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Iterative methods for optimization problems and image restoration

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ABSTRACT. In this paper, we introduce a new accelerated iterative method for finding a common fixed point of a countable family of nonexpansive mappings in the Hilbert spaces framework. Using our main result, we obtain a new accelerated image restoration iterative method for solving a minimization problem in the form of the sum of two proper lower semi-continuous and convex functions. As applications, we apply our algorithm to solving image restoration problems.

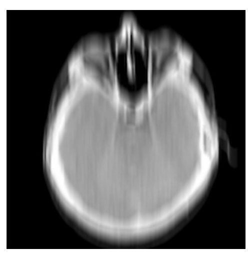
1. INTRODUCTION

Optimization theory is widely used as it can be used to solve many practical problems such as engineering, economics, computer science, and applied science. The study of optimization theory is based on functional analysis, nonlinear analysis, and convex analysis. From an operational perspective, numerical analysis was considered to solve the optimization problem and to show the convergence of the sequence. It is studied or solved using numerical iterative methods, most of which are used in everyday life.

At present, there is various photography. Be it from a camera or a scan, perhaps a blurry image, such as a blurry, blurred, or dark spot. That makes the picture distort the truth as in the Figure 1.


(A) Kitkuan


(B) Butterfly


(C) HeadCT

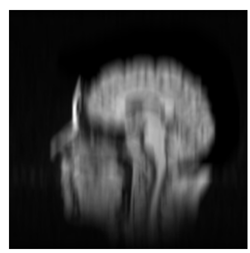

(D) Brain

FIGURE 1. Test images

From the above problem can be written in mathematical models as shown in the Figure 2.

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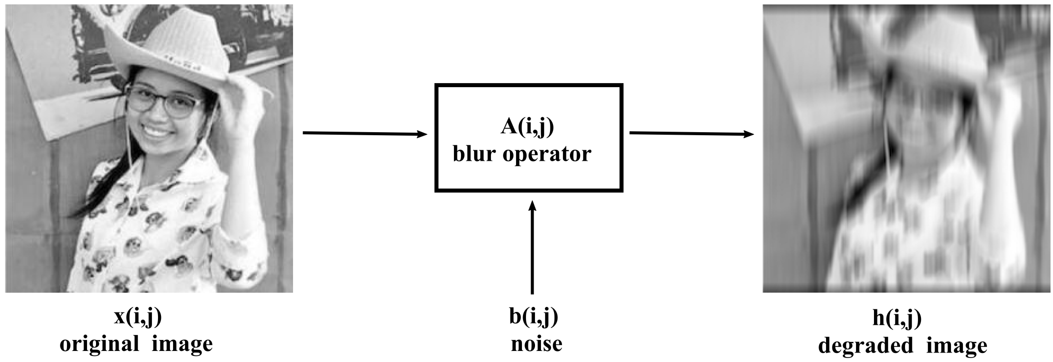


FIGURE 2. Degradation model

We can be written as an equation as follows:

$$h = Ax + b, \tag{1.1}$$

where $x \in \mathbb{R}^n$ is original image, A is blur operator, b is noise, and h is degraded image. In order to solve the problem (1.1), Tibshirani in [19], introduced the least absolute shrinkage and selection operator (LASSO) for solving the following minimization problem:

$$\min_x \frac{1}{2}\|Ax - h\|_2^2 + \mu\|x\|_1, \tag{1.2}$$

where $\mu > 0$ is a regularization parameter, $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$. In the theory of optimal optimization, concrete problems can be solved in many ways:

$$\begin{aligned} &\text{minimize } f(x) + g(x) \\ &\text{subject to } x \in \mathbb{R}^n, \end{aligned} \tag{1.3}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex smooth function and $g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a proper convex lower semi-continuous and nonsmooth function. The solution of (1.3) can be characterized by Theorem 16.3 of Bauschke and Combettes [1] as follows:

$$w \text{ is a minimizer of } (f + g) \text{ if and only if } 0 \in \partial g(w) + \nabla f(w),$$

where ∂g is subdifferential of g and ∇f is the gradient of f. The subdifferential of g at w, denoted by $\partial g(w)$, is defined by

$$\partial g(w) := \{z : g(x) \geq \langle z, x - w\rangle + g(w) \text{ for all } x\}.$$

It is well-known that the subdifferential operator ∂g is maximal monotone, see [3] for more details. For solving (1.3) is characterized by the following fixed point problem:

$$w \text{ is a minimizer of } (f + g) \text{ if and only if } w = \operatorname*{prox}_{rg}(w - r\nabla f(w)),$$

for any $r > 0$ and prox_{rg} is the proximity operator of g defined by

$$\text{prox}_g(x) = \arg\min_z \left\{ g(z) + \frac{\|x - z\|^2}{2} \right\}.$$

Moudafi and Oliny [14] introduced a method called Inertial method, which is another name for the heavy ball method, as follows:

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \text{prox}_{rg}(z_n - r_n\nabla f(x_n)), \quad \forall n \geq 1 \end{cases} \tag{1.4}$$

and proved that this algorithm converts to the minimum of $f+g$ which $r_n < \dfrac{1}{L}$, where L is the lipchitz constant of ∇f.

Beck and Teboulle [2] introduced a so-called FISTA (fast iterative shrinkage-thresholding algorithm) as follows:

$$\begin{cases} x_1 = z_0, t_0 = 1, \\ z_n = \text{prox}_{r_n g}(x_n - r_n \nabla f(x_n)), \\ t_{n+1} = \dfrac{1+\sqrt{4t_n^2+1}}{2}, \\ \theta_n = \dfrac{t_n - 1}{t_{n+1}}, \\ x_{n+1} = z_n + \theta_n(z_n - z_{n-1}), \quad \forall n \geq 1. \end{cases} \tag{1.5}$$

Recently, Verma and Shukla [22] introduced a new accelerated proximal gradient algorithm (NAGA) as follow:

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = T_n((1-\delta_n)z_n + \delta_n T_n z_n), \quad \forall\ n \geq 1, \end{cases} \tag{1.6}$$

where $x_0, x_1 \in \mathbb{R}^n$, T_n is the forward-backward operator of f and g with respect to $r_n \in (0, 2/L)$. They proved the convergence of the NAGA and applied to solving the convex minimization problem with sparsity-inducing regularizes for multitask learning framework.

Motivated by those works mentioned above, in this paper, a new accelerated fixed point algorithms for solving (1.3) by Ishikawa type with the inertial step for a countable family of nonexpansive mappings.

2. Preliminaries

The fundamentals of the Hilbert space are studied, as well as definitions and theorems. It is used to prove it as follows:

Definition 2.1. [19] The two real-value functions $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ are called the inner product (inner product) on the X vector space for members $x, y, z \in X$ and the constants $\alpha, \beta \in \mathbb{R}$ satisfy the following conditions

(i) $\langle x, x \rangle \geq 0$ for all $x \in X$;
(ii) $\langle x, x \rangle = 0$ if and only if $x = 0$;
(iii) $\langle x, y \rangle = \langle y, x \rangle$;
(iv) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$.

