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Dedicated to the memory of Academician Mitrofan M. Choban (1942-2021)

Generalized Bernstein Kantorovich operators: Voronovskaya type results, convergence in variation

ANA MARIA ACU[1], ALI ARAL[2] and IOAN RAŞA[3]

ABSTRACT. This paper includes Voronovskaya type results and convergence in variation for the exponential Bernstein Kantorovich operators. The Voronovskaya type result is accompanied by a relation between the mentioned operators and suitable auxiliary discrete operators. Convergence of the operators with respect to the variation seminorm is obtained in the space of functions with bounded variation. We propose a general framework covering the results provided by previous literature.

1. INTRODUCTION

In Approximation Theory one is often interested in the convergence of a sequence of operators. This can be seen by taking a direct limit as well as by obtaining quantitative theorems that measure the degree of convergence. Other outstanding tools to achieve the measurement of effectiveness of the approximation are Voronovskaya type theorems and variation diminishing properties of the operators. In this paper, we focus on these properties of the operators given in [6]. But firstly we recall the sequence G_n of linear positive operators introduced and studied in [5], that are called exponential Bernstein operators; under suitable conditions they perform better than the classical Bernstein operators. These operators fix the exponential functions $\exp(\mu t)$ and $\exp(2\mu t)$, $\mu > 0$, and are defined by

$$G_n f(x) = G_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) e^{-\mu k/n} e^{\mu x} p_{n,k}(a_n(x)), \quad x \in [0,1], \ n \in \mathbb{N}, \tag{1.1}$$

where

$$a_n(x) = \frac{e^{\mu x/n} - 1}{e^{\mu/n} - 1}. \tag{1.2}$$

They have close connection with the Bernstein operators that is given by

$$G_n f(x) = \exp_\mu(x) B_n\left(\frac{f}{\exp_\mu}; a_n(x)\right), \tag{1.3}$$

where for a fixed real parameter $\mu > 0$, the exponential function is defined as $\exp_\mu(x) = e^{\mu x}$. We denote respectively by $\exp$ and $\log$ the natural exponential and logarithmic functions, although we are also writing e^t for the value of $\exp(t)$. As usual, we denote by e_i the polynomial functions defined by $e_i(t) = t^i$.

It is known that an integral version of the above operators can be introduced by replacing in G_n the sample values by the mean values

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Corresponding author: Ana Maria Acu; anamaria.acu@ulbsibiu.ro

$(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt$, for any locally integrable function f. This way we get the operators

$$\overline{K}_n f(x) := (n+1)e^{\mu x}\sum_{k=0}^{n} e^{-\mu k/n} p_{n,k}(a_n(x)) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt.$$

The main advantage that can be achieved by using the operators $\overline{K}_n$ is that also not necessarily continuous functions can be approximated. But, here our construction is different from the classical one. In order to furnish an approximation process for the space of integrable functions on the interval $[0,1]$, integral modification of the operators G_n are defined in [6] by setting for $n \geq 1$, $\mu \geq 0$ and $x \in [0,1]$

$$\widetilde{K}_n = D_\mu \circ G_{n+1} \circ I_\mu, \tag{1.4}$$

where the operators $D_\mu : C^1[0,1] \to C[0,1]$ and $I_\mu : C[0,1] \to C^1[0,1]$, are defined by

$$I_\mu(f,x) = e^{\mu x}\int_0^x e^{-\mu t} f(t)\,dt, \quad f \in C[0,1] \text{ and } x \in [0,1], \tag{1.5}$$

$$D_\mu(f,x) = f'(x) - \mu f(x), \quad f \in C^1[0,1] \text{ and } x \in [0,1]. \tag{1.6}$$

Using the integral operators in (1.4) which are different from the integral versions of G_n mentioned above, we have

$$\widetilde{K}_n f(x) = a_{n+1}'(x)(n+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_\mu(t)\,dt, \tag{1.7}$$

where $a_{n+1}(x)$ is given in (1.2) and $f_\mu(t) = e^{-\mu t} f(t)$. To represent the operators in (1.7), we sometimes use the notation $\widetilde{K}_n(f;x)$.

In more recent years, exponential type operators using the above idea have been object of investigation by several mathematicians. For example, in the continuation of mentioned papers, results on quantitative uniform and pointwise estimates for exponential Szász operators were obtained in [9] and [10]. Other contributions can be found in [16], [17] and recently [14]. Also, the variation detracting property has been intensively studied. Although this topic was first studied by Lorentz [19], important contributions were given in [12]. Similar contributions can be found in [18] and [11].

The present note is motivated by the papers [5] and [6] where the operators G_n and $\widetilde{K}_n$ are investigated. We deal with Voronovskaya type results and convergence in variation for the exponential Bernstein Kantorovich operators. Section 2 is devoted to some results regarding the discrete operators $\widetilde{G}_n$ and $\overline{G}_n$. We establish estimates for the difference between $\widetilde{K}_n$ and $\widetilde{G}_n$, $\overline{K}_n$ and $\overline{G}_n$, as well as the Voronovskaya type formula for the operators $\widetilde{G}_n$. In Section 3 we give and application of Voronovskaya type result for $\widetilde{K}_n$. Section 4 is devoted to the convergence in variation of the operators considered in the previous sections. The last section contains some conclusions and perspectives.

