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Tzitzeica equations and Tzitzeica surfaces in separable coordinate systems and the Ricci flow tensor field

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ABSTRACT. The Tzitzeica equation and two well-known Tzitzeica surfaces are studied in the separable coordinate systems on the plane and space respectively. We study also Tzitzeica graphs with a parameter and interpret the induced class of first fundamental forms as a Riemannian flow. Consequently, we introduce a tensor field which measures how far is a given Riemannian flow to be a Ricci one. This tensor field is explicitly computed for the case of a initial isothermic metric and a flow of convex type.

1. INTRODUCTION

Let $M \subset \mathbb{R}^3$ be a regular and orientable surface in the Euclidean 3-dimensional space with the usual Cartesian coordinates (x, y, z) for which we denote by $K(p)$ the Gaussian curvature at the point $p \in M$. From a historical point of view the first centro-affine invariant of M was introduced by Georges Tzitzeica in [17] as the function $Tzitzeica(M) : M \to \mathbb{R}$:

$$Tziteica(M)(p) := \frac{K(p)}{d^4(p)} \tag{1.1}$$

where $d(p) := d(O, T_pM)$ is the Euclidean distance from the origin O to tangent space T_pM; the historical details are presented in [1]. Hence, he introduced a class of surfaces (and later hypersurfaces in the same manner) by asking the constancy of this function (the constant being called below as *Tzitzeica value*) and these are called *Tzitzeica surfaces* from a long time. This class of surfaces is intimately related to two classes of remarkable partial differential equations (PDEs):

1) *Monge-Ampère equations* since for an explicit expression of M, namely $z = z(x, y)$, the right-hand-side of (1.1) is a Monge-Ampère expression:

$$Tzitzeica(M)(x, y, z) = \frac{z_{xx}z_{yy} - z_{xy}^2}{(xz_x + yz_y - z)^4}(= constant), \tag{1.2}$$

It follows that if $z = z(x, y)$ is a Tzitzeica graph then a linear deformation (equivalently centro-affine transformation) $\tilde{z}(x, y) := z(x, y) + \alpha x + \beta y$ with $\alpha, \beta \in \mathbb{R}$ is also a Tzitzeica graph with the same Tzitzeica value. We remark here and in Example 4.2 that not all Tzitzeica surfaces are expressed as a graph.

2) the so-called *Tzitzeica equation* for M given in asymptotic coordinates (u, v) (for the hyperbolic case $K < 0$ i.e. $Tzitzeica(M) < 0$) since then the compatibility relation of the Gauss-Weingarten equations is an equation in a function $h = h(u, v)$:

$$(\ln h)_{uv} = h - h^{-2}. \tag{1.3}$$

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Although the Tzitzeica equation (1.3) was derived by Tzitzeica himself and extensively studied, especially form a solitonic point of view ([3]), there are few examples of Tzitzeica surfaces, see Chapter 13 of [13]; also we fix (1.3) as Tzitzieca equation since, as one referee notifies: "throughout the mathematics literature, there are a few equations that are referred to as the Tzitzeica equation depending on how the surface is defined". Our study below restricts to two surfaces also found by Tzitzeica:

$$M_1 : xyz = 1; \quad M_2 : z(x^2 + y^2) = 1 \tag{1.4}$$

which are generalized in arbitrary dimension in [8]. Their Tzitzeica value is:

$$Tzitzeica(M_1) = \frac{1}{27} > 0; \quad Tzitzeica(M_2) = -\frac{4}{27} < 0. \tag{1.5}$$

Another very interesting Tzitzeica surface was introduced in [2]:

$$M_3 : z = -\frac{3 + xy}{x + y}, \quad Tzitzeica(M_3) = -\frac{1}{108} < 0 \tag{1.6}$$

which can be called as *Euler-Tzitzeica surface* due to its relationship with the Euler line in triangle geometry. Let us remark also that Tzitzeica himself gives at page 1258 of [17] the generalization of M_1 with arbitrary coefficients:

$$M_1^{general} : (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)(a_3x + b_3y + c_3z) = 1 \tag{1.7}$$

which is an algebraic surface of order 3.

The previous coordinate system (x, y, z), respectively (u, v), is without any geometrical or physical significance; for a Lagrangian point of view regarding Tzitzeica surfaces see [4] while for indicatrices of Tzitzeica type in Lagrangian and Hamiltonian geometries see [9]-[10]. But is has been known that for some constant curvature spaces there exist *orthogonal separable coordinate systems*, namely coordinate systems for which a given Hamiltonian system in classical mechanics and Schrödinger equation of the quantum mechanics admit solutions via separation of variables; hence these coordinate systems are related to the superintegrability problem as it is pointed out in [12] and [14]. For example, in the real 2D space there are four such systems, while for the complex plane there are six. Also, the 3D Euclidean space has 11 separable systems, while for 2D sphere there are only two.

