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New classes of certain analytic functions concerned with subordinations

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ABSTRACT. Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ which satisfy $f(0) = 0$ and $f'(0) = 1$. Applying the extremal function for the subclass $\mathcal{S}^*(\alpha)$ of $\mathcal{A}$, new classes $\mathcal{P}^*(\alpha)$ and $\mathcal{Q}^*(\alpha)$ are considered using certain subordinations. The object of the present paper is to discuss some interesting properties for $f(z)$ belonging to the classes $\mathcal{P}^*(\alpha)$ and $\mathcal{Q}^*(\alpha)$.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. If $f(z) \in \mathcal{A}$ satisfies $f(z_1) \neq f(z_2)$ for any $z_1 \in \mathbb{U}$ and $z_2 \in \mathbb{U}$ with $z_1 \neq z_2$, then $f(z)$ is said to be univalent in $\mathbb{U}$ and denoted by $f(z) \in \mathcal{S}$. If a function $f(z) \in \mathcal{S}$ maps $\mathbb{U}$ onto a starlike domain with respect to the origin, then $f(z)$ is said to be starlike in $\mathbb{U}$ and denoted by $f(z) \in \mathcal{S}^*$. We say that $f(z)$ is starlike of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.1}$$

for some real α $(0 \leqq \alpha < 1)$. We also denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions $f(z)$ of order α in $\mathbb{U}$. Furthermore, we call that $f(z)$ is convex of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}$ satisfies $zf'(z) \in \mathcal{S}^*(\alpha)$ for some real α $(0 \leqq \alpha < 1)$ and denote by $\mathcal{K}(\alpha)$. From the definitions of these classes, we know that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$ and that $f(z) \in \mathcal{S}^*(\alpha)$ if and only if $\displaystyle\int_0^z \frac{f(t)}{t}\,dt \in \mathcal{K}(\alpha)$. The function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n}(j-2\alpha)}{(n-1)!} z^n \tag{1.2}$$

is the extremal function for the class $\mathcal{S}^*(\alpha)$, and the function $f(z)$ given by

$$f(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1} = z + \displaystyle\sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n}(j-2\alpha)}{n!} z^n & \left(\alpha \neq \dfrac{1}{2}\right) \\[2ex] -\log(1-z) = z + \displaystyle\sum_{n=2}^{\infty} \frac{1}{n} z^n & \left(\alpha = \dfrac{1}{2}\right) \end{cases} \tag{1.3}$$

is the extremal function for the class $\mathcal{K}(\alpha)$ (see [2] or [6]).
Considering the principal value for $\sqrt{z}$, we consider a function $f(z)$ given by

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$$f(z) = \frac{z}{(1-\sqrt{z})^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n}(j-2\alpha)}{(n-1)!} z^{\frac{n+1}{2}}. \tag{1.4}$$

Then, $f(z)$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(\alpha + \frac{1-\alpha}{1-\sqrt{z}}\right) > \frac{1+\alpha}{2} \quad (z \in \mathbb{U}). \tag{1.5}$$

Therefore, $f(z)$ given by (1.4) is said to be starlike of order $\frac{1+\alpha}{2}$ in $\mathbb{U}$.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisfying $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$ and $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}). \tag{1.6}$$

This subordination is used for many papers of univalent function theory [1], [4], [8], [9] and [10].

Now, with the function $f(z)$ given by (1.4), we introduce a new class of $f(z)$ as follows. Let $\mathcal{A}^*$ be the class of functions $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_{\frac{n+1}{2}} z^{\frac{n+1}{2}} \qquad (z \in \mathbb{U}) \tag{1.7}$$

which are analytic in $\mathbb{U}$, where we consider the principal value for $\sqrt{z}$. If $f(z) \in \mathcal{A}^*$ satisfies the following subordination

$$f(z) \prec g(z) = \frac{z}{(1-\sqrt{z})^{2(1-\alpha)}} \qquad (z \in \mathbb{U}) \tag{1.8}$$

for some real $\alpha (0 \leqq \alpha < 1)$, then we say that $f(z) \in \mathcal{P}^*(\alpha)$. Also, if $f(z) \in \mathcal{A}^*$ satisfies $zf'(z) \in \mathcal{P}^*(\alpha)$, then we say that $f(z) \in \mathcal{Q}^*(\alpha)$.

