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Convergence of Tseng-type self-adaptive algorithms for variational inequalities and fixed point problems

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ABSTRACT. In this paper, we present a Tseng-type self-adaptive algorithm for solving a variational inequality and a fixed point problem involving pseudomonotone and pseudocontractive operators in Hilbert spaces. A weak convergent result for such algorithm is proved under a weaker assumption than sequentially weakly continuous imposed on the pseudomonotone operator. Some corollaries are also included.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed and convex subset of H.

In this paper, our work is closely related to a classical variational inequality:

$$\text{find } x^\dagger \in C \text{ such that } \langle f(x^\dagger), x - x^\dagger\rangle \geq 0, \forall x \in C, \tag{1.1}$$

where $f : H \to H$ is a nonlinear operator. Here, use $Sol(f, C)$ to denote the solution set of (1.1). Throughout, assume that $Sol(f, C)$ is nonempty.

Variational inequalities are theoretically and algorithmically applied in various fields like particular cases convex optimization problems ([3, 4]), linear and monotone complementarity problems ([2]), equilibrium problems ([28]), fixed point problems ([27]), etc. For more information, please refer to [5, 11, 20, 21, 24].

A survey of algorithms for variational inequalities can be found in [12]. If $f(x) = \nabla F(x)$ for some convex function $F : C \to C$, variational inequality (1.1) is equivalent to $\min_C F(x)$. This fact indicates a natural extension of the projection gradient algorithm ([17, 18, 19, 22]) for the constrained optimization, i.e., an iterate with the form

$$u_{n+1} = proj_C[u_n - \tau_n f(u_n)] \tag{1.2}$$

where $\tau_n > 0$ is stepsize and $proj_C$ means the orthogonal projection from H onto C.

This algorithm (1.2) is convergent under quite strong assumptions, in which f must be strongly monotone and Lipschitz continuous. To avoid these difficulties, Korpelevich suggested in [16] an extragradient algorithm of the form

$$\begin{cases} v_n = proj_C[u_n - \tau_n f(u_n)], \\ u_{n+1} = proj_C[u_n - \tau_n f(v_n)]. \end{cases} \tag{1.3}$$

Extragradient algorithm (1.3) affords an available method for solving a classical monotone variational inequality. Consequently, extragradient algorithm (1.3) was applied by many scholars, who implemented it in a variety of forms; see, e.g., [7, 9, 13, 14, 15, 23]. Especially, Ceng, Teboulle and Yao [6] established the weak convergence of extragradient algorithm for solving the pseudomonotone variational inequality and fixed point problem under the

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additional hypothesis of the sequentially weak-to-strong continuity of f. However, this additional hypothesis is not satisfied even for the identity operator. Recently, Vuong [26] weaken this hypothesis to the sequentially weak-to-weak continuity of f.

At the same time, an inevitable drawback of extragradient algorithm is the need to calculate two projections onto the closed convex set C in each iteration. For solving this flaw, as a transformation of extragradient algorithm (1.3) is the following remarkable procedure introduced by Tseng [25]

(1.4)
$$\begin{cases} v_n = proj_C[u_n - \tau_n f(u_n)], \\ u_{n+1} = v_n + \tau_n[f(u_n) - f(v_n)]. \end{cases}$$

Here, a natural problem arises: could we extend Tseng's algorithm for solving some common problems related to variational inequalities under some weaker conditions imposed on f?

It is our main purpose in this paper that we further investigate iterative algorithm for solving pseudomonotone variational inequality and fixed point problem of pseudocontractive operators under the weaker assumption imposed on f. Our method bases on Tseng's algorithm and self-adaptive technique which is independent of the Lipschitz constant of f. We prove that the proposed algorithm weakly converges to a common solution of the pseudomonotone variational inequality and of the fixed point problem for the pseudocontractive operator g.

2. PRELIMINARIES

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{u_n\}$ be a sequence in H. $u_n \rightharpoonup z^\dagger$ denotes the weak convergence of u_n to $z^\dagger$. $\omega_w(u_n)$ denotes the set of all weak cluster points of $\{u_n\}$, i.e., $\omega_w(u_n) = \{u^\dagger : \exists\{u_{n_i}\} \subset \{u_n\}\ such\ that\ u_{n_i} \rightharpoonup u^\dagger (i \to \infty)\}$. Recall that an operator $f : H \to H$ is said to be