We will need the following inequality involving the classical modulus of continuity: if $\delta > 0$, $f \in C[a,b]$ and $x, y \in [a,b]$, then

$$|f(x) - f(y)| \leq \left(1 + \min\left\{\frac{|x-y|}{\delta}, \frac{(x-y)^2}{\delta^2}\right\}\right)\omega(f,\delta). \tag{1.8}$$

This is a slight extension of [13, Proposition 1.22 (v)].

2. DIFFERENCES OF OPERATORS

In order to present our results, we will focus our attention on a sequence $\widetilde{G}_n$ of linear positive auxiliary operators

$$\widetilde{G}_n\left(f;x\right)=a_{n+1}'\left(x\right)e^{\mu x}\sum_{k=0}^{n}p_{n,k}\left(a_{n+1}\left(x\right)\right)f_\mu\left(\frac{2k+1}{2(n+1)}\right) \tag{2.9}$$

for $f\in C\left[0,1\right]$. In fact, $\widetilde{G}_n$ is a discrete operator associated with $\widetilde{K}_n$: one replaces the functional $f_\mu\to(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f_\mu\left(t\right)dt$ by the point evaluation at $\frac{2k+1}{2(n+1)}$ (see also [4], [20]). We consider also the discrete operators

$$\overline{G}_nf(x):=e^{\mu x}\sum_{k=0}^{n}e^{-\mu k/n}p_{n,k}(a_n(x))f\left(\frac{2k+1}{2(n+1)}\right),$$

associated with $\overline{K}_nf(x)$.

The following exponential moments and limits will be used in order to obtain the Voronovskaya type result for $\widetilde{G}_n$. They can be derived by direct calculations, or with the use of some mathematical software like Maple.

Lemma 2.1. *For the operator $(\widetilde{G}_n)_{n\geq1}$ we have*

1. $\widetilde{G}_n\left(e_0;x\right)=a_{n+1}'\left(x\right)e^{\mu x-\mu}e^{\frac{\mu}{2(n+1)}}\left(1-e^{\frac{\mu x}{n+1}}+e^{\frac{\mu}{n+1}}\right)^n,$
2. $\widetilde{G}_n\left(\exp_\mu;x\right)=a_{n+1}'\left(x\right)e^{\mu x},$
3. $\widetilde{G}_n\left(\exp_\mu^2;x\right)=a_{n+1}'\left(x\right)e^{\frac{\mu(2nx+1)}{2(n+1)}x}e^{\mu x},$
4. $\widetilde{G}_n\left(\exp_\mu^3;x\right)=a_{n+1}'\left(x\right)e^{\mu x}e^{\frac{\mu}{n+1}}\left(e^{\frac{\mu(x+1)}{n+1}}+e^{\frac{\mu x}{n+1}}-e^{\frac{\mu}{n+1}}\right)^n,$
5. $\widetilde{G}_n\left(\exp_\mu^4;x\right)=a_{n+1}'\left(x\right)e^{\mu x}e^{\frac{3\mu}{2(n+1)}}$
$\cdot\left(e^{\frac{\mu x}{n+1}}\left(1+e^{\frac{\mu}{n+1}}+e^{\frac{2\mu}{n+1}}\right)-e^{\frac{\mu}{n+1}}\left(1+e^{\frac{\mu}{n+1}}\right)\right)^n.$

Theorem 2.1. *If $f\in C[0,1]$, then $\tilde{G}_nf$ converges to f uniformly on* $[0,1]$.

Proof. Since $\left\{e_0,exp_\mu,exp_\mu^2\right\}$ is an extended complete Chebyshev system, from Korovkin's theorem and Lemma 2.1 the theorem is proved. □

Concerning the images of e_0 under $\widetilde{G}_n$, for $x\in[0,1]$, using Lemma 2.1 one finds that

$$\lim_{n\to\infty}n\left(\widetilde{G}_n\left(e_0;x\right)-1\right)=-\mu(1-(2+\mu)x+\mu x^2), \tag{2.10}$$

$$\lim_{n\to\infty}n\left(\widetilde{G}_n\left(\exp_\mu;x\right)-e^{\mu x}\widetilde{G}_n\left(e_0;x\right)\right)=\frac{1}{2}e^{\mu x}\mu(2\mu x^2-2\mu x-2x+1), \tag{2.11}$$

and

$$\lim_{n\to\infty}n\left(\widetilde{G}_n\left(\exp_\mu^2;x\right)-2e^{\mu x}\widetilde{G}_n\left(\exp_\mu;x\right)+e^{2\mu x}\widetilde{G}_n\left(e_0;x\right)\right)=-e^{2\mu x}\mu^2x(x-1). \tag{2.12}$$

Investigating the difference of positive linear operators is an active area of research: see [1, 2, 3, 4, 7, 8]. The following lemma provides an estimate of the difference between $\widetilde{K}_n$ and $\widetilde{G}_n$.