The vector space X and the internal product $\langle \cdot, \cdot \rangle$ can be written together $(X, \langle \cdot, \cdot \rangle)$, which is called the inner product space.

In addition, various features of the convergence sequence in the Hilbert space $\mathcal{H}$ are investigated, beginning with the naming and proof of the following theorem.

Lemma 2.1. [19] *Let $\mathcal{H}$ be a Hilbert space. The following statement holds in $\mathcal{H}$:*

$$\|\alpha x + \rho z\|^2 = \alpha(\alpha+\rho)\|x\|^2 + \rho(\alpha+\rho)\|z\|^2 - \rho\alpha\|x - z\|^2,\ \forall x, z \in \mathcal{H}, \forall \alpha, \rho \in \mathbb{R}.$$

Let $\mathcal{H}$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, and C be a nonempty closed convex subset of $\mathcal{H}$.

A nonlinear operator $T : C \to C$ is called

(i) L-Lipschitz operator, if there exists $L > 0$ such that

$$\|Tx - Tz\| \leq L\|x - z\|, \quad \text{for all } x, z \in C;$$

(ii) nonexpansive, if

$$\|Tx - Tz\| \leq \|x - z\|, \quad \text{for all } x, z \in C.$$

Next, we denote by $Fix(T)$ the set of all fixed points of T, $Fix(T) := \{w \in C : Tw = w\}$, $\omega_w(x_n)$ denote the set of all weak-cluster points of a bounded sequence $\{x_n\}$ in C, $\{T_n\}$ and Ψ be families of nonexpansive operators of C into itself such that $\Omega := \bigcap_{k=1}^{\infty} Fix(T_n) \supset Fix(\Psi) \neq \emptyset$, where $Fix(\Psi)$ is the set of all common fixed points of Ψ.

Nakajo et al. [15] introduced the NST-condition (I) with Ψ. A sequence $\{T_n\}$ is said to satisfy the NST if for every bounded sequence $\{x_n\}$ in C,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 \quad \text{implies} \quad \lim_{n\to\infty} \|x_n - Tx_n\| = 0, \quad \forall\, T \in \Psi.$$

Nakajo et al. [16] introduced the NST*-condition which is more general than that of NST-condition. A sequence $\{T_n\}$ is said to satisfy the NST*-condition if for every bounded sequence $\{x_n\}$ in C,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0 \quad \text{implies} \quad \omega_w(x_n) \subset \Omega.$$

Lemma 2.2. [14] *Let $\mathcal{H}$ be a Hilbert space and $\{x_n\}$ be a sequence in $\mathcal{H}$ such that there exists a nonempty set $\Omega \subset \mathcal{H}$ satisfying*

(i) *for every $x \in \Omega$, $\lim_{n\to\infty} \|x_n - x\|$ exists*;

(ii) *each weak-cluster point of the sequence $\{x_n\}$ is in Ω.*

Then there exists $w \in \Omega$ such that $\{x_n\}$ weakly converges to w.

Lemma 2.3. [4] *For a real Hilbert space $\mathcal{H}$, let $g : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a proper convex and lower semi-continuous function, and $f : \mathcal{H} \to \mathbb{R}$ be convex differentiable with gradient ∇f being L-Lipschitz constant for some $L > 0$. If $\{T_n\}$ is the forward-backward operator of f and g with respect to $r_n \in (0, 2/L)$ such that r_n converges to r, then $\{T_n\}$ satisfies NST-condition (I) with T, where T is the forward-backward operator of f and g with respect to $r \in (0, 2/L)$.*

Lemma 2.4. [11] *Let $\{\upsilon_n\}, \{\delta_n\}$ and $\{\theta_n\}$ are sequence in $[0, \infty)$ such that*

$$\upsilon_{n+1} \leq \upsilon_n + \theta_n(\upsilon_n - \upsilon_{n-1}) + \delta_n,\ \forall n \geq 1, \sum_{n=1}^{\infty} \delta_n < \infty$$

and there exists θ which $0 \leq \theta_n \leq \theta < 1$ for all $n \in \mathbb{N}$.
So the following statement hold:

(i) $\sum_{n=1}^{\infty} [\upsilon_n - \upsilon_{n-1}]_+ < \infty$, *where* $[t]_+ := \max\{t, 0\}$;

(ii) *there exists* $\upsilon^* \in [0, \infty)$, *where* $\lim_{n\to\infty} \upsilon_n = \upsilon^*$.

3. Main results

Study of the convergence theory of repeat methodology. To the answer to the problem of optimal optimization in the Hilbert space $\mathcal{H}$ and from the theorem constructions and improvements. Obtain important new knowledge with key content as follows:

Theorem 3.1. *Let $\{T_n : \mathcal{H} \to \mathcal{H}\}$ be a family of nonexpansive operators. Suppose $\{T_n\}$ satisfies NST*-condition and $\Omega := \bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. Let the sequence $\{x_n\}$ in $\mathcal{H}$ be generated by choosing $x_0, x_1 \in \mathcal{H}$ and using the recursion*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1 - \tau_n)x_n + \tau_n\left((1 - \gamma_n)w_n + \gamma_n T_n w_n\right), \\ x_{n+1} = (1 - \delta_n)x_n + \delta_n\left((1 - \lambda_n)z_n + \lambda_n T_n z_n\right), \end{cases} \tag{3.7}$$

where $\{\tau_n\}, \{\gamma_n\}, \{\delta_n\}, \{\lambda_n\}$ and $\{\theta_n\}$ are sequences such that

(i) $0 \leq \theta_n \leq \theta_{n+1} \leq 1$;
(ii) $0 < \tau \leq \tau_n \leq \tau_{n+1} \leq \dfrac{1}{2+\alpha} := \epsilon, \alpha > 0$;
(iii) $0 < \gamma \leq \gamma_n \leq \rho < 1$;
(iv) $0 < \delta \leq \delta_n \leq \delta_{n+1} \leq \dfrac{1}{2+\beta} := \varepsilon, \beta > 0$;
(v) $0 < \lambda \leq \lambda_n \leq \iota < 1$.