- monotone if
$$\langle f(x) - f(x^\dagger), x - x^\dagger \rangle \geq 0, \forall x, x^\dagger \in H.$$
- strongly monotone if there exists some constant $\gamma > 0$ such that
$$\langle f(x) - f(x^\dagger), x - x^\dagger \rangle \geq \gamma \|x - x^\dagger\|^2, \forall x, x^\dagger \in H.$$
- pseudomonotone if
$$\langle f(x^\dagger), x - x^\dagger \rangle \geq 0 \text{ implies that } \langle f(x), x - x^\dagger \rangle \geq 0, \forall x, x^\dagger \in H;$$
- L-Lipschitz continuous if there exists some constant $L > 0$ such that
$$\|f(x) - f(x^\dagger)\| \leq L\|x - x^\dagger\|, \text{ for all } x, x^\dagger \in H.$$
- sequently weakly continuous if $x_n \rightharpoonup \tilde{x}$ implies that $f(x_n) \rightharpoonup f(\tilde{x})$.

Recall that an operator $g : C \to C$ is said to be pseudocontractive if
$$\|g(x) - g(x^\dagger)\|^2 \leq \|x - x^\dagger\|^2 + \|(I - g)x - (I - g)x^\dagger\|^2$$
for all $x, x^\dagger \in C$.

Here, we use $Fix(g)$ to denote the fixed points set of g.

For fixed $x \in H$, there exists a unique $x^\dagger \in C$ satisfying $\|x - x^\dagger\| = \inf\{\|x - \tilde{x}\| : \tilde{x} \in C\}$. Denote $x^\dagger$ by $proj_C[x]$. The projection $proj_C$ has the following basic property: for given $x \in H$,

(2.5)
$$\langle x - proj_C[x], y - proj_C[x] \rangle \leq 0,\ \forall y \in C.$$

Applying this characteristic inequality, we have the following equivalence relation

(2.6)
$$x^\dagger \in Sol(f, C) \Leftrightarrow x^\dagger = proj_C[x^\dagger - \tau f(x^\dagger)], \forall \tau > 0.$$

In a Hilbert space H, we have

$$\|\alpha u + (1-\alpha)u^\dagger\|^2 = \alpha\|u\|^2 + (1-\alpha)\|u^\dagger\|^2 - \alpha(1-\alpha)\|u-u^\dagger\|^2, \tag{2.7}$$

$\forall u, u^\dagger \in H$ and $\forall \alpha \in [0,1]$.

Lemma 2.1 ([28]). *Let C be a nonempty, convex and closed subset of a Hilbert space H. Assume that $g : C \to C$ is an L-Lipschitz pseudocontractive operator. Then, for all $\tilde{u} \in C$ and $u^\dagger \in Fix(g)$, we have*

$$\|u^\dagger - g[(1-\mu)\tilde{u} + \mu g(\tilde{u})]\|^2 \leq \|\tilde{u} - u^\dagger\|^2 + (1-\mu)\|\tilde{u} - g[(1-\mu)\tilde{u} + \mu g(\tilde{u})]\|^2,$$

where $0 < \mu < \frac{1}{\sqrt{1+L^2}+1}$.

Lemma 2.2 ([27]). *Let C be a nonempty, convex and closed subset of a Hilbert space H. Let $g : C \to C$ be a continuous pseudocontractive operator. Then,*

(i) *$Fix(g) \subset C$ is closed and convex;*
(ii) *g is demi-closedness, i.e., $u_n \rightharpoonup \tilde{z}$ and $g(u_n) \to z^\dagger$ imply that $g(\tilde{z}) = z^\dagger$.*

Lemma 2.3 ([8]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : H \to H$ be a continuous and pseudomonotone operator. Then $x^\dagger \in Sol(f,C)$ iff $x^\dagger$ solves the following dual variational inequality*

$$\langle f(u^\dagger), u^\dagger - x^\dagger\rangle \geq 0, \ \forall u^\dagger \in C.$$

Lemma 2.4 ([1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\} \subset H$ be a sequence. If the following assumptions are satisfied*

(i) *$\forall \tilde{x} \in C$, $\lim_{n\to\infty} \|x_n - \tilde{x}\|$ exists;*
(ii) *$\omega_w(x_n) \subset C$,*

then $x_n \rightharpoonup u \in C$.

3. Main results

In this section, we first propose a Tseng-type algorithm for solving pseudomonotone variational inequality (1.1) and the fixed point problem for the pseudocontractive operator g by using a self-adaptive stepsize search. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f, g : H \to H$ be two nonlinear operators. Let $\{\gamma_n\}$ and $\{\mu_n\}$ be two sequences in $(0,1)$. Let $\alpha \in (0,1]$ and $\delta \in (0,1)$ be two constants.

Algorithm 3.1. Initialization: Take $u_0 \in C$ and $\tau_0 > 0$. Set $n = 0$.