2. Some properties

We would like to study some properties of functions $f(z) \in \mathcal{A}^*$ concerned with the classes $\mathcal{P}^*(\alpha)$ and $\mathcal{Q}^*(\alpha)$.

Theorem 2.1. *If $f(z) \in \mathcal{A}^*$ satisfies*

$$\sum_{n=2}^{\infty} (n-\alpha) \left| a_{\frac{n+1}{2}} \right| \leqq 1-\alpha \tag{2.9}$$

for some real α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{P}^*(\alpha)$. The result is sharp for $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\alpha)\varepsilon}{n(n-1)(n-\alpha)} z^{\frac{n+1}{2}} \tag{2.10}$$

with $|\varepsilon| = 1$.

Proof. It is easy to know that if $f(z) \in \mathcal{A}^*$ satisfies

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{1-\alpha}{2} \qquad (z \in \mathbb{U}) \tag{2.11}$$

for some real α $(0 \leqq \alpha < 1)$, then

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \frac{1+\alpha}{2} \qquad (z \in \mathbb{U}), \tag{2.12}$$

that is, that $f(z) \in \mathcal{P}^*(\alpha)$. In order to get (2.11), we notice that we have

$$\left| \frac{zf'(z) - f(z)}{f(z)} \right| = \left| \frac{\sum_{n=2}^{\infty} \frac{n-1}{2} a_{\frac{n+1}{2}} z^{\frac{n+1}{2}}}{1 + \sum_{n=2}^{\infty} a_{\frac{n+1}{2}} z^{\frac{n+1}{2}}} \right| < \frac{1-\alpha}{2} \qquad (z \in \mathbb{U}) \tag{2.13}$$

if $f(z)$ satisfies

$$\sum_{n=2}^{\infty} \frac{n-1}{2} \left| a_{\frac{n+1}{2}} \right| \leqq \frac{1-\alpha}{2} \left(1 - \sum_{n=2}^{\infty} \left| a_{\frac{n+1}{2}} \right| \right), \tag{2.14}$$

which is equivalent to

$$\sum_{n=2}^{\infty} (n-\alpha) \left| a_{\frac{n+1}{2}} \right| \leqq 1 - \alpha \tag{2.15}$$

for some real α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{P}^*((\alpha)$.

Further, if we consider a function $f(z)$ given by (2.10), then

$$a_{\frac{n+1}{2}} = \frac{(1-\alpha)\varepsilon}{n(n-1)(n-\alpha)} \qquad (|\varepsilon| = 1). \tag{2.16}$$

This shows us that

$$\sum_{n=2}^{\infty} (n-\alpha) \left| a_{\frac{n+1}{2}} \right| = \sum_{n=2}^{\infty} \frac{1-\alpha}{n(n-1)} \tag{2.17}$$

$$= (1-\alpha) \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1 - \alpha.$$

□

Taking $\alpha = 0$ in Theorem 2.1, we have

Corollary 2.1. *If $f(z) \in \mathcal{A}^*$ satisfies*

$$\sum_{n=2}^{\infty} n \left| a_{\frac{n+1}{2}} \right| \leqq 1, \tag{2.18}$$

then $f(z) \in \mathcal{P}^*(0)$. The result is sharp for

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\varepsilon}{n^2(n-1)} z^{\frac{n+1}{2}} \qquad (|\varepsilon| = 1). \tag{2.19}$$

Noting that $f(z) \in \mathcal{Q}^*(\alpha)$ if and only if $zf'(z) \in \mathcal{P}^*(\alpha)$, we have

Theorem 2.2. *If $f(z) \in \mathcal{A}^*$ satisfies*

$$\sum_{n=2}^{\infty}(n+1)(n-\alpha)\left|a_{\frac{n+1}{2}}\right| \leqq 2(1-\alpha) \tag{2.20}$$

for some real α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{Q}^*(\alpha)$. The result is sharp for $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\alpha)\varepsilon}{n(n^2-1)(n-\alpha)} z^{\frac{n+1}{2}} \tag{2.21}$$

with $|\varepsilon| = 1$.