In order to enlarge the study of Tzitzeica two-dimensional geometries the aim of this note is twofold:
1) to express the Tzitzeica equation (1.3) in the separable coordinate systems of the $\mathbb{R}^2$ and study it with a solitonic expression and with separation of variables: both multiplicative and additive,
2) to express and draw (with Mathematica) the Tzitzeica surfaces (1.4) using the separable coordinate systems of the $\mathbb{R}^3$. Some of the new surfaces remain Tzitzeica while other are no longer Tzitzeica, since these transformations of coordinates are not of centro-affine type.
To express in other words, we consider the centro-affine geometry in 2D and 3D separable coordinate systems, relating this study to centro-affine shape analysis and geometric design.

We treat also Tzitzeica surfaces depending on a parameter for which we study the variation of the metric (=first fundamental form of these surfaces) inspired by the well-known Ricci flow. Also, we introduce a tensor field which is a measure of how far away is a given Riemannian flow from being a Ricci flow and this tensor field is computed for the surfaces with an isothermal initial metric and supports a flow of convex type. Our interest in this subject (somehow outside of the present work) has as starting point the following remark of [7, p. 504] "The reader may think of the Ricci flow on surfaces as a toy model for developing techniques for the Ricci flow and Kähler-Ricci flow in higher dimensions". We finish this study with some issues concerning the present work.

2. THE ORTHOGONAL COMPANIONS OF THE TZITZEICA EQUATION AND THE ASSOCIATED SOLITONIC ODES

In this section we start with the transform $h \to \exp(h)$ and the Tzitzeica equation becomes:

$$h_{uv} = \exp(h) - \exp(-2h)) \tag{2.8}$$

known as Tzitzeica-Dodd-Bullough or Dodd-Bullough-Mikhailov equation, [16]. In [19, p. 150] the unknown function h is supposed to be of solitonic type (or solitary wave type) $h = \varphi(t) = \varphi(u + \alpha v)$ with $\alpha \in \mathbb{R}$. Then (2.8) becomes:

$$\alpha\varphi''(t) = \exp(\varphi(t)) - \exp(-2\varphi(t)) \tag{2.9}$$

which can be called *the solitonic Tzitzeica equation*. By multiplying with $\varphi'(t)$ and integrating we obtain an ODE of first order:

$$\alpha(\varphi'(t))^2 = 2\exp(\varphi(t)) + \exp(-2\varphi(t)) + C \tag{2.10}$$

with C a real constant.

In the following we consider the Tzitzeica equation (2.8) in the separable coordinate systems of $\mathbb{R}^2$.

1) Cartesian coordinates (u, v) form the first separable coordinate system of the plane. With the separation of variables $h = f(x)g(y)$ we have:

$$f'(x)g'(y) = \exp(f(x)g(y)) - \exp(-2f(x)g(y)), \tag{2.11}$$

while the separation of variables $h = f(x)+g(y)$ yields the constant solution $h = 0$, which is studied at the end of this section.

The following three systems are:

2) polar coordinates: $x = u\cos v$, $y = u\sin v$. The Tzitzeica equation becomes *the polar Tzitzeica equation*:

$$x(h_y + xh_{xy} - yh_{xx}) + y(-h_x + xh_{yy} - yh_{xy}) = \sqrt{x^2 + y^2}(\exp(h) - \exp(-2h)). \tag{2.12}$$

2.1 By searching for h of the form $h = \varphi(t) = \varphi(x + \alpha y)$ we get the *polar solitonic Tzitzieca equation*:

$$(\alpha x-y)\varphi'(t)+(\alpha(x^2-y^2)+(\alpha^2-1)xy)\varphi''(t) = \sqrt{x^2 + y^2}(\exp(\varphi(t))-\exp(-2\varphi(t))). \tag{2.13}$$

2.2 By searching for h of the form $h = f(x)g(y)$ we obtain:

$$x(fg' + xf'g' - yf''g) + y(xfg'' - f'g - yf'g') = \sqrt{x^2 + y^2}(\exp(fg) - \exp(-2fg)). \tag{2.14}$$

2.3 By searching for h of the form $h = f(x) + g(y)$ we derive:

$$x(g' - yf'') + y(xg'' - f') = \sqrt{x^2 + y^2}(\exp f \exp g - \exp(-2f)\exp(-2g)). \tag{2.15}$$

3) parabolic coordinates: $\xi = \frac{1}{2}(u^2 - v^2)$, $\eta = uv$. The Euclidian metric $g = d\xi^2 + d\eta^2$ becomes a Liouville one: $g = (u^2 + v^2)(du^2 + dv^2)$ and the Tzitzeica equation is now *the parabolic Tzitzeica equation*:

$$h_\eta + \eta(h_{\eta\eta} - h_{\xi\xi}) + 2\xi h_{\xi\eta} = \exp(h) - \exp(-2h). \tag{2.16}$$