Step 1. (Fixed point step) For known u_n, compute

$$v_n = (1-\gamma_n)u_n + \gamma_n g[(1-\mu_n)u_n + \mu_n g(u_n)]. \tag{3.8}$$

Step 2. (Tseng-type step) For known τ_n, compute

$$w_n = proj_C[v_n - \tau_n f(v_n)], \tag{3.9}$$

and

$$u_{n+1} = (1-\alpha)v_n + \alpha w_n + \alpha\tau_n[f(v_n) - f(w_n)]. \tag{3.10}$$

Step 3. (self-adaptive step) Compute

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \frac{\delta\|w_n - v_n\|}{\|f(w_n) - f(v_n)\|}\right\}, & \text{if } f(w_n) \neq f(v_n), \\ \tau_n, & \text{if } f(w_n) = f(v_n). \end{cases} \tag{3.11}$$

Step 4. Set $n := n+1$ and return to step 1.

Remark 3.1. If at some step $w_n = v_n = proj_C[v_n - \tau_n f(v_n)]$, by the equivalence relation (2.6), we deduce that $v_n \in Sol(f,C)$.

Remark 3.2. If choose $\alpha = 1$ in (3.10), then Step 2 can be rewritten as

$$\begin{cases} w_n = proj_C[v_n - \tau_n f(v_n)], \\ u_{n+1} = w_n + \tau_n[f(v_n) - f(w_n)], \end{cases}$$

which is exactly Tseng's method.

Remark 3.3 ([3]). By (3.11), we know that τ_n is monotonically decreasing. Moreover, by the κ-Lipschitz continuity of f, we deduce that $\frac{\delta\|w_n - v_n\|}{\|f(w_n)-f(v_n)\|} \geq \frac{\delta}{\kappa}$, which together with (3.11) implies that $\tau_n \geq \min\{\tau_0, \frac{\delta}{\kappa}\}$. Thus, the limit $\lim_{n\to\infty} \tau_n$ exists, denoted by $\tau^\dagger$. It is obviously that $\tau^\dagger > 0$ which ensures τ_n strictly greater than zero at each iterative step.

Remark 3.4. If $f(w_n) = f(v_n)$, then the next iterate u_{n+1} is independent of the stepsize τ_n. In this case, we can choose τ_{n+1} to be any number between $\tau^\dagger$ and τ_n.

In the sequel, we assume that the operator f satisfies the following property (F): For given a sequence $\{u_n\} \subset H$, if $u_n \rightharpoonup u \in H$ and $\liminf_{n\to\infty} \|f(u_n)\| = 0$, then $f(u) = 0$.

Remark 3.5. It is obviously that if f is sequentially weakly continuous, then f satisfies the above property (F).

Next, we prove the convergence of Algorithm 3.1.

Theorem 3.1. *Assume that f is a pseudomonotone and κ-Lipschitz continuous operator satisfying property (F). Assume that g is a pseudocontractive and L-Lipschitz continuous operator. Suppose that $\Gamma := Sol(f, C) \cap Fix(g) \neq \emptyset$ and $0 < \underline{\gamma} < \gamma_n < \overline{\gamma} < \mu_n < \overline{\mu} < \frac{1}{\sqrt{1+L^2}+1} (\forall n \geq 0)$. Then the sequence $\{u_n\}$ generated by Algorithm* (3.10) *converges weakly to some point in* Γ.

Proof. Let $p \in \Gamma$. By the property (2.5) of $proj_C$ and (3.9), we have

$$\langle w_n - v_n + \tau_n f(v_n), w_n - p\rangle \leq 0. \tag{3.12}$$

Since $p \in Sol(C, f)$, $\langle f(p), w_n - p\rangle \geq 0$. This together with the pseudomonotonicity of f implies that

$$\langle f(w_n), w_n - p\rangle \geq 0. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\langle w_n - v_n, w_n - p\rangle + \tau_n\langle f(v_n) - f(w_n), w_n - p\rangle \leq 0.$$

It follows that

$$\frac{1}{2}(\|w_n - v_n\|^2 + \|w_n - p\|^2 - \|v_n - p\|^2) + \tau_n\langle f(v_n) - f(w_n), w_n - p\rangle \leq 0,$$

which yields that

$$\|w_n - p\|^2 \leq \|v_n - p\|^2 - 2\tau_n\langle f(v_n) - f(w_n), w_n - p\rangle - \|w_n - v_n\|^2. \tag{3.14}$$