Then the sequence $\{x_n\}$ generated by (3.7) *converges weakly to a point $w \in \Omega$.*

Proof. Let $w \in \Omega$, $\zeta_n = (1-\gamma_n)w_n + \gamma_n T_n w_n$ and $\vartheta_n = (1-\lambda_n)z_n + \lambda_n T_n z_n$. Using Lemma 2.1, we obtain

$$\begin{aligned} \|\zeta_n - w\|^2 &= \|(1-\gamma_n)w_n + \gamma_n T_n w_n - w\|^2 \\ &= \|(1-\gamma_n)(w_n - w) + \gamma_n (T_n w_n - w)\|^2 \\ &= (1-\gamma_n)\|w_n - w\|^2 + \gamma_n \|T_n w_n - w\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\|T_n w_n - w_n\|^2 \\ &\leq \|w_n - w\|^2 - \gamma_n(1-\gamma_n)\|T_n w_n - w_n\|^2. \end{aligned} \tag{3.8}$$

Hence,

$$\|\zeta_n - w\| \leq \|w_n - w\| \tag{3.9}$$

and

$$\begin{aligned} \|\vartheta_n - w\|^2 &= \|(1-\lambda_n)z_n + \lambda_n T_n z_n - w\|^2 \\ &= \|(1-\lambda_n)(z_n - w) + \lambda_n (T_n z_n - w)\|^2 \\ &= (1-\lambda_n)\|z_n - w\|^2 + \lambda_n \|T_n z_n - w\|^2 \\ &\quad - \lambda_n(1-\lambda_n)\|T_n z_n - z_n\|^2 \\ &\leq \|z_n - w\|^2 - \lambda_n(1-\lambda_n)\|T_n z_n - z_n\|^2. \end{aligned} \tag{3.10}$$

So,

$$\|\vartheta_n - w\| \leq \|z_n - w\|. \tag{3.11}$$

From (3.9), we get

$$\begin{aligned} \|x_{n+1} - w\|^2 &= \|(1-\delta_n)x_n + \delta_n \vartheta_n - w\|^2 \\ &= \|(1-\delta_n)(x_n - w) + \delta_n(\vartheta_n - w)\|^2 \\ &= (1-\delta_n)\|x_n - w\|^2 + \delta_n \|\vartheta_n - w\|^2 \\ &\quad - \delta_n(1-\delta_n)\|x_n - \vartheta_n\|^2. \end{aligned} \tag{3.12}$$

From $z_n = (1-\tau_n)x_n + \tau_n \zeta_n$ and $x_{n+1} = (1-\delta_n)x_n + \delta_n \vartheta_n$, we obtain

$$\zeta_n - x_n = \frac{1}{\tau_n}(z_n - x_n) \tag{3.13}$$

and

$$\vartheta_n - x_n = \frac{1}{\delta_n}(x_{n+1} - x_n). \tag{3.14}$$

Using (3.12), (3.13), and (3.14), we obtain

$$\begin{aligned}\|x_{n+1}-w\|^2 &\leq (1-\delta_n)\|x_n-w\|^2+\delta_n\|\vartheta_n-w\|^2\\ &\quad -\delta_n(1-\delta_n)\|x_n-\vartheta_n\|^2\\ &=(1-\delta_n)\|x_n-w\|^2+\delta_n\|z_n-w\|^2\\ &\quad -\frac{(1-\delta_n)}{\delta_n}\|x_{n+1}-x_n\|^2.\end{aligned} \tag{3.15}$$

From (3.14), we get

$$\begin{aligned}\|z_n-w\|^2 &= \|(1-\tau_n)x_n+\tau_n\zeta_n-w\|^2\\ &=\|(1-\tau_n)(x_n-w)+\tau_n(\zeta_n-w)\|^2\\ &=(1-\tau_n)\|x_n-w\|^2+\tau_n\|\zeta_n-w\|^2\\ &\quad -\tau_n(1-\tau_n)\|x_n-\zeta_n\|^2\\ &=(1-\tau_n)\|x_n-w\|^2+\tau_n\|\zeta_n-w\|^2\\ &\quad -\frac{(1-\tau_n)}{\tau_n}\|z_n-x_n\|^2.\end{aligned} \tag{3.16}$$

From (3.9), we get

$$\begin{aligned}\|z_n-w\|^2 &\leq (1-\tau_n)\|x_n-w\|^2+\tau_n\|w_n-w\|^2\\ &\quad -\frac{(1-\tau_n)}{\tau_n}\|z_n-x_n\|^2.\end{aligned} \tag{3.17}$$

Using (3.15), (3.16), and (3.17), we obtain

$$\begin{aligned}\|x_{n+1}-w\|^2 &\leq (1-\delta_n)\|x_n-w\|^2+\delta_n(1-\tau_n)\|x_n-w\|^2+\delta_n\tau_n\|w_n-w\|^2\\ &\quad -\frac{\delta_n(1-\tau_n)}{\tau_n}\|z_n-x_n\|^2\\ &\quad -\frac{(1-\delta_n)}{\delta_n}\|x_{n+1}-x_n\|^2\\ &=(1-\delta_n\tau_n)\|x_n-w\|^2+\delta_n\tau_n\|w_n-w\|^2\\ &\quad -\frac{\delta_n(1-\tau_n)}{\tau_n}\|z_n-x_n\|^2\\ &\quad -\frac{(1-\delta_n)}{\delta_n}\|x_{n+1}-x_n\|^2\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\|w_n-w\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-w\|^2\\ &=\|(1+\theta_n)(x_n-w)-\theta_n(x_{n-1}-w)\|^2\\ &=(1+\theta_n)\|x_n-w\|^2-\theta_n\|x_{n-1}-w\|^2\\ &\quad +\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{3.19}$$

Using (3.18) and (3.19), we obtain

$$
\begin{aligned}
\|x_{n+1} - w\|^2 &\le (1 - \delta_n \tau_n)\|x_n - w\|^2 + \delta_n \tau_n (1 + \theta_n)\|x_n - w\|^2 \\
&\quad - \delta_n \tau_n \theta_n \|x_{n-1} - w\|^2 + \delta_n \tau_n \theta_n (1 + \theta_n)\|x_n - x_{n-1}\|^2 \\
&\quad - \frac{\delta_n (1 - \tau_n)}{\tau_n}\|z_n - x_n\|^2 \\
&\quad - \frac{(1 - \delta_n)}{\delta_n}\|x_{n+1} - x_n\|^2 \\
&= (1 + \delta_n \tau_n \theta_n)\|x_n - w\|^2 \\
&\quad - \delta_n \tau_n \theta_n \|x_{n-1} - w\|^2 + \delta_n \tau_n \theta_n (1 + \theta_n)\|x_n - x_{n-1}\|^2 \\
&\quad - \frac{\delta_n (1 - \tau_n)}{\tau_n}\|z_n - x_n\|^2 \\
&\quad - \frac{(1 - \delta_n)}{\delta_n}\|x_{n+1} - x_n\|^2.
\end{aligned} \tag{3.20}
$$