Proposition 2.1. *For any $n \in \mathbb{N}$, $x \in [0,1]$ and $f \in C[0,1]$, the following inequality holds:*

$$\left|\widetilde{K}_n f(x) - \widetilde{G}_n f(x)\right| \leq a_{n+1}^{'}(x) e^{\mu x}\left(1+\frac{1}{\mu^2}\left(1+\left(e^{\frac{\mu x}{n+1}}-1\right)\left(e^{\frac{\mu}{n+1}}+1\right)\right)^n\right)\omega(f_\mu;h), \tag{2.13}$$

where

$$h=\left[e^{\frac{\mu}{n+1}}-\frac{n+1}{2\mu}\left(1-e^{\frac{\mu}{n+1}}\right)\left(1+e^{\frac{\mu}{n+1}}-4e^{\frac{\mu}{2(n+1)}}\right)\right]^{1/2}.$$

Proof. Formulas (2.9) and (1.7) imply that

$$\begin{aligned}
&\widetilde{K}_n f(x) - \widetilde{G}_n f(x)\\
&= a_{n+1}^{'}(x)(n+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left[f_\mu(t)-f_\mu\left(\frac{2k+1}{2(n+1)}\right)\right]dt.
\end{aligned}$$

Using (1.8) and then the mean value theorem, we obtain with $h>0$

$$\begin{aligned}
\left|f_\mu(t)-f_\mu\left(\frac{2k+1}{2(n+1)}\right)\right| &\leq \left(1+\frac{1}{h^2}\left(t-\frac{2k+1}{2(n+1)}\right)^2\right)\omega(f_\mu;h)\\
&\leq \left(1+\frac{1}{\mu^2h^2}\left(e^{\mu t}-e^{\frac{\mu(2k+1)}{2(n+1)}}\right)^2\right)\omega(f_\mu;h).
\end{aligned} \tag{2.14}$$

Then by using the above inequality, we have

$$\begin{aligned}
&\left|\widetilde{K}_n f(x) - \widetilde{G}_n f(x)\right| \leq a_{n+1}^{'}(x) e^{\mu x}\\
&\times\left(1+\frac{(n+1)}{\mu^2h^2}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left(e^{\mu t}-e^{\frac{\mu(2k+1)}{2(n+1)}}\right)^2dt\right)\omega(f_\mu;h).
\end{aligned}$$

Simple computations show that

$$\begin{aligned}
&(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left(e^{\mu t}-e^{\frac{\mu(2k+1)}{2(n+1)}}\right)^2dt\\
&= e^{\frac{2k\mu}{n+1}}\left[e^{\frac{\mu}{n+1}}-\frac{n+1}{2\mu}\left(1-e^{\frac{\mu}{n+1}}\right)\left(1+e^{\frac{\mu}{n+1}}-4e^{\frac{\mu}{2(n+1)}}\right)\right]
\end{aligned}$$

and

$$\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))e^{\frac{2\mu k}{n+1}}=\left(1+\left(e^{\frac{\mu x}{n+1}}-1\right)\left(e^{\frac{\mu}{n+1}}+1\right)\right)^n.$$

Taking into account the above relations, we get

$$\begin{aligned}
&\left|\widetilde{K}_n f(x) - \widetilde{G}_n f(x)\right|\\
\leq\ & a_{n+1}^{'}(x)e^{\mu x}\left(1+\frac{1}{\mu^2h^2}\left(1+\left(e^{\frac{\mu x}{n+1}}-1\right)\left(e^{\frac{\mu}{n+1}}+1\right)\right)^n\right.\\
&\left.\times\left[e^{\frac{\mu}{n+1}}-\frac{n+1}{2\mu}\left(1-e^{\frac{\mu}{n+1}}\right)\left(1+e^{\frac{\mu}{n+1}}-4e^{\frac{\mu}{2(n+1)}}\right)\right]\right)\omega(f_\mu;h).
\end{aligned}$$

Setting

$$h=\left[e^{\frac{\mu}{n+1}}-\frac{n+1}{2\mu}\left(1-e^{\frac{\mu}{n+1}}\right)\left(1+e^{\frac{\mu}{n+1}}-4e^{\frac{\mu}{2(n+1)}}\right)\right]^{1/2}$$

we have the desired result. ∎

Proposition 2.2. *(i) The following estimates hold for $f \in C^2[0,1]$:*

$$\left|\tilde{K}_n f(x) - \tilde{G}_n f(x)\right| \leq a'_{n+1}(x) e^{\mu x} \frac{\|f''_\mu\|_\infty}{24(n+1)^2}, \tag{2.15}$$

$$\left|\overline{K}_n f(x) - \overline{G}_n f(x)\right| \leq e^{\mu x}\left(1 - e^{-\frac{\mu}{n}}\left(e^{\frac{\mu x}{n}} - 1\right)\right)^n \frac{\|f''_\mu\|_\infty}{24(n+1)^2}. \tag{2.16}$$

(ii) Besides the estimate (2.16)*, the following one holds for $f \in C[0,1]$:*

$$\left|\overline{K}_n f(x) - \overline{G}_n f(x)\right| \leq \frac{13}{12} e^{\mu x}\left(1 - e^{-\frac{\mu}{n}}\left(e^{\frac{\mu x}{n}} - 1\right)\right)^n \omega\left(f; \frac{1}{n+1}\right). \tag{2.17}$$