By (3.10), we have

$$\begin{aligned} \|u_{n+1} - p\|^2 &= \|(1-\alpha)(v_n - p) + \alpha(w_n - p) + \alpha\tau_n[f(v_n) - f(w_n)]\|^2 \\ &= \|(1-\alpha)(v_n - p) + \alpha(w_n - p)\|^2 + \alpha^2\tau_n^2\|f(v_n) - f(w_n)\|^2 \\ &\quad + 2\alpha(1-\alpha)\tau_n\langle v_n - p, f(v_n) - f(w_n)\rangle \\ &\quad + 2\alpha^2\tau_n\langle w_n - p, f(v_n) - f(w_n)\rangle. \end{aligned} \tag{3.15}$$

From (2.7) and (3.15), we derive

$$\begin{aligned}\|u_{n+1}-p\|^2 &= (1-\alpha)\|v_n-p\| + \alpha\|w_n-p\|^2 - \alpha(1-\alpha)\|v_n-w_n\|^2 \\ &\quad + \alpha^2\tau_n^2\|f(v_n)-f(w_n)\|^2 + 2\alpha^2\tau_n\langle w_n-p, f(v_n)-f(w_n)\rangle \\ &\quad + 2\alpha(1-\alpha)\tau_n\langle v_n-p, f(v_n)-f(w_n)\rangle.\end{aligned} \tag{3.16}$$

According to (3.14) and (3.16), we obtain

$$\begin{aligned}\|u_{n+1}-p\|^2 &\le \|v_n-p\| - \alpha(2-\alpha)\|v_n-w_n\|^2 + \alpha^2\tau_n^2\|f(v_n)-f(w_n)\|^2 \\ &\quad + 2\alpha(1-\alpha)\tau_n\langle v_n-w_n, f(v_n)-f(w_n)\rangle \\ &\le \|v_n-p\| - \alpha(2-\alpha)\|v_n-w_n\|^2 + \alpha^2\tau_n^2\|f(v_n)-f(w_n)\|^2 \\ &\quad + 2\alpha(1-\alpha)\tau_n\|v_n-w_n\|\|f(v_n)-f(w_n)\|.\end{aligned} \tag{3.17}$$

Thanks to (3.11), $\|f(w_n)-f(v_n)\| \le \frac{\delta\|w_n-v_n\|}{\tau_{n+1}}$. It follows from (3.17) that

$$\begin{aligned}\|u_{n+1}-p\|^2 &\le \|v_n-p\| - \alpha(2-\alpha)\|v_n-w_n\|^2 + \alpha^2\delta^2\frac{\tau_n^2}{\tau_{n+1}^2}\|w_n-v_n\|^2 \\ &\quad + 2\alpha(1-\alpha)\delta\frac{\tau_n}{\tau_{n+1}}\|v_n-w_n\|^2 \\ &= \|v_n-p\| - \alpha\left[2-\alpha-\alpha\delta^2\frac{\tau_n^2}{\tau_{n+1}^2} - 2(1-\alpha)\delta\frac{\tau_n}{\tau_{n+1}}\right]\|v_n-w_n\|^2.\end{aligned} \tag{3.18}$$

By Remark 3.3, we deduce

$$\lim_{n\to\infty}\left[2-\alpha-\alpha\delta^2\frac{\tau_n^2}{\tau_{n+1}^2} - 2(1-\alpha)\delta\frac{\tau_n}{\tau_{n+1}}\right] = 2-\alpha-\alpha\delta^2-2(1-\alpha)\delta > 0.$$

So, there exists $\theta > 0$ and N such that

$$2-\alpha-\alpha\delta^2\frac{\tau_n^2}{\tau_{n+1}^2} - 2(1-\alpha)\delta\frac{\tau_n}{\tau_{n+1}} \ge \theta$$

when $n \ge N$.

In combination with (3.18), we get

$$\|u_{n+1}-p\|^2 \le \|v_n-p\| - \alpha\theta\|v_n-w_n\|^2. \tag{3.19}$$

Set $t_n = (1-\mu_n)u_n + \mu_n g(u_n)$ for all $n \ge 0$. By (3.8) and (2.7), we obtain

$$\begin{aligned}\|v_n-p\|^2 &= \|(1-\gamma_n)(u_n-p) + \gamma_n[g(t_n)-p]\|^2 \\ &= (1-\gamma_n)\|u_n-p\|^2 + \gamma_n\|g(t_n)-p\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\|u_n-g(t_n)\|^2.\end{aligned} \tag{3.20}$$