3.1 With $h = \varphi(t) = \varphi(\xi + \alpha\eta)$ we derive *the parabolic solitonic Tzitzeica equation*:

$$\alpha\varphi'(t) + [(\alpha^2 - 1)\eta + 2\alpha\xi]\varphi''(t) = exp(\varphi(t)) - \exp(-2\varphi(t)). \tag{2.17}$$

3.2 For $h = f(\xi)g(\eta)$ we get:

$$fg' + \eta(fg'' - f''g) + 2\xi f'g' = \exp(fg) - \exp(-2fg). \tag{2.18}$$

3.3 For $h = f(x) + g(y)$ we obtain:

$$g' + \eta(g'' - f'') = \exp f \exp g - \exp(-2f)\exp(-2g). \tag{2.19}$$

4) elliptic coordinates: $x^2 = c^2(u-1)(v-1)$, $y^2 = -c^2uv$. The Euclidian metric $g = dx^2 + dy^2$ becomes a Lorentzian one: $g = \frac{c^2(v-u)}{4}\left(\frac{du^2}{u(u-1)} - \frac{dv^2}{v(v-1)}\right)$ and the Tzitzeica equation is now *the elliptic Tzitzeica equation*:

$$c^2\left(h_{xx} - h_{yy} + \frac{h_x}{x} - \frac{h_y}{y}\right) - \frac{c^2}{xy}(x^2 - y^2 - c^2)h_{xy} = 4\exp(h) - 4\exp(-2h). \tag{2.20}$$

We point out that Tzitzeica surfaces in Minkowski 3D geometry are studied in [5].
4.1 With $h = \varphi(t) = \varphi(x + \alpha y)$ we obtain *the elliptic solitonic Tzitzeica equation*:
(2.21)

$$c^2\left(\frac{1}{x} - \frac{\alpha}{y}\right)\varphi'(t) + c^2\left[1 - \alpha^2 - \frac{\alpha(x^2 - y^2 - c^2)}{xy}\right]\varphi''(t) = 4\exp(\varphi(t)) - 4\exp(-2\varphi(t)).$$

4.2 With $h = f(x)g(y)$ we get:

$$xy(f''g - fg'') + yf'g - xfg' - (x^2 - y^2 - c^2)f'g' = \frac{4xy}{c^2}[\exp(fg) - \exp(-2fg)]. \tag{2.22}$$

4.3 With $h = f(x) + g(y)$ we obtain:

$$xy(f'' - g'') + yf' - xg' = \frac{4xy}{c^2}[\exp(f)\exp(g) - \exp(-2f)\exp(-2g)]. \tag{2.23}$$

Returning to the general case we derive the following *solitonic character* of the Tzitzeica graphs:

Proposition 2.1. *Let $z = z(x, y)$ be a Tzitzeica graph and $\alpha, \beta \in \mathbb{R}$. Then:*

$$z^{\alpha}(x, y) := z(x + \alpha y, y), \quad z^{\beta}(x, y) = z(x, y + \beta x) \tag{2.24}$$

are also Tzitzeica graphs with the same Tzitzeica value.

Proof. From $z^{\alpha}_x = z_x$, $z^{\beta}_y = \alpha z_x + z_y$, $z^{\alpha}_{xx} = z_{xx}$, $z^{\alpha}_{xy} = \alpha z_{xx} + z_{xy}$ and $z^{\alpha}_{yy} = \alpha(\alpha z_{xx} + z_{xy}) + \alpha z_{xy} + z_{yy}$ it results:

$$Tzitzeica(z^{\alpha}(x, y)) = Tzitzeica(z(x + \alpha y, y)). \tag{2.25}$$

A similar equation holds for z^{β}. In fact the conclusion is just the well-known result that the Tzitzeica equation is invariant under centro-affine transformations. □

At the end of this section we remark that a large study of the Tzitzeica equation from a solitonic point of view is the Chapter 3 of [15]. Here, the constant solution $h = 1$ of (1.3) is called *Jonas hexenut* and its asymptotic parametric expression given at page 105:
(2.26)

$$\bar{r}(u, v) = \left(\cos[\frac{\sqrt{3}}{2}(u - v)]\exp[-\frac{1}{2}(u + v)], \sin[\frac{\sqrt{3}}{2}(u - v)]\exp[-\frac{1}{2}(u + v)], \exp(u + v)\right)$$

corresponds to our M_2. By using the exponential expression of the complex numbers it follows:

$$M_2 : \bar{r}(u, v) = \left(\exp(\frac{\sqrt{3}}{2}(u - v) - \frac{i}{2}(u + v)), \exp(u + v)\right) \subset \mathbb{C} \times \mathbb{R}. \tag{2.27}$$

3. TZITZEICA SURFACES IN THE 3D SEPARABLE COORDINATE SYSTEMS

In this section we provide the pictures of the Tzitzeica surfaces M_1, M_2 in the separable coordinate systems of the Euclidean 3D space:

1) *Cartesian Coordinates*: (x, y, z). 2) *Cylindrical polar*: $(\rho > 0, \varphi \in [0, 2\pi), z)$ with the transformation rule:

$$x = \rho \cos\varphi, \quad y = \rho \sin\varphi, \quad z = z. \tag{3.28}$$

M_1^1: $z\rho^2 \sin 2\varphi = 2$ with $Tzitzeica(M_1^1) = \frac{2\rho^2 \sin^4 2\varphi(3+\cos^2 2\varphi)}{(3\sin 2\varphi + 2\varphi\cos 2\varphi)^4}$.