Proof. Using [2, Lemma 1] we get

$$\begin{aligned} &\left|(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt - f\left(\frac{2k+1}{2(n+1)}\right)\right| \\ &\leq \frac{\|f''\|_\infty}{2}\left[(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t^2 dt - \left(\frac{2k+1}{2(n+1)}\right)^2\right] = \frac{\|f''\|_\infty}{24(n+1)^2}. \end{aligned} \tag{2.18}$$

In the sequel we will estimate $|\tilde{K}_n f - \tilde{G}_n f|$. We have

$$\begin{aligned} &|\tilde{K}_n f(x) - \tilde{G}_n f(x)| \\ &= a'_{n+1}(x) e^{\mu x} \sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) \left|(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_\mu(t)dt - f_\mu\left(\frac{2k+1}{2(n+1)}\right)\right| \\ &\leq a'_{n+1}(x) e^{\mu x} \frac{\|f''_\mu\|_\infty}{24(n+1)^2}. \end{aligned}$$

In the following we will estimate $\left|\overline{K}_n f(x) - \overline{G}_n f(x)\right|$ with the technique from Proposition 2.1. Using (2.18) we can write

$$\begin{aligned} \left|\overline{K}_n f(x) - \overline{G}_n f(x)\right| &\leq e^{\mu x} \sum_{k=0}^{n} e^{-\mu k/n} p_{n,k}(a_n(x)) \frac{\|f''\|_\infty}{24(n+1)^2} \\ &= e^{\mu x} \frac{\|f''\|_\infty}{24(n+1)^2} \sum_{k=0}^{n} \binom{n}{k} (1 - a_n(x))^{n-k} (e^{-\mu/n} a_n(x))^k \\ &= e^{\mu x}\left(1 - e^{-\frac{\mu}{n}}\left(e^{\frac{\mu x}{n}} - 1\right)\right)^n \frac{\|f''\|_\infty}{24(n+1)^2}. \end{aligned}$$

We have

$$\begin{aligned} &\left|\overline{K}_n f(x) - \overline{G}_n f(x)\right| \\ &\leq (n+1) e^{\mu x} \sum_{k=0}^{n} e^{-\frac{\mu k}{n}} p_n(a_n(x)) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left|f(t) - f\left(\frac{2k+1}{2(n+1)}\right)\right| dt. \end{aligned}$$

Using the relation (1.8) we get

$$\left|f(t) - f\left(\frac{2k+1}{2(n+1)}\right)\right| \leq \left(1 + \frac{1}{h^2}\left(t - \frac{2k+1}{2(n+1)}\right)^2\right) \omega(f; h).$$

Therefore,

$$\begin{aligned}&\left|\overline{K}_n f(x)-\overline{G}_n f(x)\right|\\&\leq (n+1)e^{\mu x}\omega(f;h)\sum_{k=0}^{n} e^{-\frac{\mu k}{n}} p_n(a_n(x))\left(\frac{1}{n+1}+\frac{1}{12h^2(n+1)^3}\right).\end{aligned}$$

For $h=\dfrac{1}{n+1}$, we get

$$\begin{aligned}\left|\overline{K}_n f(x)-\overline{G}_n f(x)\right| &= e^{\mu x}\omega\left(f;\frac{1}{n+1}\right)\frac{13}{12}\sum_{k=0}^{n}\binom{n}{k}(1-a_n(x))^{n-k}\left(e^{-\frac{\mu}{n}}a_n(x)\right)^k\\&=\frac{13}{12}e^{\mu x}\omega\left(f;\frac{1}{n+1}\right)\left(1-e^{-\mu/n}\left(e^{\mu x/n}-1\right)\right)^n.\end{aligned}$$

□

The Voronovskaya formula for the operators $\widetilde{G}_n$ is presented in

Proposition 2.3. *If $f\in C^2[0,1]$, then*

$$\begin{aligned}\lim_{n\to\infty} n\left(\widetilde{G}_n(f;x)-f(x)\right) &= \frac{x(1-x)}{2}\left(f''(x)-3\mu f'(x)+2\mu^2 f(x)\right)\\&\quad+\frac{\mu}{2}(4x-1)f(x)-xf'(x).\end{aligned}$$

Proof. Denote the inverse function of $\exp_\mu$ by $\log_\mu$, i.e. $\log_\mu$ is the logarithmic function with base e^μ. Applying the Taylor's theorem to the function $\left(f\circ\log_\mu\right)(e^{\mu\cdot})$ on the interval $[x,u]$, $x,u\in[0,1]$, we get

$$\begin{aligned}f(u)&=f(x)+\left(f\circ\log_\mu\right)'(e^{\mu x})(e^{\mu u}-e^{\mu x})+\frac{1}{2}\left(f\circ\log_\mu\right)''(e^{\mu x})(e^{\mu u}-e^{\mu x})^2\\&\quad+h_x(u)(e^{\mu u}-e^{\mu x})^2,\end{aligned}$$