Let $\Upsilon_n = \|x_n - w\|^2 - \delta_n \tau_n \theta_n \|x_{n-1} - w\|^2 + \delta_n \tau_n \theta_n (1 + \theta_n)\|x_n - x_{n-1}\|^2$.
Since $\delta_n \le \delta_{n+1}, \quad \tau_n \le \tau_{n+1}$ and $\theta_n \le \theta_{n+1}$ then $\delta_n \tau_n \theta_n \le \delta_{n+1} \tau_{n+1} \theta_{n+1}$. Hence,

$$
\begin{aligned}
\Upsilon_{n+1} - \Upsilon_n &\le \|x_{n+1} - w\|^2 - (1 + \delta_{n+1} \tau_{n+1} \theta_{n+1})\|x_n - w\|^2 \\
&\quad + \delta_n \tau_n \theta_n \|x_{n-1} - w\|^2 + \delta_{n+1} \tau_{n+1} \theta_{n+1} (1 + \theta_{n+1})\|x_{n+1} - x_n\|^2 \\
&\quad - \delta_n \tau_n \theta_n (1 + \theta_n)\|x_n - x_{n-1}\|^2 \\
&\le \|x_{n+1} - w\|^2 - (1 + \delta_n \tau_n \theta_n)\|x_n - w\|^2 \\
&\quad + \delta_n \tau_n \theta_n \|x_{n-1} - w\|^2 + \delta_{n+1} \tau_{n+1} \theta_{n+1} (1 + \theta_{n+1})\|x_{n+1} - x_n\|^2 \\
&\quad - \delta_n \tau_n \theta_n (1 + \theta_n)\|x_n - x_{n-1}\|^2.
\end{aligned} \tag{3.21}
$$

Using (3.20) and (3.21), we obtain

$$
\begin{aligned}
\Upsilon_{n+1} - \Upsilon_n &\le -\frac{(1 - \delta_n)}{\delta_n}\|x_{n+1} - x_n\|^2 + \delta_{n+1} \tau_{n+1} \theta_{n+1} (1 + \theta_{n+1})\|x_{n+1} - x_n\|^2 \\
&= -\left(\frac{1 - \delta_n}{\delta_n} - \delta_{n+1} \tau_{n+1} \theta_{n+1} (1 + \theta_{n+1})\right)\|x_{n+1} - x_n\|^2.
\end{aligned} \tag{3.22}
$$

Using conditions (i), (ii), and (iv), we obtain

$$
\begin{aligned}
\frac{1 - \delta_n}{\delta_n} - \delta_{n+1} \tau_{n+1} \theta_{n+1} (1 + \theta_{n+1}) &= \frac{1}{\delta_n} - 1 - \delta_{n+1} \tau_{n+1} \theta_{n+1} (1 + \theta_{n+1}) \\
&\ge 2 + \beta - 1 - \frac{2}{(2 + \beta)(2 + \alpha)} \\
&\ge \beta.
\end{aligned} \tag{3.23}
$$

Using (3.22) and (3.23), we obtain

$$
\Upsilon_{n+1} - \Upsilon_n \le -\beta\|x_{n+1} - x_n\|^2. \tag{3.24}
$$

Hence, the sequence $\{\Upsilon_n\}$ is non-increasing. Similarly,

$$
\begin{aligned}
\Upsilon_n &= \|x_n - w\|^2 - \delta_n \tau_n \theta_n \|x_{n-1} - w\|^2 + \delta_n \tau_n \theta_n (1 + \theta_n)\|x_n - x_{n-1}\|^2 \\
&\ge |x_n - w\|^2 - \delta_n \tau_n \theta_n \|x_{n-1} - w\|^2,
\end{aligned} \tag{3.25}
$$

where

$$
\delta_n \tau_n \theta_n \le \frac{1}{(2 + \beta)(2 + \alpha)} = \varepsilon \epsilon < 1.
$$

From (3.24), we get

$$\begin{aligned}
\|x_n - w\|^2 &\le \delta_n \tau_n \theta_n \|x_{n-1} - w\|^2 + \Upsilon_n \\
&\le \varepsilon\epsilon \|x_{n-1} - w\|^2 + \Upsilon_n \\
&\le \varepsilon\epsilon \|x_{n-1} - w\|^2 + \Upsilon_1 \\
&\vdots \\
&\le (\varepsilon\epsilon)^n \|x_0 - w\|^2 + (1 + \cdots + (\varepsilon\epsilon)^{n-1})\Upsilon_1 \\
&\le (\varepsilon\epsilon)^n \|x_0 - w\|^2 + \frac{\Upsilon_1}{1-\varepsilon\epsilon}
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
\Upsilon_{n+1} &= \|x_{n+1} - w\|^2 - \delta_{n+1}\tau_{n+1}\theta_{n+1}\|x_n - w\|^2 \\
&\quad + \delta_{n+1}\tau_{n+1}\theta_{n+1}(1+\theta_{n+1})\|x_{n+1} - x_n\|^2 \\
&\ge |x_n - w\|^2 - \delta_n\tau_n\theta_n\|x_{n-1} - w\|^2.
\end{aligned} \tag{3.27}$$

From (3.27) and (3.26), we get

$$\begin{aligned}
-\Upsilon_{n+1} &\le \delta_{n+1}\tau_{n+1}\theta_{n+1}\|x_n - w\|^2 \\
&\le \varepsilon\epsilon\|x_n - w\|^2 \\
&\vdots \\
&\le (\varepsilon\epsilon)^{n+1}\|x_0 - w\|^2 + \frac{\varepsilon\epsilon\Upsilon_1}{1-\varepsilon\epsilon}.
\end{aligned} \tag{3.28}$$

Using (3.24) and (3.28), we obtain

$$\begin{aligned}
\beta\sum_{n=1}^{k}\|x_{n+1} - x_n\|^2 &\le \Upsilon_1 - \Upsilon_{k+1} \\
&\le (\varepsilon\epsilon)^{k+1}\|x_0 - w\|^2 + \frac{\Upsilon_1}{1-\varepsilon\epsilon}.
\end{aligned} \tag{3.29}$$

This implies

$$\sum_{n=1}^{\infty}\|x_{n+1} - x_n\|^2 \le \frac{\Upsilon_1}{\beta(1-\varepsilon\epsilon)} < \infty. \tag{3.30}$$

Hence, $\lim_{n\to\infty}\|x_{n+1} - x_n\|^2 = 0$, we get

$$\begin{aligned}
\|w_n - x_n\|^2 &= \theta_n\|x_n - x_{n-1}\|^2 \\
&\le \|x_n - x_{n-1}\|^2 \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.31}$$

From (3.20), we get

$$\|x_{n+1} - w\|^2 \le (1+\delta_n\tau_n\theta_n)\|x_n - w\|^2 - \delta_n\tau_n\theta_n\|x_{n-1} - w\|^2 + 2\|x_n - x_{n-1}\|^2. \tag{3.32}$$