Applying Lemma 2.1, we derive

$$\begin{aligned}\|g(t_n)-p\|^2 &= \|g[(1-\mu_n)u_n+\mu_n g(u_n)]-p\|^2 \\ &\le \|u_n-p\|^2 + (1-\mu_n)\|u_n-g(t_n)\|^2.\end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21), we obtain

$$\|v_n-p\|^2 \le \|u_n-p\|^2 + (\gamma_n-\mu_n)\gamma_n\|u_n-g(t_n)\|^2, \tag{3.22}$$

which results, together with (3.19), that

$$\|u_{n+1}-p\|^2 \le \|u_n-p\|^2 - (\mu_n-\gamma_n)\gamma_n\|u_n-g(t_n)\|^2 - \alpha\theta\|v_n-w_n\|^2, \tag{3.23}$$

which can be transformed into

$$(\mu_n-\gamma_n)\gamma_n\|u_n-g(t_n)\|^2 + \alpha\theta\|v_n-w_n\|^2 \le \|u_n-p\|^2 - \|u_{n+1}-p\|^2. \tag{3.24}$$

From inequalities (3.23) and (3.24), we can conclude the following conclusions:

(r1): The sequence $\{\|u_n - p\}$ is monotonically decreasing and hence $\lim_{n\to\infty} \|u_n - p\|$ exists. Thus, the sequence $\{u_n\}$ is bounded.

(r2): $\lim_{n\to\infty} \|u_n - g(t_n)\| = 0$ and so $\lim_{n\to\infty} \|v_n - u_n\| = \lim_{n\to\infty} \gamma_n \|u_n - g(t_n)\| = 0.$

(r3): $\lim_{n\to\infty} \|v_n - w_n\| = 0$ and thus $\lim_{n\to\infty} \|f(v_n) - f(w_n)\| = 0$ due to the Lipschitz continuity of f.

By the boundedness of the sequence $\{u_n\}$, we obtain the following results:

(r4): the sequence $\{v_n\}$ is bounded by (3.22) and $\gamma_n < \mu_n$.

(r5): the sequence $\{w_n\}$ is bounded because of $\|w_n\| \le \|v_n\| + \tau_n \|f(v_n)\|$ by (3.9).

Since f is κ-Lipschitz continuous, we have

$$\begin{aligned}\|u_n - g(u_n)\| &\le \|u_n - g(t_n)\| + \|g(t_n) - g(u_n)\| \\ &\le \|u_n - g(t_n)\| + \kappa\mu_n \|u_n - g(u_n)\|.\end{aligned}$$

It follows that

$$\|u_n - g(u_n)\| \le \frac{1}{1-\kappa\mu_n} \|u_n - g(t_n)\| \to 0,$$

and thus,

$$\lim_{n\to\infty} \|u_n - g(u_n)\| = 0. \tag{3.25}$$

By virtue of (3.10) and (r3), we have

$$\lim_{n\to\infty} \|u_{n+1} - v_n\| = 0. \tag{3.26}$$

Next, we show that $\omega_w(u_n) \subset \Gamma$. Pick up any $p^\dagger \in \omega_w(u_n)$. Then, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup p^\dagger$ as $i \to \infty$. Consequently, $v_{n_i} \rightharpoonup p^\dagger$ and $w_{n_i} \rightharpoonup p^\dagger$ based on (r2) and (r3), respectively.

On account of (3.25) and Lemma 2.2, we acquire that $p^\dagger \in Fix(g)$. Now, we only need to prove that $p^\dagger \in Sol(f, C)$. In view of (2.5) and $w_{n_i} = proj_C[v_{n_i} - \tau_{n_i} f(v_{n_i})]$, we achieve

$$\langle w_{n_i} - v_{n_i} + \tau_{n_i} f(v_{n_i}), w_{n_i} - u\rangle \le 0, \forall u \in C.$$

It follows that

$$\frac{1}{\tau_{n_i}} \langle v_{n_i} - w_{n_i}, u - w_{n_i}\rangle + \langle f(v_{n_i}), w_{n_i} - v_{n_i}\rangle \le \langle f(v_{n_i}), u - v_{n_i}\rangle,\ \forall u \in C. \tag{3.27}$$

Noting that from (r3), we have $\lim_{i\to\infty} \|v_{n_i} - w_{n_i}\| = 0$. Then, by (3.27), we deduce

$$\liminf_{i\to\infty} \langle f(v_{n_i}), u - v_{n_i}\rangle \ge 0. \tag{3.28}$$

Next, we consider two possible cases.

Case 1. $\liminf_{i\to\infty} \|f(v_{n_i})\| = 0$. By $v_{n_i} \rightharpoonup p^\dagger$ and f satisfying property (F), we deduce that $f(p^\dagger) = 0$. Consequently, $p^\dagger \in Sol(f, C)$.