where $h_x(u)$ is a continuous function and $\lim_{u\to x}h_x(u)=0$. Applying the operator $\widetilde{G}_n(f)$, we obtain

$$\begin{aligned}&\widetilde{G}_n(f;x)-f(x)=\\&(\widetilde{G}_n(e_0;x)-1)f(x)+\left(f\circ\log_\mu\right)'(e^{\mu x})\left(\widetilde{G}_n\left(\exp_\mu;x\right)-e^{\mu x}\widetilde{G}_n(e_0;x)\right)\\&+\frac{1}{2}\left(f\circ\log_\mu\right)''(e^{\mu x})\left(\widetilde{G}_n\left(\exp_\mu^2;x\right)-2e^{\mu x}\widetilde{G}_n\left(\exp_\mu;x\right)+e^{2\mu x}\widetilde{G}_n(e_0;x)\right)\\&+\widetilde{G}_n\left(h_x\left(exp_\mu-exp_\mu(x)\right)^2;x\right).\end{aligned}$$

Since $\left(f\circ\log_\mu\right)'(e^{\mu x})=f'(x)e^{-\mu x}\mu^{-1}$ and $\left(f\circ\log_\mu\right)''(e^{\mu x})$ $=e^{-2\mu x}\left(\mu^{-2}f''(x)-\mu^{-1}f'(x)\right)$ as a consequence of Lemma 2.1, we get

$$\begin{aligned}&\lim_{n\to\infty}n\left(\widetilde{G}_n(f;x)-f(x)\right)\\&=-\mu(\mu x^2-\mu x-2x+1)f(x)+\frac{1}{2}\left(2\mu x^2-2\mu x-2x+1\right)f'(x)\\&+\frac{1}{2}\left(f''(x)-\mu f'(x)\right)(1-x)x+\lim_{n\to\infty}n\widetilde{G}_n\left(h_x\left(exp_\mu-exp_\mu(x)\right)^2;x\right).\end{aligned}$$

It remains to prove that

$$\lim_{n\to\infty}n\widetilde{G}_n\left(h_x\left(exp_\mu-exp_\mu(x)\right)^2;x\right)=0.$$

From Cauchy-Schwarz inequality we get

$$\begin{aligned} & n\left|\widetilde{G}_n\left(h_x\left(exp_\mu(x)-exp_\mu(x)\right)^2;x\right)\right| \\ & \leq \sqrt{\widetilde{G}_n\left(h_x^2;x\right)}\sqrt{n^2\widetilde{G}_n\left(\left(exp_\mu-exp_\mu(x)\right)^4;x\right)}. \end{aligned}$$

From Lemma 2.1 we have

$$\lim_{n\to\infty} n^2\widetilde{G}_n\left(\left(exp_\mu-exp_\mu(x)\right)^4;x\right)=3\mu^4e^{4\mu x}(x-1)^2x^2$$

and

$$\lim_{n\to\infty}\widetilde{G}_n\left(h_x;x\right)=h_x\left(x\right)=0.$$

This completes the proof. □

3. A RELATION BETWEEN $\tilde{K}_n$ AND $\tilde{G}_n$, AND A VORONOVSKAYA TYPE RESULT

In this section we present a relation between $\tilde{K}_n$ and $\tilde{G}_n$. Then we give an application of the Voronovskaya type result for $\tilde{K}_n$.

Theorem 3.2. *For $f\in C^r\left[0,1\right]$ and $x\in\left[0,1\right]$, we have*

$$\begin{aligned} & \left|\tilde{K}_nf(x)-\tilde{G}_n\left(\sum_{j=0}^{r-1}\frac{1+(-1)^j}{2^{j+1}(n+1)^j(j+1)!}D_\mu^{(j)}f\right)(x)\right| \\ & \leq \|D_\mu^{(r)}f\|_\infty a'_{n+1}(1)\frac{e^\mu}{(r+1)!}\frac{1}{2^r(n+1)^r}. \end{aligned}$$

Proof. Using (1.6), we can write

$$\left(\frac{f\left(x\right)}{e^{\mu x}}\right)^{(j)}=\frac{D_\mu^{(j)}f\left(x\right)}{e^{\mu x}},\quad j\in\mathbb{N}.$$

For $f\in C^r\left[0,1\right]$, we use the following version of the Taylor formula:

$$\begin{aligned} \frac{f\left(t\right)}{e^{\mu t}} &= \sum_{j=0}^{r-1}\frac{1}{j!}\left(\frac{f\left(x\right)}{e^{\mu x}}\right)^{(j)}\left(t-x\right)^j+\frac{1}{r!}\left(\frac{f}{\exp_\mu}\right)^{(r)}\left(\xi_{t,x}\right)\left(t-x\right)^r \\ &= \sum_{j=0}^{r-1}\frac{1}{j!}\frac{D_\mu^{(j)}f\left(x\right)}{e^{\mu x}}\left(t-x\right)^j+\frac{1}{r!}\frac{D_\mu^{(r)}f\left(\xi_{t,x}\right)}{e^{\mu\xi_{t,x}}}\left(t-x\right)^r \end{aligned}$$

where $t,x\in\left[0,1\right]$ and $\xi_{t,x}$ is between t and x.