Using Lemma 2.4, we obtain

$$\lim_{n\to\infty}\|x_n - w\|^2 = q < \infty. \tag{3.33}$$

Using condition (i), we get $\lim_{n\to\infty}\theta_n$ exists. Suppose $\lim_{n\to\infty}\theta_n = \theta \in [0,1]$ then $\lim_{n\to\infty}\theta_n\|x_n - w\|^2 = \theta q$. Similarly, $\lim_{n\to\infty}\theta_n\|x_{n-1} - w\|^2 = \theta q$ and from (3.19), we get

$$\begin{aligned}
\lim_{n\to\infty}\|w_n - w\|^2 &= \lim_{n\to\infty}\big[(1+\theta_n)\|x_n - w\|^2 - \theta_n\|x_{n-1} - w\|^2 \\
&\quad + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2\big] = q.
\end{aligned} \tag{3.34}$$

Hence, $\lim_{n\to\infty}[(1+\theta_n)\|x_n - w\|^2 - \theta_n\|x_{n-1} - w\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2]$ exists. Using (3.8) and (3.18), we obtain

$$\begin{aligned}\|x_{n+1} - w\|^2 &\le (1-\delta_n\tau_n)\|x_n - w\|^2 + \delta_n\tau_n\|w_n - w\|^2 \\ &\quad - \delta_n\tau_n\gamma_n\|T_n w_n - w_n\|^2.\end{aligned} \tag{3.35}$$

So, we get

$$\begin{aligned}\delta\tau\gamma\|T_n w_n - w_n\|^2 &\le \delta_n\tau_n\gamma_n\|T_n w_n - w_n\|^2 \\ &\le \|x_n - w\|^2 - \|x_{n+1} - w\|^2 - \delta_n\tau_n\|x_n - w\|^2 \\ &\quad + \delta_n\tau_n\|w_n - w\|^2.\end{aligned} \tag{3.36}$$

By condition (ii) and (iv), we get $\lim_{n\to\infty}\delta_n$ and $\lim_{n\to\infty}\tau_n$ exists. From

$$\lim_{n\to\infty}\delta_n\tau_n\|x_n - w\|^2 = \lim_{n\to\infty}\delta_n\tau_n\|w_n - w\|^2$$

and (3.36), we get $\limsup_{n\to\infty}\|T_n w_n - w_n\|^2 \le 0$. Then,

$$\lim_{n\to\infty}\|T_n w_n - w_n\| = 0.$$

Hence,

$$\begin{aligned}\|x_n - T_n x_n\| &\le \|x_n - w_n\| + \|w_n - T_n w_n\| + \|T_n w_n - T_n x_n\| \\ &\le 2\|x_n - w_n\| + \|w_n - T_n w_n\| \to 0 \text{ as } n\to\infty.\end{aligned}$$

Since $\{T_n\}$ satisfies NST*-condition, we get $\omega_w(x_n) \subset \Omega := \bigcap_{n=1}^{\infty} Fix(T_n)$. Hence, by Lemma 2.2, we obtain that $\{x_n\}$ converges weakly to a point $w \in \Omega$. □

Finally, we apply Algorithm (3.7) for solving the minimization problem (1.3) by setting $T_n = \mathrm{prox}_{r_n g}(I - r_n\nabla f)$, the forward-backward operator of f and g with respect to r_n, where $g : \mathcal{H} \to \mathbb{R}\cup\{\infty\}$ is proper convex and lower semi-continuous, and $f : \mathcal{H} \to \mathbb{R}$ is a continuously differentiable convex function, whose gradient is Lipschitz continuous ($L > 0$).

Theorem 3.2. *Let $\mathcal{H}$ be a Hilbert space, $f : \mathcal{H} \to \mathbb{R}$ be a convex and differentiable function such that ∇f is $\frac{1}{L}$-Lipschitz continuous and let $g : \mathcal{H} \to \mathbb{R}\cup\{\infty\}$ be a proper convex and lower semi-continuous function. We define a sequence $\{x_n\}$ by the iterative scheme, for any $x_0, x_1 \in \mathcal{H}$,*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1-\tau_n)x_n + \tau_n\left((1-\gamma_n)w_n + \gamma_n \mathrm{prox}_{r_n g}(w_n - r_n\nabla f(w_n))\right), \\ x_{n+1} = (1-\delta_n)x_n + \delta_n\left((1-\lambda_n)z_n + \lambda_n \mathrm{prox}_{r_n g}(z_n - r_n\nabla f(z_n))\right), \end{cases} \tag{3.37}$$

where $\{\tau_n\}, \{\gamma_n\}, \{\delta_n\}, \{\lambda_n\}, \{r_n\}$ and $\{\theta_n\}$ are sequences such that

(i) $0 \le \theta_n \le \theta_{n+1} \le 1$;
(ii) $0 < \tau \le \tau_n \le \tau_{n+1} \le \dfrac{1}{2+\alpha} := \epsilon, \alpha > 0$;
(iii) $0 < \gamma \le \gamma_n \le \rho < 1$;
(iv) $0 < \delta \le \delta_n \le \delta_{n+1} \le \dfrac{1}{2+\beta} := \varepsilon, \beta > 0$;
(v) $0 < \lambda \le \lambda_n \le \iota < 1$;
(vi) $0 < r_n < \dfrac{2}{L}$.

Then the sequence $\{x_n\}$ generated by (3.37) *converges weakly to a point $w \in \arg\min(f+g)$.*

Proof. Let T be the forward-backward operator of f and g with respect to r, and T_n be the forward-backward operator of f and g with respect to r_n, that is $T := \mathrm{prox}_{rg}(I - r\nabla f)$ and $T_n := \mathrm{prox}_{r_n g}(I - r_n\nabla f)$. Then T and $\{T_n\}$ are nonexpansive operators for all n, and $Fix(T) = \bigcap_{n=1}^{\infty} Fix(T_n) = \arg\min(f+h)$(see Proposition 26.1 in [1].) Using Lemma 2.3, we have that $\{T_n\}$ satisfies the NST*-condition. Hence, we obtain the required result directly by Theorem 3.1. □

4. NUMERICAL EXPERIMENTS

In this section, we will discuss the results of some of the tests using our proposed algorithm in Theorem 3.2 to solve the problem of image restoration. In Theorem 3.2, we set $f(x) = \frac{1}{2}\|Ax-h\|_2^2$ and $g(x) = \mu\|x\|_1$, it is easy to see that f is a smooth function with L-Lipschitz continuous gradient $\nabla f(x) = A^*(Ax-h)$, where $L = \|A^*A\|$. The 1-norm is "simple", as its proximal operator is a soft thresholding: $\text{prox}_{rg}(x_n) = \max\left(0, 1-\frac{\mu r}{|x_n|}\right)x_n$. We consider two blurring functions from MATLAB: a Gaussian blur (Matlab function is, "fspecial('gaussian',7,7)") and a motion blur (Matlab function is, "fspecial('motion',15,45)") respectively.