Case 2. $\liminf_{i\to\infty} \|f(v_{n_i})\| > 0$. In terms of (3.28), we obtain

$$\liminf_{i\to\infty} \langle (f(v_{n_i}))^0, u - v_{n_i}\rangle \ge 0, \tag{3.29}$$

where $(f(v_{n_i}))^0$ means the unit vector of $f(v_{n_i})$, that is, $(f(v_{n_i}))^0 = \frac{f(v_{n_i})}{\|f(v_{n_i})\|}$ (note that for each $i \ge 0$, $f(v_{n_i}) \ne 0$, otherwise, $v_{n_i} \in Sol(f, C)$ and $p^\dagger \in Sol(f, C)$).

Thanks to (3.29), we can choose a positive real numbers sequence $\{\epsilon_i\}$ satisfying $\epsilon_i \to 0$ as $i \to \infty$. For each ϵ_i, there exists the smallest positive integer N_i such that

$$\langle (f(v_{n_i}))^0, u - v_{n_i}\rangle + \epsilon_i \ge 0,\ \forall i \ge N_i.$$

It follows that

$$\langle f(v_{n_i}), u - v_{n_i}\rangle + \epsilon_i \|f(v_{n_i})\| \geq 0,\ \forall i \geq N_i. \tag{3.30}$$

Set $\hat{v}_{n_i} = \frac{f(v_{n_i})}{\|f(v_{n_i})\|^2}$. Thus, we have $\langle f(v_{n_i}), \hat{v}_{n_i}\rangle = 1$ for each i. From (3.30), we deduce

$$\langle f(v_{n_i}), u + \epsilon_i \|f(v_{n_i})\|\hat{v}_{n_i} - v_{n_i}\rangle \geq 0,\ \forall i \geq N_i. \tag{3.31}$$

Since f is pseudomonotone, it follows from (3.31) that

$$\langle f(u + \epsilon_i \|f(v_{n_i})\|\hat{v}_{n_i}), u + \epsilon_i \|f(v_{n_i})\|\hat{v}_{n_i} - v_{n_i}\rangle \geq 0,\ \forall i \geq N_i. \tag{3.32}$$

Note that $\lim_{i\to\infty} \|\epsilon_i \|f(v_{n_i})\|\hat{v}_{n_i}\| = \lim_{i\to\infty} \epsilon_i = 0$. Thus, taking the limit as $i \to \infty$ in (3.32), we obtain

$$\langle f(u), u - p^\dagger\rangle \geq 0. \tag{3.33}$$

Applying Lemma 2.1 to (3.33), we conclude that $p^\dagger \in Sol(f, C)$.

Finally, we show that the entire sequence $\{u_n\}$ converges weakly to $p^\dagger$. As a matter of fact, we have the following facts in hand:

(i) $\forall p \in \Gamma$, $\lim_{n\to\infty} \|u_n - p\|$ exists;
(ii) $w_\omega(u_n) \subset \Gamma$;
(iii) $p^\dagger \in w_\omega(u_n)$.

Thus, by Lemma 2.4, we deduce that the sequence $\{u_n\}$ weakly converges to $p^\dagger \in \Gamma$. This completes the proof. □

Remark 3.6. It is obviously that monotonicity implies pseudo-monotonicity. Hence, our theorem holds when the involved operator f is monotone.

Based on Algorithm 3.1 and Theorem 3.1, we can obtain the following algorithms and the corresponding corollaries.

Algorithm 3.2. Initialization: Take $u_0 \in C$ and $\tau_0 > 0$. Set $n = 0$.

Step 1. For known u_n and τ_n, compute

$$w_n = proj_C[u_n - \tau_n f(u_n)],$$

and

$$u_{n+1} = (1-\alpha)u_n + \alpha w_n + \alpha\tau_n[f(u_n) - f(w_n)].$$

Step 2. Compute

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \frac{\delta\|w_n - u_n\|}{\|f(w_n) - f(u_n)\|}\right\}, & \text{if } f(w_n) \neq f(u_n), \\ \tau_n, & \text{else.} \end{cases}$$

Step 3. Set $n := n+1$ and return to step 1.

Corollary 3.1. *Assume that f is a pseudomonotone and κ-Lipschitz continuous operator satisfying property (F). Suppose that $Sol(f, C) \neq \emptyset$. Then the sequence $\{u_n\}$ generated by Algorithm 3.2 converges weakly to some point in $Sol(f, C)$.*

Algorithm 3.3. Initialization: Take $u_0 \in C$ and $\tau_0 > 0$. Set $n = 0$.

Step 1. For known u_n, compute

$$u_{n+1} = (1-\gamma_n)u_n + \gamma_n g[(1-\mu_n)u_n + \mu_n g(u_n)].$$

Step 2. Set $n := n+1$ and return to step 1.