We have

$$\left|\frac{f(t)}{e^{\mu t}}-\sum_{j=0}^{r-1}\frac{1}{j!}\frac{D_\mu^{(j)}f(x)}{e^{\mu x}}(t-x)^j\right|\leq\frac{1}{r!}\|D_\mu^{(r)}f\|_\infty|t-x|^r.$$

For $x=\dfrac{2k+1}{2(n+1)}$ it follows

$$\begin{aligned} & \left|\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f_\mu(t)dt-\sum_{j=0}^{r-1}\frac{1}{(j+1)!}\frac{D_\mu^{(j)}f\left(\frac{2k+1}{2(n+1)}\right)}{e^{\mu\frac{2k+1}{2(n+1)}}}\frac{1+(-1)^j}{2^{j+1}(n+1)^{j+1}}\right| \\ & \leq\left\|D_\mu^{(r)}f\right\|_\infty\frac{1}{(r+1)!2^r(n+1)^{r+1}}. \end{aligned}$$

Consequently,

$$\begin{aligned}
&\Bigg| a_{n+1}'(x)(n+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_\mu(t)dt \\
&-a_{n+1}'(x)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\sum_{j=0}^{r-1}\frac{1}{(j+1)!}\frac{D_\mu^{(j)}f\left(\frac{2k+1}{2(n+1)}\right)}{e^{\mu\frac{2k+1}{2(n+1)}}}\frac{1+(-1)^j}{2^{j+1}(n+1)^j}\Bigg| \\
&\leq a_{n+1}'(x)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\frac{1}{(r+1)!}\|D_\mu^{(r)}f\|_\infty\frac{1}{2^r(n+1)^r}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\left|\tilde{K}_n f(x)-\sum_{j=0}^{r-1}\frac{1+(-1)^j}{2^{j+1}(n+1)^j(j+1)!}\sum_{k=0}^{n} a_{n+1}'(x)e^{\mu x}p_{n,k}(a_{n+1}(x))\frac{D_\mu^{(j)}f\left(\frac{2k+1}{2(n+1)}\right)}{e^{\mu\frac{2k+1}{2(n+1)}}}\right| \\
&\leq \|D_\mu^{(r)}f\|_\infty a_{n+1}'(1)\frac{e^\mu}{(r+1)!}\frac{1}{2^r(n+1)^r},
\end{aligned}$$

which proves the theorem. □

Theorem 3.3. *Suppose that* $f\in C^2[0,1]$. *If*

$$\lim_{n\to\infty} n\left(\widetilde{K}_n f(x)-f(x)\right)=\mu(2x-1)f(x)-\left(x-\frac{1}{2}\right)f'(x), \tag{3.19}$$

$x\in(0,1)$, *then* $f(x)=ae^{\mu x}+be^{2\mu x}$ *for some* $a,b\in\mathbb{R}$.

Proof. According to [6],

$$\begin{aligned}
\lim_{n\to\infty} n\left(\tilde{K}_n f(x)-f(x)\right)=&\frac{1}{2\mu}\left[\mu x-1-x(\mu x-2)\right]\cdot\left[f''(x)-3\mu f'(x)+2\mu^2 f(x)\right] \\
&+(2x-1)\left[f'(x)-\frac{1}{2\mu}f''(x)\right].
\end{aligned}$$

Combined with (3.19), this yields the differential equation

$$\begin{aligned}
&\frac{1}{2\mu}\left[\mu x-1-x(\mu x-2)\right]\left[f''(x)-3\mu f'(x)+2\mu^2 f(x)\right] \\
&+(2x-1)\left[f'(x)-\frac{1}{2\mu}f''(x)\right]=\mu(2x-1)f(x)-\frac{2x-1}{2}f'(x).
\end{aligned}$$

It is easy to check that it reduces to $f''(x)-3\mu f'(x)+2\mu^2 f(x)=0$, and $e^{\mu x},e^{2\mu x}$ are solution of it; this concludes the proof. □

4. Convergence in variation

Convergence of Bernstein polynomials in variation seminorm has been developed and studied in details in [12]. After this fundamental study, the topic has become classical and important in Approximation Theory. An essential role is played by $AC[0,1]$, the space of absolutely continuous functions, which is a closed subspace of $TV[0,1]$, the space of all functions of bounded variation on $[0,1]$ with the seminorm

$$\|f\|_{TV_\mu}:=V_{[0,1]}\left[\frac{f}{exp_\mu}\right]$$

and by $BV[0,1]$, which is $TV[0,1]$ with the norm

$$\|f\|_{BV_\mu} := V_{[0,1]}\left[\frac{f}{exp_\mu}\right] + |f(0)|.$$

First, we show that the operator G_n is a bounded operator with respect to BV_μ-norm.