Numerical experiments presented in this section aim at proving the validity of our proposed algorithm compared with NAGA proposed in (1.6). We set $\theta_n = 0.99$, $\mu = 10^{-4}$, $\tau_n = \gamma_n = \delta_n = \lambda_n = 0.91$. In our paper, the comparison is done in terms of the relative error defined as

$$\frac{\|x_n - x\|_2^2}{\|x\|_2^2},$$

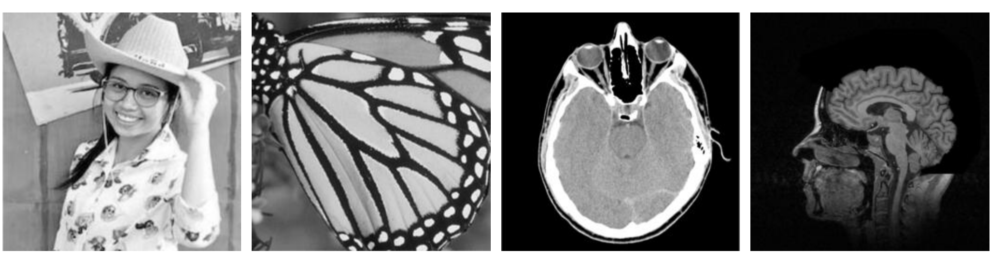
the quality of image recovery is measured by the improvement in signal to noise ratio (ISNR). Note that ISNR defined as

$$\text{ISNR} = 10\log\frac{\|x-h\|_2^2}{\|x-x_n\|_2^2},$$

where x, h, and x_n are the original image, the observed image, and estimated image at iteration n, respectively. All algorithms are implemented under Windows 10 and MATLAB 2017b running on a Dell laptop with Intel(R) Core(TM) i5 CPU and 4 GB of RAM. The stopping criterion of the algorithm is

$$\frac{\|x_{n+1}-x_n\|_2}{\|x_{n+1}\|_2} < 10^{-4}.$$

The test images are Kitkuan(256×256), Butterfly(256×256), HeadCT(256×256) and Brain(256×256), which show in Figure 3.



(A) Kitkuan (B) Butterfly (C) HeadCT (D) Brain

FIGURE 3. Test images

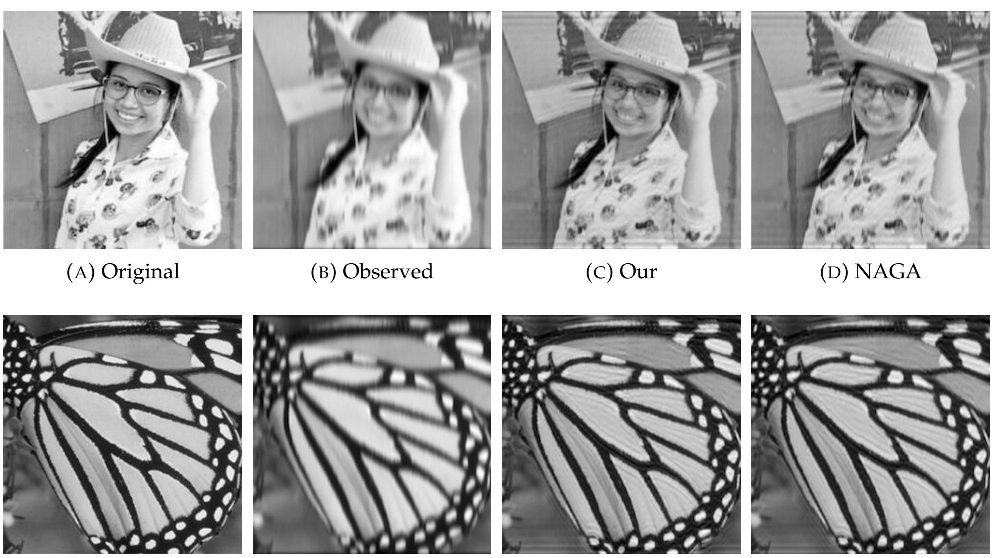
We show numerical results for case Gaussian blur in Table 1, restoration image results for case Gaussian blur in Figure 4, ISNR and relative error results for case Gaussian blur in Figure 5, numerical results for case Motion blur in Table 2, restoration image results for case Motion blur in Figure 6, ISNR results for case Motion blur in Figure 7

TABLE 1. Numerical results in case Gaussian blur

Figure	Our algorithm				NAGA			
	Iter.	Time	ISNR	Relative error	Iter	Time	ISNR	Relative error
Kitkuan	149	1.73	6.14	2.49×10^{-3}	362	3.79	4.70	3.48×10^{-3}
Butterfly	268	4.29	7.74	5.68×10^{-3}	733	10.66	6.82	7.03×10^{-3}
HeadCT	277	2.38	9.10	2.82×10^{-3}	893	7.68	7.54	4.05×10^{-3}
Brain	264	2.99	6.51	5.71×10^{-3}	753	7.90	5.71	7.08×10^{-3}

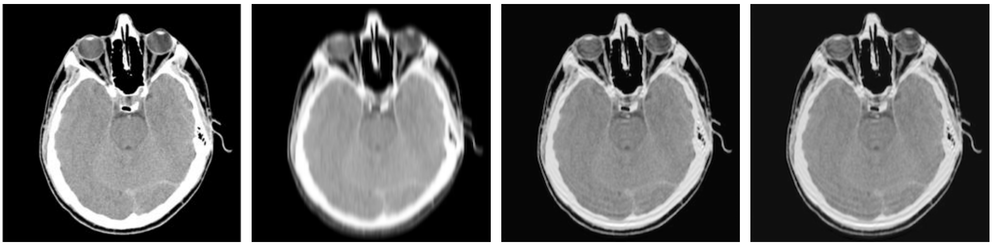
TABLE 2. Numerical results in case Motion blur

Figure	Our algorithm				NAGA			
	Iter.	Time	ISNR	Relative error	Iter	Time	ISNR	Relative error
Kitkuan	157	1.51	11.51	1.63×10^{-3}	606	5.48	10.04	2.09×10^{-3}
Butterfly	321	3.15	14.22	4.42×10^{-3}	1155	10.51	13.89	4.77×10^{-3}
HeadCT	277	2.52	18.99	7.29×10^{-4}	1088	23.65	17.69	9.85×10^{-4}
Brain	288	2.80	16.01	1.85×10^{-3}	1023	8.56	15.55	2.05×10^{-3}

