of (4.17).

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References

[1] Aizicovici, S.; McKibben, M. Existence results for a class of abstract nonlocal Cauchy problems. *Nonlinear Anal.* **39** (2000), 649–668.

[2] Aizicovici, S.; Lee, H. Nonlinear nonlocal Cauchy problems in Banach spaces. *Appl. Math. Lett.* **18** (2005) 401–407.

[3] Aizicovici, S.; Lee, H. Existence results for nonautonomous evolution equations with nonlocal initial conditions. *Commun. Appl. Anal.* **11** (2007) 285–297.

[4] Aleroev, T. S.; Erokhin, S. Some solutions of the nonhomogeneous Bagley-Torvik equation. *International Journal of Modeling, Simulation, and Scientific Computing* **10** (2019), no. 1, 1941002

[5] Benson, D. A.; Wheatcraft, S. W.; Meerschaert, M. M., Application of a fractional advection-dispersion equation. *Water Resour. Res.* **36** (2000), no. 6, 1403–1412.

[6] Benson, D. A.; Wheatcraft, S. W.; Meerschaert, M. M. The fractional-order governing equation of Lévy motion. *Water Resour. Res.* **3** (2000), no. 6, 1413–1423.

[7] Byszewski, L. Theorems about the existence and uniqueness of solution of semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* **162** (1991), 494–505.

[8] Byszewski, L. Existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problem. *Zeszyty Nauk. Politech. Rzeszowskiej Mat. Fiz.* **18** (1993), 109–112.

[9] Byszewski, L.; Lakshmikantham, V. Theorem about the existence and uniqueness of solutions of a nonlocal Cauchy problem in a Banach space. *Appl. Anal.* **40** (1990), 11–19.

[10] Diethelm, K. *The Analysis of Fractional Differential Equations*. Springer-Verlag , Berlin, 2010

[11] El-Nabulsi, R. A. Fractional nonlocal Newton's law of motion and emergence of Bagley-Torvik equation. *J. Peridyn. Nonlocal Model.* (2020) doi:10.1007/s42102-019-00018-6

[12] Fazli, H.; Nieto, J. J. An investigation of fractional Bagley-Torvik equation. *Open Math.***17** (2019), 499–512.

[13] Fazli, H.; Nieto, J. J. Fractional Langevin equation with anti-periodic boundary conditions. *Chaos, Solitons and Fractals* **114** (2018), 332–337.

[14] Fazli, H.; Sun, H.; Aghchi, S. Existence of extremal solutions of fractional Langevin equation involving nonlinear boundary conditions. *Int. J. Comput. Math.* (2020) doi.org/10.1080/00207160.2020.1720662.

[15] Fazli, H.; Sun, H.; Nieto, J. J. Fractional Langevin equation involving two fractional orders: existence and uniqueness revisited. *Mathematics* **8** (2020), 743.

[16] Kilbas, A. A.; Srivastava, H. M.; Trujillo, J. J. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, **204**. *Elsevier Science B. V., Amsterdam*, 2006

[17] Ladde, G. S.; Lakshmikanthan, V.; Vastala, A. S. Monotone Iterative Techniques for Nonlinear Differential Equations. *Pitman, Boston*, 1985.

[18] Lazarevic, M. P.; Spasic, A. M. Finite-time stability analysis of fractional order time delay systems: Gronwall's approach. *Math. Comput. Modelling* **49** (2009), 475–481.

[19] Nieto, J. J. An abstract monotone iterative technique. *Nonlinear Anal.* **28** (1997), 1923–1933.

[20] Nieto, J. J. Maximum principles for fractional differential equations derived from Mittag-Leffler functions. *Appl. Math. Lett.* **23** (2010), 1248–1251

[21] Pang, D.; Jiang, W.; Du, J.; Niazi, A. U. K. Analytical solution of the generalized Bagley-Torvik equation. *Adv. Difference Equ.* **2019**, Paper No. 207, 13 pp.

[22] Podlubny, I. Fractional Differential Equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering, 198. *Academic Press, Inc., San Diego, CA*, 1999.

[23] Schwan, L.; Umnova, O.; Boutin, C.; Groby, J. Nonlocal boundary conditions for corrugated acoustic metasurface with strong near-field interactions. *J. Appl. Phys.* **123** (2018), 091712, doi.org/10.1063/1.5011385

[24] Torvik, P. J.; Bagley, R. L. On the appearance of the fractional derivative in the behavior of real materials. *J. Appl. Mech.* **51** (1984), no. 2, 294–298.

[25] Zafar, A. A.; Kudra, G.; Awrejcewicz, J. An investigation of fractional Bagley-Torvik equation. *Entropy* **22** (2020), no. 1, Paper No. 28, 13 pp.

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