Theorem 4.4. *If $f \in TV[0,1]$, then for all $n \in \mathbb{N}$, we have*

$$\|G_n f\|_{BV_\mu} \leq \|f\|_{BV_\mu}.$$

Proof. Since

$$\left(\frac{G_n f}{\exp_\mu}\right)'(x) = n a_n'(x) \sum_{k=0}^{n-1} \left[\frac{f}{\exp_\mu}\left(\frac{k+1}{n}\right) - \frac{f}{\exp_\mu}\left(\frac{k}{n}\right)\right] p_{n-1,k}(a_n(x)),$$

we have

$$\begin{aligned} V_{[0,1]}\left[\frac{G_n f}{\exp_\mu}\right] &= \int_0^1 \left|\left(\frac{G_n f}{\exp_\mu}\right)'(x)\right| dx \\ &\leq n \sum_{k=0}^{n-1} \left|\frac{f}{\exp_\mu}\left(\frac{k+1}{n}\right) - \frac{f}{\exp_\mu}\left(\frac{k}{n}\right)\right| \int_0^1 p_{n-1,k}(a_n(x))\, a_n'(x)\, dx \\ &\leq \sum_{k=0}^{n-1} \left|\frac{f}{\exp_\mu}\left(\frac{k+1}{n}\right) - \frac{f}{\exp_\mu}\left(\frac{k}{n}\right)\right| \\ &\leq V_{[0,1]}\left[\frac{f}{\exp_\mu}\right]. \end{aligned}$$

On the other hand

$$\begin{aligned} \left\|\frac{G_n f}{\exp_\mu}\right\|_{BV_\mu} &= V_{[0,1]}\left[\frac{G_n f}{\exp_\mu}\right] + |G_n f(0)| \\ &\leq V_{[0,1]}\left[\frac{f}{\exp_\mu}\right] + |f(0)| = \left\|\frac{f}{\exp_\mu}\right\|_{BV_\mu}. \end{aligned}$$

□

Theorem 4.5. *If $f \in TV[0,1]$, then for all $n \in \mathbb{N}$, we have*

$$\|\tilde{K}_n f\|_{TV_\mu} \leq e^{\frac{\mu}{n+1}} \|f\|_{TV_\mu} + \frac{\mu}{n+1}\left\|\frac{f}{exp_\mu}\right\|_\infty.$$

Proof. Since

$$F_{k,n} := (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_\mu(t)\, dt = \int_0^1 f_\mu\left(\frac{k+v}{n+1}\right) dv$$

we can write

$$\begin{aligned} \left(\frac{\widetilde{K}_n f}{\exp_\mu}\right)'(x) &= a_{n+1}''(x) \sum_{k=0}^{n} F_{k,n} p_{n,k}(a_{n+1}(x)) \\ &\quad + \left(a_{n+1}'(x)\right)^2 n \sum_{k=0}^{n-1} \left(F_{k+1,n} - F_{k,n}\right) p_{n-1,k}(a_{n+1}(x)). \end{aligned}$$

Using the equality $a_{n+1}^{''}(x) = \frac{\mu}{n+1} a_{n+1}^{'}(x)$, we have

$$\begin{aligned}
V_{[0,1]}\left[\frac{\widetilde{K}_n f}{\exp_\mu}\right] &= \int_0^1 \left|\left(\frac{\widetilde{K}_n f}{\exp_\mu}\right)^{'}(x)\right| dx \\
&= \int_0^1 \Bigg| \frac{\mu}{n+1} a'_{n+1}(x) \sum_{k=0}^{n} F_{k,n} p_{n,k}\left(a_{n+1}(x)\right) \\
&+ \left(a'_{n+1}(x)\right) n \sum_{k=0}^{n-1} \left(F_{k+1,n} - F_{k,n}\right) p_{n-1,k}\left(a_{n+1}(x)\right) a'_{n+1}(x) \Bigg| dx \\
&\leq \frac{\mu}{n+1} \sum_{k=0}^{n} |F_{k,n}| \int_0^1 a'_{n+1}(x) p_{n,k}\left(a_{n+1}(x)\right) dx \\
&+ \frac{\mu}{n+1} \frac{e^{\frac{\mu x}{n+1}}}{e^{\frac{\mu}{n+1}} - 1} n \sum_{k=0}^{n-1} |F_{k+1,n} - F_{k,n}| \int_0^1 p_{n-1,k}(a_{n+1}(x)) \cdot a'_{n+1}(x) dx \\
&\leq \frac{\mu}{(n+1)^2} \sum_{k=0}^{n} |F_{k,n}| + \frac{\mu}{n+1} \frac{e^{\frac{\mu x}{n+1}}}{e^{\frac{\mu}{n+1}} - 1} \sum_{k=0}^{n-1} |F_{k+1,n} - F_{k,n}|.
\end{aligned}$$

But

$$\frac{\mu}{n+1} \frac{e^{\frac{\mu x}{n+1}}}{e^{\frac{\mu}{n+1}} - 1} \leq e^{\frac{\mu}{n+1}},$$

$$|F_{k,n}| \leq (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} e^{-\mu t} |f(t)| dt \leq \left\| \frac{f}{exp_\mu} \right\|_\infty$$

Therefore,

$$V_{[0,1]}\left[\frac{\tilde{K}_n f}{exp_\mu}\right] \leq \frac{\mu}{n+1} \left\| \frac{f}{\exp_\mu} \right\|_\infty + e^{\frac{\mu}{n+1}} \sum_{k=0}^{n-1} |F_{k+1,n} - F_{k,n}|$$