M_2^1: $z\rho^2 = 1$ with $Tzitzeica(M_2^1) = 0$. It belongs to $M_1^{general}$ for $a_1 = a_2 = c_3 = 1$.

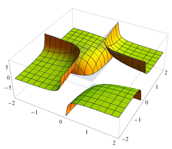


FIGURE 1. M_1

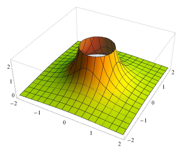


FIGURE 2. M_2

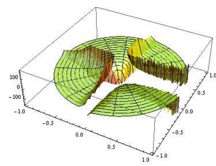


FIGURE 3. M_1^1

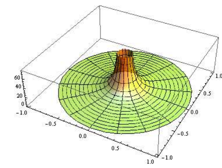


FIGURE 4. M_2^1

3) *Cylindrical elliptic*: $(e_1 < \mu_1 < e_2 < \mu_2, z \in \mathbb{R})$ with the transformation rule:

$$x^2 = \frac{(\mu_1 - e_1)(\mu_2 - e_1)}{e_2 - e_1}, \quad y^2 = \frac{(\mu_1 - e_2)(\mu_2 - e_2)}{e_1 - e_2}, \quad z = z. \tag{3.29}$$

M_1^2: $z\sqrt{(\mu_1 - e_1)(e_2 - \mu_1)(\mu_2 - e_1)(\mu_2 - e_2)} = e_2 - e_1$ with a non-constant $Tzitzeica$ function.

M_2^2: $z[\mu_1 + \mu_2 - (e_1 + e_2)] = 1$ with $Tzitzeica(M_2^2) = 0$. It is a hyperbolic cylinder.

In order to picture these surfaces, we take $e_1 = 0$ and $e_2 = 1$ and hence, with a re-notation $\mu_1 = x$, $\mu_2 = y$:

$$M_1^2 : z\sqrt{xy(1-x)(y-1)} = 1, \quad M_2^2 : z(x + y - 1) = 1. \tag{3.30}$$

4) *Cylindrical parabolic*: $(x \in \mathbb{R}, \xi, \eta \geq 0)$ with the transformation rule:

$$x = x, \quad y = \xi\eta, \quad z = \frac{1}{2}(\xi^2 - \eta^2). \tag{3.31}$$

M_1^3: $x\xi\eta(\xi^2 - \eta^2) = 2$, which is an algebraic quintic surface.

M_2^3: $(\xi^2 - \eta^2)(\xi^2\eta^2 + x^2) = 2$, which is an algebraic surface of order 6.

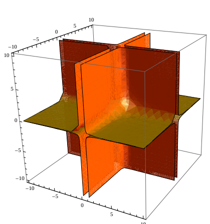


FIGURE 5. M_1^2

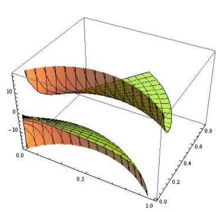


FIGURE 6. M_2^2

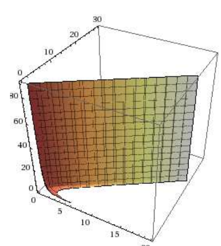


FIGURE 7. M_1^3

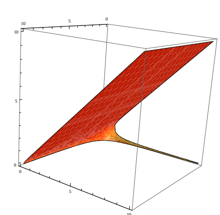


FIGURE 8. M_2^3

5) *Spherical*: $(r > 0, \varphi \in [0, 2\pi), \theta \in [0, \pi])$ with the transformation rule:

$$x = r\cos\theta\cos\varphi, \quad y = r\cos\theta\sin\varphi, \quad z = r\sin\theta. \tag{3.32}$$

M_1^4: $r^3\cos^2\theta\sin\theta\sin 2\varphi = 2$.

M_2^4: $r^3\sin\theta\cos^2\theta = 1$, with $Tzitzeica(M_2^4) = 0$.