Corollary 3.2. *Assume that g is a pseudocontractive and L-Lipschitz continuous operator. Suppose that $Fix(g) \neq \emptyset$ and $0 < \underline{\gamma} < \gamma_n < \overline{\gamma} < \mu_n < \overline{\mu} < \frac{1}{\sqrt{1+L^2}+1} (\forall n \geq 0)$. Then the sequence $\{u_n\}$ generated by Algorithm 3.3 converges weakly to some point in $Fix(g)$.*

4. APPLICATION TO COMPUTING DYNAMIC USER EQUILIBRIA

In this section, we apply Algorithm 3.2 to compute dynamic user equilibria ([10]).

Let $\mathcal{P}$ be set of paths in the network. $\mathcal{W}$ be set of O-D pairs in the network, Q_{ij} be fxed O-D demand between $(i,j) \in \mathcal{W}$, $\mathcal{P}_{ij}$ be subset of paths that connect O-D pair (i,j), t be continuous time parameter in a fxed time horizon $[t_0, t_1]$, $h_p(t)$ be departure rate along path p at time t, $h(t)$ be complete vector of departure rates $h(t) = (h_p(t) : p \in \mathcal{P})$, $\Psi_p(t,h)$ be travel cost along path p with departure time t, under departure profile h, $v_{ij}(h)$ be minimum travel cost between O-D pair (i,j) for all paths and departure times.

Assume that $h_p(\cdot) \in L^2_+[t_0,t_1]$ and $h(\cdot) \in (L^2_+[t_0,t_1])^{|\mathcal{P}|}$. Define the effective delay operator $\Psi : (L^2_+[t_0,t_1])^{|\mathcal{P}|} \to (L^2_+[t_0,t_1])^{|\mathcal{P}|}$ as follows:

$$h(\cdot) = \{h_p(\cdot), p \in \mathcal{P}\} \mapsto \Psi(h) = \{\Psi_p(\cdot, h), p \in \mathcal{P}\}$$

The travel demand satisfaction constraint satisfies

$$Q_{ij} = \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_1} h_p(t)dt, \forall (i,j) \in \mathcal{W}.$$

Then, the set of feasible path departure vector can be expressed as

$$\Lambda = \{h \geq 0 : \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_1} h_p(t)dt, \forall (i,j) \in \mathcal{W}\} \subset (L^2[t_0,t_1])^{|\mathcal{P}|}.$$

Recall that a vector of departures $h^* \in \Lambda$ is a dynamic user equilibrium with simultaneous route and departure time choice if

$$h^*_p(t) > 0, p \in \mathcal{P}_{ij} \Rightarrow \Psi_p(t,h^*) = v_{ij}(h^*), \text{ for almost every} t \in [t_0,t_1]. \tag{4.34}$$

Note that (4.34) is equivalent to the following variational inequality ([10])

$$\langle \Psi(h^*), h - h^* \rangle \geq 0, \forall h \in \Lambda. \tag{4.35}$$

Based on Algorithm 3.2, we have the following algorithm.

Algorithm 4.1. Initial path flow $u_0 \in (L^2[t_0,t_1])^{|\mathcal{P}|}$ and $\tau_0 > 0$. Set $n = 0$.

Step 1. For known u_n and τ_n, compute the effective path delays $\Psi_p(t,u_n)$ and

$$w_n = proj_\Lambda[u_n - \tau_n \Psi(u_n)].$$

Step 2. Compute the effective path delays $\Psi_p(t,w_n)$ and

$$u_{n+1} = (1-\alpha)u_n + \alpha w_n + \alpha \tau_n[\Psi(u_n) - \Psi(w_n)].$$

Step 3. Compute

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \frac{\delta\|w_n - u_n\|}{\|\Psi(w_n)-\Psi(u_n)\|}\right\}, & \text{if } \Psi(w_n) \neq \Psi(u_n), \\ \tau_n, & \text{else.} \end{cases}$$

Step 4. Set $n := n+1$ and return to step 1.

If the delay operator Ψ is Lipschitz continuous and pseudomonotone, then we can apply Algorithm 4.1 to compute dynamic user equilibria. It should be pointed out that Algorithm 4.1 requires two evaluations of the delay operator Ψ. It is clear that this procedure is the most costly step in the implementation of Algorithm 4.1.

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REFERENCES

[1] Abbas, B., Attouch, H.; Svaiter, B. F. Newton-like dynamics and forward-backward methods for structured monotone inclusions in Hilbert spaces. *J. Optim. Theory Appl.* **161** (2014), 331–360.

[2] Bauschke, H. H.; Combettes, P. L., *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Springer, Berlin 2011.

[3] Bot, R. I.; Csetnek, E. R.; Vuong, P. T. The forward-backward-forward method from continuous and discrete perspective for pseudo-monotone variational inequalities in Hilbert spaces. *Eur. J. Oper. Res.* **287** (2020), 49–60.