Since

$$\begin{aligned}
\sum_{k=0}^{n-1} |F_{k+1,n} - F_{k,n}| &\leq \int_0^1 \sum_{k=0}^{n-1} \left| f_\mu\left(\frac{k+1+v}{n+1}\right) - f_\mu\left(\frac{k+v}{n+1}\right) \right| d\nu \\
&\leq V_{[0,1]}\left[\frac{f}{\exp_\mu}\right] = \|f\|_{TV_\mu},
\end{aligned}$$

we get

$$V_{[0,1]}\left[\frac{\tilde{K}_n f}{exp_\mu}\right] \leq \frac{\mu}{n+1} \left\| \frac{f}{exp_\mu} \right\|_\infty + e^{\frac{\mu}{n+1}} \|f\|_{TV_\mu}.$$

□

Theorem 4.6. *If $f \in TV[0,1]$, then we have*

$$\lim_{n\to\infty} \|G_n f - f\|_{TV_\mu} = 0. \tag{4.20}$$

Moreover, if (4.20) *holds for $f \in C[0,1]$, then $f \in AC[0,1]$.*

Proof. We have

$$\begin{aligned}
\left(\frac{G_{n+1}f}{\exp_\mu}\right)^{'}(x) &= a_{n+1}^{'}(x)\,(n+1)\sum_{k=0}^{n} p_{n,k}\left(a_{n+1}(x)\right)\left[f_\mu\left(\frac{k+1}{n+1}\right)-f_\mu\left(\frac{k}{n+1}\right)\right]\\
&= a_{n+1}^{'}(x)\,(n+1)\sum_{k=0}^{n} p_{n,k}\left(a_{n+1}(x)\right)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_\mu^{'}(t)\,dt\\
&= \left(\frac{\widetilde{K}_n}{\exp_\mu}\right)\left(D_\mu f\right)(x),
\end{aligned}$$

and

$$\begin{aligned}
V_{[0,1]}\left[\frac{G_{n+1}f-f}{\exp_\mu}\right] &= \int_0^1\left|\left(\frac{G_{n+1}f}{\exp_\mu}\right)^{'}(x)-\left(\frac{f}{\exp_\mu}\right)^{'}(x)\right|dx\\
&= \int_0^1\left|\left(\frac{\widetilde{K}_n}{\exp_\mu}\right)\left(D_\mu f\right)(x)-\left(\frac{D_\mu f}{\exp_\mu}\right)(x)\right|dx. \qquad (4.21)
\end{aligned}$$

It is known that (see [6]) when $g\in C[0,1]$, $\widetilde{K}_n g$ converges uniformly to g. It is easy to prove that for the operators $\widetilde{K}_n : L_1(0,1)\to L_1(0,1)$ one has $\|\widetilde{K}_n\|\le e^{\mu}, n\ge 1$. It follows that $\widetilde{K}_n g$ converges to g as $n\to\infty$ according to the L_1-norm when $g\in L_1(0,1)$ .Thus for $f^{'}\in L_1(0,1)$, we have $\widetilde{K}_n f^{'}\to f^{'}$, which gives $\widetilde{K}_n D_\mu f\to D_\mu f$ in the sense of $L_1(0,1)$. Considering the equality (4.21), then we have

$$\lim_{n\to\infty} V_{[0,1]}\left[\left(G_n f-f\right)/\exp_\mu\right]=0.$$

Conversely, we assume that the sequence $G_n f$ is convergent to f as $n\to\infty$ in the sense of the variation seminorm of $TV[0,1]$. Therefore, f has to belong to $AC[0,1]$, since the images $G_n f$ belong to $AC[0,1]$ and $AC[0,1]$ is a closed subspace in $TV[0,1]$ (Lemma 2.1 of [12]). Thus the proof is complete. □

5. Conclusions and perspectives

The classical positive linear operators used in Approximation Theory preserve the constants and-some of them-also the linear functions. In the last years the attention of several mathematicians was focused on operators preserving other functions, like monomials or exponentials. These new operators offer a better approximation on subintervals and/or on some classes of functions. In a similar manner the operators $\overline{K}_n$ and $\tilde{K}_n$ have new shape preserving properties and offer a better approximation (on subintervals and/or some classes of functions) than that offered by the classical Kantorovich operators. We intend to deepen these aspects in a forthcoming paper, where we will investigate operators having new invariant subspaces of functions.

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[1]DEPARTMENT OF MATHEMATICS AND INFORMATICS
LUCIAN BLAGA UNIVERSITY OF SIBIU
STR. DR. I. RATIU, NO. 5-7, RO-550012 SIBIU, ROMANIA
Email address: anamaria.acu@ulbsibiu.ro

[2] DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND ARTS
KIRIKKALE UNIVERSITY
71450, YAHSIHAN, KIRIKKALE, TURKEY
Email address: aliaral73@yahoo.com

[3]DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF CLUJ-NAPOCA
STR. MEMORANDUMULUI NO. 28, 400114 CLUJ-NAPOCA, ROMANIA
Email address: ioan.rasa@math.utcluj.ro