6) *Prolate Spheroidal*: $(e_1 < u_1 < e_2 < u_2, \varphi \in [0, 2\pi)$ with the transformation rule:

(3.33)

$$x^2 = \frac{(u_1 - e_2)(u_2 - e_2)}{e_1 - e_2}(\cos\varphi)^2, y^2 = \frac{(u_1 - e_2)(u_2 - e_2)}{e_1 - e_2}(\sin\varphi)^2, z^2 = \frac{(u_1 - e_1)(u_2 - e_2)}{e_2 - e_1}.$$

M_1^5: $(e_2 - u_1)(u_2 - e_2)^{\frac{3}{2}}\sqrt{u_1 - e_1}|\sin 2\varphi| = 2(e_2 - e_1)^{\frac{3}{2}}$.

M_2^5: $(u_1 - e_1)^{\frac{1}{2}}(u_2 - e_2)^{\frac{3}{2}}(e_2 - u_1) = (e_2 - e_1)^{\frac{3}{2}}$ with $Tzitzeica(M_2^5) = 0$.

In order to picture the above surfaces we take $e_1 = 0, e_2 = 1$ and with the re-notation $u_1 = x, u_2 = y, \varphi = z$, we get:

$$M_1^5 : (1 - x)(y - 1)^{\frac{3}{2}}\sqrt{x}|\sin 2z| = 2, \quad M_2^5 : (1 - x)(y - 1)^{\frac{3}{2}}\sqrt{x} = 1. \tag{3.34}$$

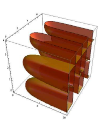


FIGURE 9. M_1^4

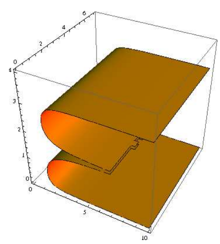


FIGURE 10. M_2^4

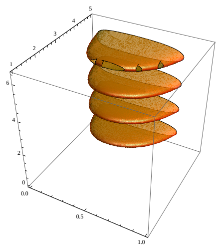


FIGURE 11. M_1^5

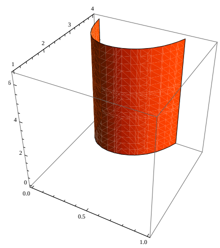


FIGURE 12. M_2^5

7) *Oblate Spheroidal*: $(e_1 < u_1 < e_2 < u_2, \varphi \in [0, 2\pi)$ with the transformation rule:

(3.35)

$$x^2 = \frac{(u_1 - e_1)(u_2 - e_1)}{e_2 - e_1}(\cos\varphi)^2, y^2 = \frac{(u_1 - e_1)(u_2 - e_1)}{e_2 - e_1}(\sin\varphi)^2, z^2 = \frac{(u_1 - e_2)(u_2 - e_2)}{e_1 - e_2}.$$

M_1^6: $(u_1 - e_1)(u_2 - e_1)\sqrt{(e_2 - u_1)(u_2 - e_2)}|\sin 2\varphi| = 2(e_2 - e_1)^{\frac{3}{2}}$.

M_2^6: $(u_1 - e_1)(u_2 - e_1)\sqrt{(e_2 - u_1)(u_2 - e_2)} = (e_2 - e_1)^{\frac{3}{2}}$ with $Tzitzeica(M_6^2) = 0$.

In the above pictures we take $e_1 = 0, e_2 = 1$ and with $u_1 \to x, u_2 \to y, \varphi \to z$, we get:

$$M_1^6 : xy\sqrt{(1 - x)(y - 1)}|\sin 2z| = 2, \quad M_2^6 : xy\sqrt{(1 - x)(y - 1)} = 1. \tag{3.36}$$

8) *Sphero-Conical*: $(r \geq 0, e_1 < \rho_1 < e_2 < \rho_2 < e_3)$ with the transformation rule:

$$x^2 = r^2\frac{(\rho_1 - e_1)(\rho_2 - e_1)}{(e_1 - e_2)(e_1 - e_3)}, y^2 = r^2\frac{(\rho_1 - e_2)(\rho_2 - e_2)}{(e_2 - e_1)(e_2 - e_3)}, z^2 = r^2\frac{(\rho_1 - e_3)(\rho_2 - e_3)}{(e_3 - e_2)(e_3 - e_1)}. \tag{3.37}$$

M_1^7: $r^3\sqrt{(\rho_1 - e_1)(\rho_2 - e_1)(e_2 - \rho_1)(\rho_2 - e_2)(e_3 - \rho_1)(e_3 - \rho_2)} = (e_2 - e_1)(e_3 - e_1)(e_3 - e_2)$.

M_2^7: $r^3\sqrt{\frac{(e_3-\rho_1)(e_3-\rho_2)}{(e_3-e_1)(e_3-e_2)}}\left[\frac{(\rho_1-e_1)(\rho_2-e_1)}{e_3-e_1} + \frac{(e_2-\rho_1)(\rho_2-e_2)}{e_3-e_2}\right] = e_2 - e_1$.