[4] Cai, X.; Gu, G.; He, B. On the $O(1/t)$ convergence rate of the projection and contraction methods for variational inequalities with Lipschitz continuous monotone operators. *Comput. Optim. Appl.* **57** (2014), 339–363.

[5] Ceng, L. C.; Petruşel, A.; Qin, X.; Yao, J. C. A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems. *Fixed Point Theory* **21** (2020), 93–108.

[6] Ceng, L. C.; Teboulle, M.; Yao, J. C. Weak convergence of an iterative method for pseudomonotone variational inequalities and fixed-point problems. *J. Optim. Theory Appl.* **146** (2010), 19–31.

[7] Censor, Y.; Gibali, A.; Reich, S. Extensions of Korpelevich's extragradient method for solving the variational inequality problem in Euclidean space. *Optim.* **61** (2012), 1119–1132.

[8] Cottle, R. W.; Yao, J. C. Pseudomonotone complementarity problems in Hilbert space. *J. Optim. Theory Appl.* **75** (1992), 281–295.

[9] Dong, Q. L.; Lu, Y. Y.; Yang, J. The extragradient algorithm with inertial effects for solving the variational inequality. *Optim.* **65** (2016), 2217–2226.

[10] Friesz, T. L.; Han, K. The mathematical foundations of dynamic user equilibrium. *Transportation Research Part B: Methodological* **126** (2019), 309–328.

[11] Gibali, A.; Reich, S.; Zalas, R. Iterative methods for solving variational inequalities in Euclidean space. *J. Fixed Point Theory Appl.* **17** (2015), 775–811.

[12] Harker, P. T.; Pang, J. S. Finite dimensional variational inequalities and nonlinear complementarity problems: a survey of theory, algorithms and applications. *Math. Programming* **48** (1990), 161–220.

[13] Hieu, D. V.; Anh, P. K.; Muu, L. D. Modified extragradient-like algorithms with new stepsizes for variational inequalities. *Comput. Optim. Appl.* **73** (2019), 913–932

[14] Iusem, A. N.; Svaiter, B. F. A variant of Korpelevich's method for variational inequalities with a new search strategy. *Optim.* **42** (1997), 309–321.

[15] Khanh, P. D. A modified extragradient method for infinite-dimensional variational inequalities. *Acta. Math. Vietnam.* **41** (2016), 251–263.

[16] Korpelevich, G. M. The extragradient method for finding saddle points and other problems. *Metody* **12** (1976), 747–756.

[17] Mainge, P. E. Numerical approach to monotone variational inequalities by a one-step projected reflected gradient method with line-search procedure. *Comput. Math. Appl.* **3** (2016), 720–728.

[18] Mainge, P. E.; Gobinddass, M. L. Convergence of one-step projected gradient methods for variational inequalities. *J. Optim. Theory Appl.* **171** (2016), 146–168.

[19] Malitsky, Y. V. Projected reflected gradient method for variational inequalities. *SIAM J. Optim.* **25** (2015), 502–520.

[20] Shehu, Y.; Gibali, A.; Sagratella, S. Inertial projection-type methods for solving quasi-variational inequalities in real Hilbert spaces. *J. Optim. Theory Appl.* **184** (2020), 877–894.

[21] Shehu, Y.; Li, X. H.; Dong, Q. L. An efficient projection-type method for monotone variational inequalities in Hilbert spaces. *Numer. Algor.* **84** (2020), 365–388.

[22] Solodov, M. V.; Svaiter, B. F. A new projection method for variational inequality problems. *SIAM J. Control Optim.* **37** (1999), 765–776.

[23] Thong, D. V.; Hieu, D. V. New extragradient methods for solving variational inequality problems and fixed point problems. *J. Fixed Point Theory Appl.* **20** (2018), Art. ID 129.

[24] Thong, D. V.; Vinh, N. T.; Cho, Y. J. A strong convergence theorem for Tsengs extragradient method for solving variational inequality problems. *Optim. Lett.* **14** (2020), 1157–1175.

[25] Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control Optim.* **38** (2000), 431–446.

[26] Vuong, P. T. On the weak convergence of the extragradient method for solving pseudomonotone variational inequalities. *J. Optim. Theory Appl.* **176** (2018), 399–409.

[27] Zhou, H. Strong convergence of an explicit iterative algorithm for continuous pseudocontractions in Banach spaces. *Nonlinear Anal.* **70** (2009), 4039–4046.

[28] Zhu, L. J.; Yao, Y.; Postolache, M. Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems. *U.P.B. Sci. Bull., Series A*, in press.

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