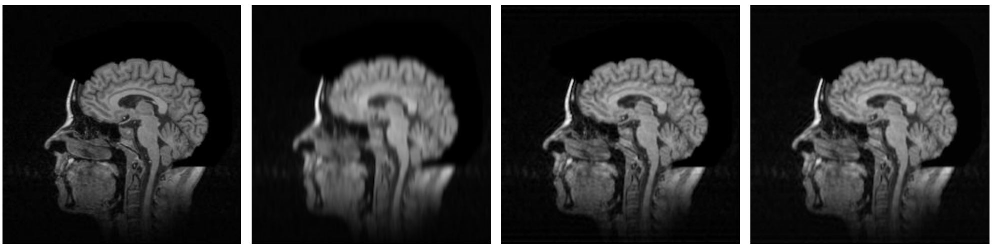


(A) Original (B) Observed (C) Our (D) NAGA

(E) Original (F) Observed (G) Our (H) NAGA

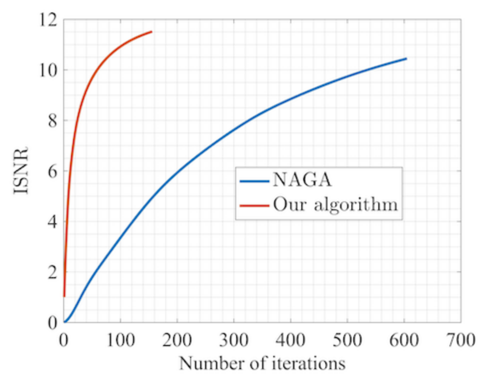


(A) Original (B) Observed (C) Our (D) NAGA

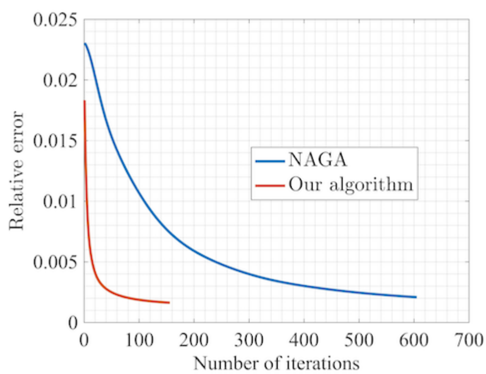


(E) Original (F) Observed (G) Our (H) NAGA

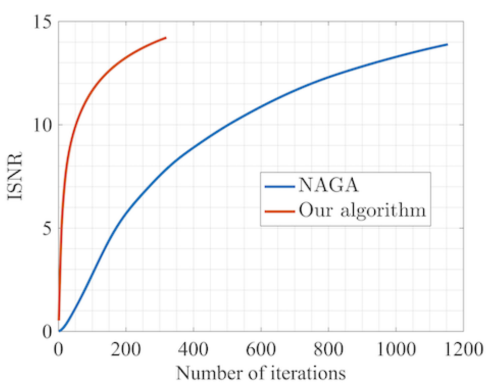
FIGURE 4. Restoration image results in case Gaussian blur



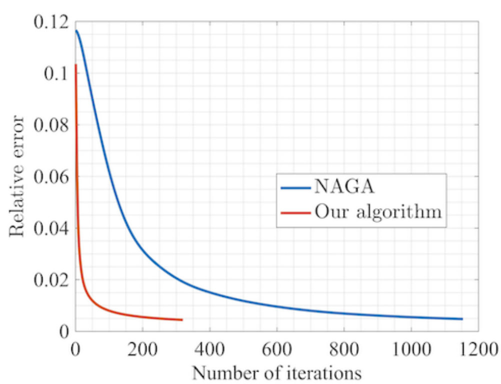
(A) Kitkuan ISNR results (Gaussian)



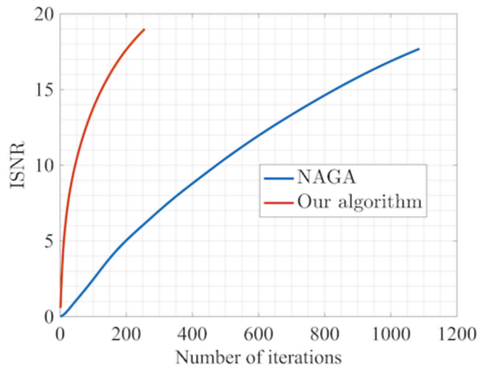
(B) Kitkuan relative error results (Gaussian)



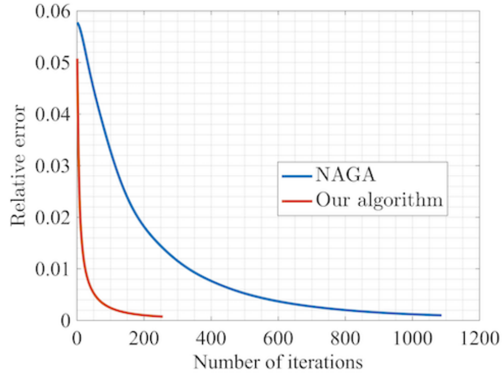
(C) Butterfly ISNR results (Gaussian)



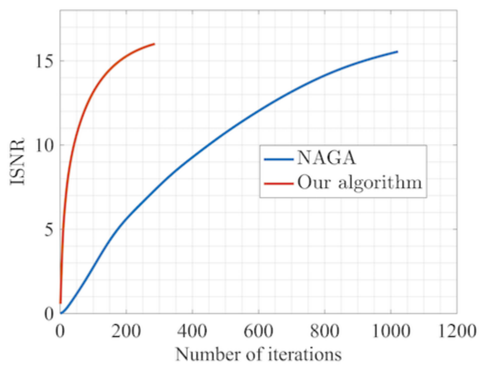
(D) Butterfly relative error results (Gaussian)



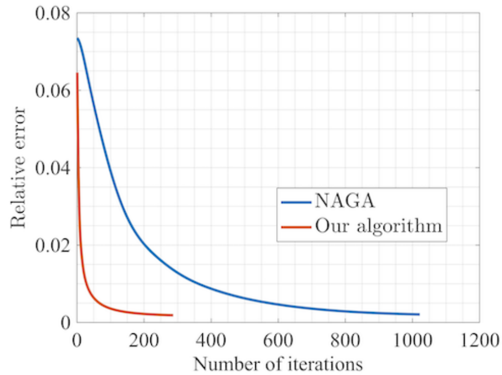
(A) HeadCT ISNR results (Gaussian)



(B) HeadCT relative error results (Gaussian)



(C) Brain ISNR results (Gaussian)



(D) Brain relative error results (Gaussian)

FIGURE 5. ISNR and relative error (Gaussian)