In the above pictures we take $e_1 = 0, e_2 = 1, e_3 = 2$, and with $\rho_1 \to x$, $\rho_2 \to y$, $r \to z$, we get:

(3.38)

$$M_1^7 : z^3\sqrt{xy(1-x)(y-1)(2-x)(2-y)} = 2, M_2^7 : z^3\sqrt{(2-x)(2-y)}[xy + 2(1-x)(y-1)] = 2\sqrt{2}.$$

9) *Parabolic*: $(\xi, \eta \geq 0, \varphi \in [0, 2\pi)$ with the transformation rule:

$$x = \xi\eta\cos\varphi, \quad y = \xi\eta\sin\varphi, \quad z = \frac{1}{2}(\xi^2 - \eta^2). \tag{3.39}$$

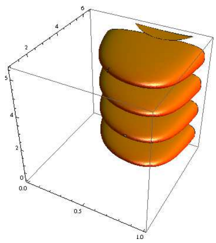


FIGURE 13. M_1^6

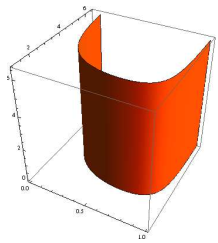


FIGURE 14. M_2^6

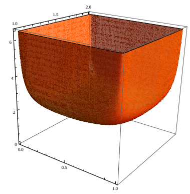


FIGURE 15. M_1^7

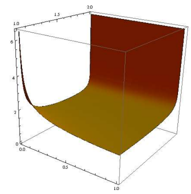


FIGURE 16. M_2^7

M_1^8: $\xi^2\eta^2(\xi^2-\eta^2)\sin 2\varphi = 4$.
M_2^8: $\xi^2\eta^2(\xi^2-\eta^2) = 2$, with $Tzitzeica(M_2^8) = 0$.

10) *Ellipsoidal*: $(a_1 < u_1 < a_2 < u_2 < a_3 < u_3)$ with the transformation rule:
(3.40)
$$x^2 = \frac{(u_1-a_1)(u_2-a_1)(u_3-a_1)}{(a_3-a_1)(a_2-a_1)}, y^2 = \frac{(u_1-a_2)(u_2-a_2)(u_3-a_2)}{(a_1-a_2)(a_3-a_2)}, z^2 = \frac{(u_1-a_3)(u_2-a_3)(u_3-a_3)}{(a_1-a_3)(a_2-a_3)}.$$

M_1^9: $\sqrt{(u_1-a_1)(a_2-u_1)(a_3-u_1)(u_2-a_1)(u_2-a_2)(a_3-u_2)(u_3-a_1)(u_3-a_2)(u_3-a_3)} =$
$= (a_2-a_1)(a_3-a_1)(a_3-a_2)$.
M_2^9: $\sqrt{\frac{(a_3-u_1)(a_3-u_2)(u_3-a_3)}{(a_3-a_1)(a_3-a_2)}}\left[\frac{(u_1-a_1)(u_2-a_1)(u_3-a_1)}{a_3-a_1} + \frac{(a_2-u_1)(u_2-a_2)(u_3-a_2)}{a_3-a_2}\right] = a_2 - a_1$.

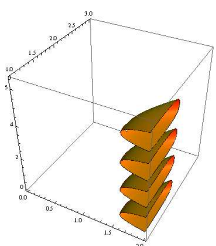


FIGURE 17. M_1^8

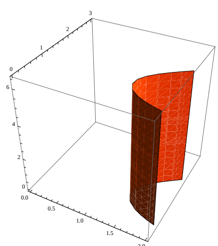


FIGURE 18. M_2^8

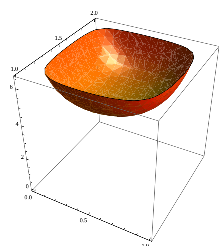


FIGURE 19. M_1^9

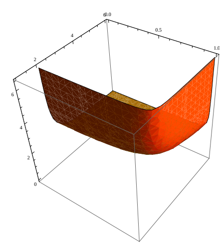


FIGURE 20. M_2^9

In the above pictures we take $a_1 = 0, a_2 = 1, a_3 = 2$, and with $u_1 \to x, u_2 \to y, u_3 \to z$, we get:

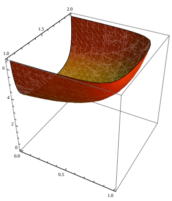
$$\text{(3.41)} \qquad \begin{cases} M_1^9 : \sqrt{x(1-x)(2-x)y(y-1)(2-y)z(z-1)(z-2)} = 2, \\ M_2^9 : \sqrt{(1-x)(2-y)(z-2)}[xyz + 2(1-x)(y-1)(z-1)] = 2\sqrt{2}. \end{cases}$$

11) *Paraboloidal*: $(0 < \eta_1 < a_2 < \eta_2 < a_3 < \eta_3)$, with the transformation rule:
(3.42)
$$x^2 = \frac{(a_3-\eta_1)(a_3-\eta_2)(\eta_3-a_3)}{a_3-a_2}, y^2 = \frac{(a_2-\eta_1)(\eta_2-a_2)(\eta_3-a_2)}{a_3-a_2}, z^2 = \frac{1}{2}(\eta_1+\eta_2+\eta_3-a_2-a_3).$$