Letting $\alpha = 0$ in Theorem 2.2, we have

Corollary 2.2. *If $f(z) \in \mathcal{A}^*$ satisfies*

$$\sum_{n=2}^{\infty} n(n+1)\left|a_{\frac{n+1}{2}}\right| \leqq 2, \tag{2.22}$$

then $f(z) \in \mathcal{Q}^*(0)$. The result is sharp for $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2\varepsilon}{n^2(n^2-1)} z^{\frac{n+1}{2}} \qquad (|\varepsilon| = 1). \tag{2.23}$$

To discuss next properties for $f(z) \in \mathcal{Q}^*(\alpha)$, we have to recall here the following lemma which is called as Carathéodory theorem (see [3], [5], [7]).

Lemma 2.1. *Let a function $p(z)$ given by*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2.24}$$

be analytic in $\mathbb{U}$ and $\mathrm{Re}\, p(z) > 0$ $(z \in \mathbb{U})$. Then

$$|c_n| \leqq 2 \qquad (n = 1, 2, 3, \cdots). \tag{2.25}$$

The equality holds true for

$$p(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n. \tag{2.26}$$

Applying Lemma 2.1, we derive

Theorem 2.3. *If $f(z) \in \mathcal{P}^*(\alpha)$, then*

$$\left|a_{\frac{n+1}{2}}\right| \leqq \frac{1}{(n-1)!}\prod_{j=2}^{n}(j-2\alpha) \tag{2.27}$$

for $n = 2, 3, 4, \cdots$. The equality holds true for

$$f(z) = \frac{z}{(1-\sqrt{z})^{2(1-\alpha)}}. \tag{2.28}$$

Proof. For $f(z) \in \mathcal{P}^*(\alpha)$, we define a function $p(z)$ by

$$p(z) = \frac{1}{1-\alpha}\left(2\frac{zf'(z)}{f(z)} - (1+\alpha)\right) \qquad (z \in \mathbb{U}) \tag{2.29}$$

with

$$p(z) = 1 + \sum_{n=1}^{\infty} p_{\frac{n}{2}} z^{\frac{n}{2}} \qquad (z \in \mathbb{U}), \tag{2.30}$$

where we consider the principal value for $\sqrt{z}$.

It follows that

$$2zf'(z) = \{(1-\alpha)p(z) + (1+\alpha)\} f(z). \tag{2.31}$$

This gives us that

(2.32)

$$\begin{aligned}\sum_{n=2}^{\infty}(n+1)a_{\frac{n+1}{2}} z^{\frac{n+1}{2}} &= \left(2a_{\frac{3}{2}} + (1-\alpha)p_{\frac{1}{2}}\right) z^{\frac{3}{2}} + \left(2a_2 + (1-\alpha)p_{\frac{1}{2}}a_{\frac{3}{2}} + (1-\alpha)p_1\right) z^2 \\ &+ \left(2a_{\frac{5}{2}} + (1-\alpha)p_{\frac{1}{2}}a_2 + (1-\alpha)p_1 a_{\frac{3}{2}} + (1-\alpha)p_{\frac{3}{2}}\right) z^{\frac{5}{2}} + \cdots \\ &+ \left(2a_{\frac{n+1}{2}} + (1-\alpha)p_{\frac{1}{2}}a_{\frac{n}{2}} + (1-\alpha)p_1 a_{\frac{n-1}{2}} + \cdots + (1-\alpha)p_{\frac{n-2}{2}}a_{\frac{3}{2}} + (1-\alpha)p_{\frac{n-1}{2}}\right) z^{\frac{n+1}{2}} + \cdots .\end{aligned}$$