(A) Original


(B) Observed


(C) Our


(D) NAGA

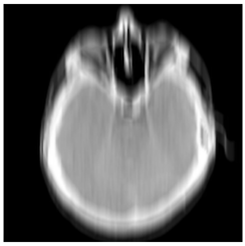

(E) Original


(F) Observed

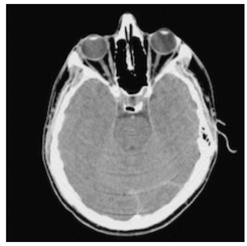

(G) Our


(H) NAGA

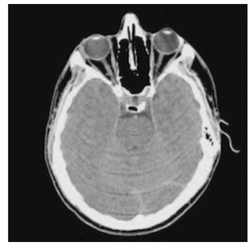
(A) Original



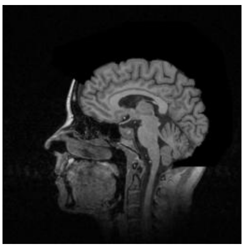
(B) Observed



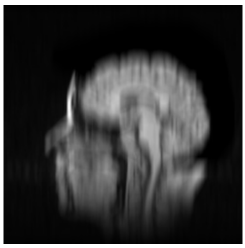
(C) Our



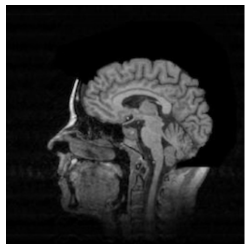
(D) NAGA



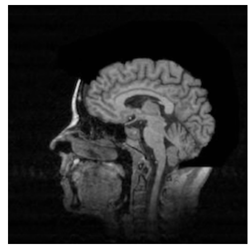
(E) Original



(F) Observed

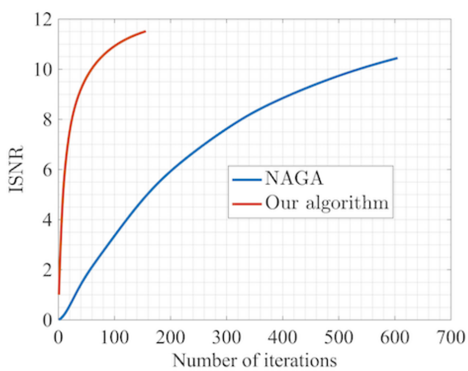


(G) Our

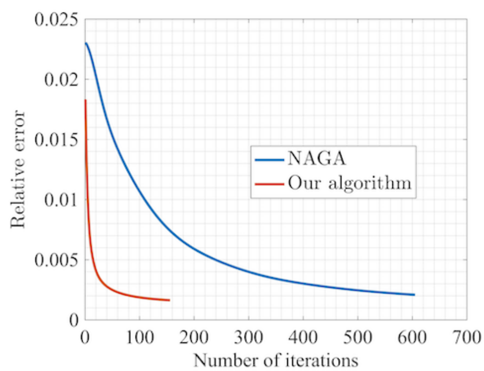


(H) NAGA

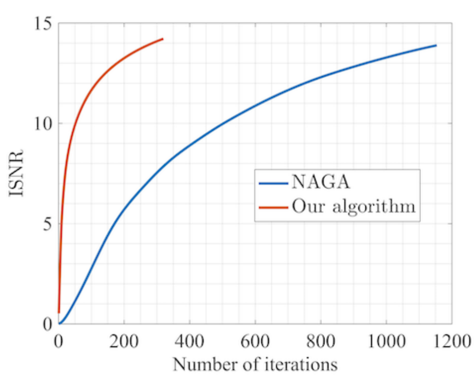
FIGURE 6. Restoration image result in case Motion blur



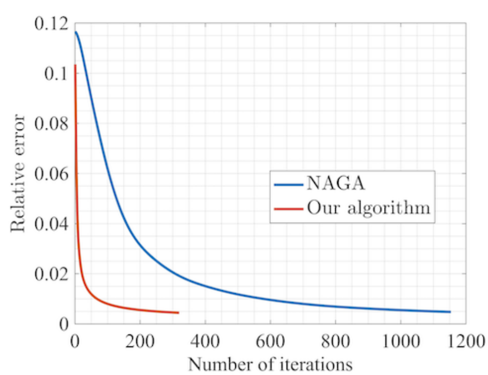
(A) Kitkuan ISNR results (Motion)



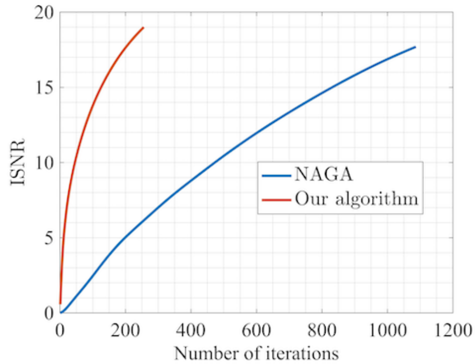
(B) Kitkuan relative error results (Motion)



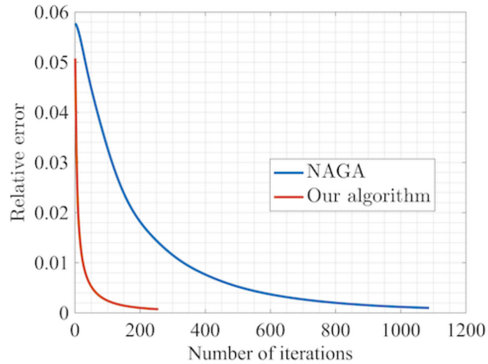
(C) Butterfly ISNR results (Motion)



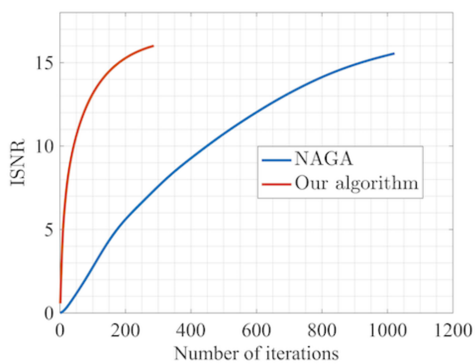
(D) Butterfly relative error results (Motion)



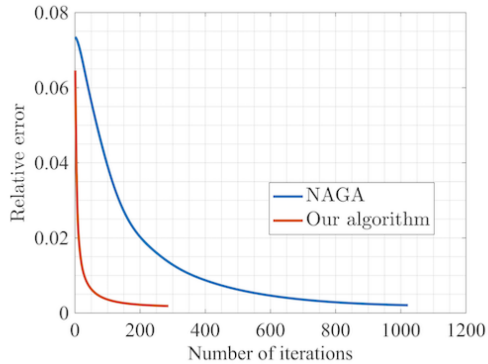
(A) HeadCT ISNR results (Motion)



(B) HeadCT relative error results (Motion)



(C) Brain ISNR results (Motion)



(D) Brain relative error results (Motion)

FIGURE 7. ISNR and relative error (Motion)

5. CONCLUSION

In this paper, we present the iterative methods using the ideas of the Ishikawa type and inertial technique to solving optimization problems and image restoration problems. A convergence theorem of our proposed methods, Theorem 3.1, is established and proved under some suitable conditions. We applied our main result to solving a minimization problem in the form of the sum of two proper lower semi-continuous and convex functions. As applications, we applied our algorithm (3.37), to solving image restoration problems. Moreover, we did some numerical experiments to illustrate the performance of the studied algorithms and show that ISNR of algorithm (3.37) is better than those of the NAGA [22].

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