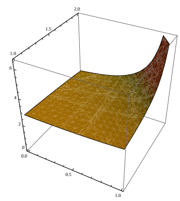
M_1^{10}: $\sqrt{(\eta_1+\eta_2+\eta_3-a_2-a_3)(a_2-\eta_1)(a_3-\eta_1)(\eta_2-a_2)(a_3-\eta_2)(\eta_3-a_2)(\eta_3-a_3)} =$
$= \sqrt{2}(a_3-a_2)$.
M_2^{10}: $\sqrt{\eta_1+\eta_2+\eta_3-a_2-a_3}[(a_3-\eta_1)(a_3-\eta_2)(\eta_3-a_3)+(a_2-\eta_1)(\eta_2-a_2)(\eta_3-a_2)] =$
$\sqrt{2}(a_3-a_2)$.

In the following pictures we take $a_2 = 1, a_3 = 2$, and with $\eta_1 \to x, \eta_2 \to y, \eta_3 \to z$, we get:

$$\left\{ \begin{array}{l} M_1^{10} : \sqrt{(x+y+z-3)(1-x)(2-x)(y-1)(2-y)(z-1)(z-2)} = \sqrt{2}, \\ M_2^{10} : \sqrt{x+y+z-3}[(2-x)(2-y)(z-2)+(1-x)(y-1)(z-1)] = \sqrt{2}. \end{array} \right. \tag{3.43}$$



21. M_1^{10}



22. M_2^{10}

Remark 3.1. *Let us point out that there are obtained several Tzitzeica cylinders:* M_2^1, M_2^2, M_2^4, M_2^5, M_2^6, M_2^8.

4. Tzitzeica surfaces with parameter and relationship with Ricci flow

Let $(M^n(t), g(t))$ be a family of n-dimensional Riemannian manifolds depending on the parameter $t \in I \subseteq \mathbb{R}$; with the title of [11] we can call it *a moving Riemannian geometry*. Due to the recent proof of the Poincaré Theorem by using the Ricci flow ([7]) we are interested in evaluating the derivative $\frac{\partial g}{\partial t}$; for details on Ricci flow on surfaces and its several applications see [20]. This section is devoted to the study of this problem for some Tzitzeica surfaces with parameter.

Let $(M, z = z(x,y))$ be a Tzitzeica graph and consider its moving family:

$$M(t) : z^t(x,y) := tz(x,y) \tag{4.44}$$

containing M at $t = 1$. A direct computation yields that $M(t)$ is also a Tzitzeica graph with $Tzitzeica(M(t)) = t^{-2} Tzitzeica(M)$ for any t. If g denotes the first fundamental form of M and I is the unit matrix of order two then:

$$g^t = I + t^2(g - I) \tag{4.45}$$

for a suitable range of t containing 1 and hence:

$$\frac{\partial g^t}{\partial t} = 2t(g - I). \tag{4.46}$$

Example 4.1. $M_1^t : xyz = t$ has the form (4.44) and the metric of M_1 is:

$$g = I_2 + \frac{1}{x^4 y^4} \begin{pmatrix} y^2 & xy \\ xy & x^2 \end{pmatrix}. \tag{4.47}$$

$M_2^t : z(x^2 + y^2) = t$ has also the form (4.44) and the metric of M_2 is:

$$g = I_2 + \frac{4}{(x^2+y^2)^4} \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}. \tag{4.48}$$

Let us remark that both M_1 and M_2 are non-compact surfaces and $\lim_{(x\to\infty, y\to\infty)} g(x,y) = I$ which can be interpreted as an asymptotic Euclidean character of g. Hence, for the metrics of M_1^t, M_2^t we have: $\lim_{(x\to\infty,y\to\infty)} \frac{\partial g^t(x,y)}{\partial t} = O$ the null matrix of order two.

Example 4.2. Not all Tzitzeica surfaces are expressed as a graph. For example, in [13, p. 320] is given the hyperbolic paraboloid $P_h : z = \sqrt{1+axy}$ as Tzitzeica surface with $Tzitzeica(P_h) = -\frac{a^2}{4}$; in fact, all quadrics with center are Tzitzeica surfaces. It is easy to express P_h in a form similar to (4.1); with $a = \frac{1}{t}$ we derive:

(4.49) $$P_h : x = t\frac{z^2-1}{2}$$

which is a graph.

A remarkable quadric with center is the hyperboloid of one sheet, which as $(1+1)$-dimensional space-time is defined as *the de Sitter space* in [6, p. 230]. Hence, we can call it *the Tzitzeica-de Sitter* surface.

Returning to the general case of moving Riemannian geometry, let us suppose that the base manifold is fixed $M(t) = M$ and endowed with an initial metric g. Inspired by (4.45) we introduce:

Definition 4.1. The Riemannian flow $g(t)$ with $t \in [0,1]$ on (M^n, g) is called *convex-Euclidean* if:

(4.50) $$g(t) = (1-t)I + tg,$$

now I being the unit matrix of order n. The manifold (M, g) is called *convex-Euclidean* if supports a convex-Euclidian flow.

Example 4.3. i) Every paralelizable manifold, in particular any Lie group, is a convex-Euclidean one.
ii) Allowing the variation of the surface M we start again with a graph $M_1 : z = z(x,y)$ and consider:

$$\tilde{M}^t : \tilde{z}^t(x,y) = \sqrt{t}z(x,y)$$

for $t \in [0,1]$. The corresponding flow $\tilde{g}(t)$ is a convex-Euclidean one as the formula (4.45) shows.

Let $\mathcal{T}^0_{2,s}(M)$ be the real linear space of symmetric tensor fields of $(0,2)$-type on M. For any $t \in I$ let us define the tensor $RicF(t) \in \mathcal{T}^0_{2,s}(M)$ provided by the Ricci flow equation:

(4.51) $$RicF(t)_{ij} := \frac{\partial g_{ij}}{\partial t} + 2Ric(t)_{ij}$$

where $Ric(t) \in \mathcal{T}^0_{2,s}(M)$ is the Ricci tensor field of $g(t)$. The tensor $RicF$ is a "measure of how far away" is $g(t)$ from being a Ricci flow.

Since in the case $n = 2$ we have $Ric = \frac{1}{2}Kg$, we get the tensor field:

(4.52) $$RicF(t) = \frac{\partial g}{\partial t} + K(t)g(t)$$

with $K(t)$ the Gaussian curvature of $g(t)$. Let us compute this quantity for an isothermal convex-Euclidean (M^2, g) being known that every smooth regular surface has an isothermal parametrization:

Proposition 4.2. *The $RicF$ tensor of the isothermal convex-Euclidean surface $(M^2, g = E(u,v)I)$ is:*

(4.53) $$RicF(t) = \left[E - 1 - \frac{t(E_{uu}+E_{vv}) + t^2[(E-1)(E_{uu}+E_{vv}) - E_u^2 - E_v^2]}{2[1+t(E-1)]^2}\right]I.$$

In particular, if E is a harmonic function, i.e. $\Delta E = 0$, then:

(4.54) $$RicF(t) = \left[E - 1 + \frac{t^2\|\nabla E\|^2}{2[1+t(E-1)]^2}\right]I.$$

Proof. The convex-Euclidean flow (4.50) becomes an isothermal one: $g(t) = [1+t(E-1)]I$. The expression of the Gaussian curvature for isothermal metrics it is well-known:

$$K(t) = -\frac{1}{2[1+t(E-1)]}\Delta_{u,v}(\ln[1+t(E-1)]) \tag{4.55}$$

where $\Delta_{u,v}$ is the usual 2D Laplacian: $\Delta_{u,v} = \partial^2_{uu} + \partial^2_{vv}$. Plugging in (4.52) we infer:

$$RicF(t) = \left[E - 1 - \frac{1}{2}\Delta_{u,v}(\ln[1+t(E-1)])\right] I \tag{4.56}$$

which yields:

$$RicF(t) = \left[E - 1 - \frac{t}{2}\left(\left(\frac{E_u}{1+t(E-1)}\right)_u + \left(\frac{E_v}{1+t(E-1)}\right)_v\right)\right] I. \tag{4.57}$$

Computing the partial derivatives we get the claimed formula. □

5. Conclusions

We finish this study with some issues concerning the present work:

0) The Romanian name of Tzitzeica was *Ţiţeica* but after his French studies in Paris he signs his papers with the French variant Tzitzeica. Details concerning his activity are on the Wiki page: http://en.wikipedia.org/wiki/Gheorghe_Ţiţeica.

1) The Tzitzeica equation and the possible associated surfaces remain largely a mystery after more a century of intense efforts to understand their structures. The Romanian mathematicians and physicists have a large contribution towards this aim.

2) We try here to develop some of its beauty by treating it in several separable coordinate systems on the plane and space respectively. So, we draw the corresponding surfaces in these new 3D coordinates making them suitable to centro-affine shape analysis and geometric design.

3) In order to connect the Tzitzeica equation with modern studies in mathematical physics we obtain in section 2 the orthogonal companions of this equation and the associated solitonic ODEs. Let us remark that the interplay between Tzitzieca geometries and solitonic theory is a continuous subject of research as it is expressed in [18].

4) Enlarging a given surface (particularly a graph) into a class depending smoothly by a parameter we point out a relationship with the Ricci flow theory. We add a new concept in this last fruitful domain by introducing a "measure of how far away" is a "time-depending" metric $g(t)$ from being a Ricci flow. This notion opens the door for similar studies concerning other remarkable classes of surfaces and Riemannian geometries.

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