Therefore, we obtain that

$$(n-1)a_{\frac{n+1}{2}} = (1-\alpha)\left(p_{\frac{n-1}{2}} + p_{\frac{n-2}{2}}a_{\frac{3}{2}} + p_{\frac{n-3}{2}}a_2 + \cdots + p_1 a_{\frac{n-1}{2}} + p_{\frac{1}{2}}a_{\frac{n}{2}}\right), n \geq 2, \tag{2.33}$$

where $a_1 = 1$. From the definition for $p(z)$, we see that $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Furthermore, noting that $\mathrm{Re}\,p(z) > 0$ $(z \in \mathbb{U})$, Lemma 2.1 gives us that

$$\left|p_{\frac{n}{2}}\right| \leqq 2 \qquad (n = 1, 2, 3, \cdots). \tag{2.34}$$

Taking $n = 2$ in (2.33), we see that

$$\left|a_{\frac{3}{2}}\right| \leqq (1-\alpha)\left|p_{\frac{1}{2}}\right| \leqq 2 - 2\alpha. \tag{2.35}$$

If we take $n = 3$ in (2.33), then we have that

$$|a_2| \leqq \frac{1-\alpha}{2}\left(\left|p_{\frac{1}{2}}\right|\left|a_{\frac{3}{2}}\right| + |p_1|\right) \leqq (1-\alpha)(3-2\alpha) = \frac{1}{2}(2-2\alpha)(3-2\alpha). \tag{2.36}$$

Further, letting $n = 4$ in (2.33), we obtain that

$$\begin{aligned}\left|a_{\frac{5}{2}}\right| &\leqq \frac{1-\alpha}{3}\left(\left|p_{\frac{1}{2}}\right||a_2| + |p_1|\left|a_{\frac{3}{2}}\right| + \left|p_{\frac{3}{2}}\right|\right) \\ &\leqq \frac{2}{3}(1-\alpha)(2-\alpha)(3-2\alpha) = \frac{1}{6}(2-2\alpha)(3-2\alpha)(4-2\alpha).\end{aligned} \tag{2.37}$$

In view of the above, we assume that

$$\left|a_{\frac{n+1}{2}}\right| \leqq \frac{1}{(n-1)!}\prod_{j=2}^{n}(j-2\alpha) \tag{2.38}$$

for $j = 2, 3, 4, \cdots, n$. Then, we see that

$$\begin{aligned}\left|a_{\frac{n+2}{2}}\right| &\leq \frac{2(1-\alpha)}{n}\left(1 + \left|a_{\frac{3}{2}}\right| + |a_2| + \left|a_{\frac{5}{2}}\right| + \cdots + \left|a_{\frac{n}{2}}\right| + \left|a_{\frac{n+1}{2}}\right|\right) \\ &\leqq \frac{1}{n!}\prod_{j=2}^{n+1}(j-2\alpha).\end{aligned} \tag{2.39}$$

Thus, applying the mathematical induction, we complete the proof of the theorem. □

Theorem 2.4. *If $f(z) \in \mathcal{Q}^*(\alpha)$, then*

$$\left| a_{\frac{n+1}{2}} \right| \leqq \frac{1}{n!} \prod_{j=2}^{n} (j - 2\alpha) \tag{2.40}$$

for $n = 2, 3, 4, \cdots$. The equality holds true for $f(z)$ satisfying

$$f'(z) = \frac{1}{(1 - \sqrt{z})^{2(1-\alpha)}}. \tag{2.41}$$

3. Distortion inequalities

In this section, we consider some distortion inequalities for $f(z)$ in $\mathcal{P}^*(\alpha)$ and $\mathcal{Q}^*(\alpha)$.

Theorem 3.5. *If $f(z) \in \mathcal{P}^*(\alpha)$, then*

$$\frac{|z|}{(1 + \sqrt{|z|})^{2(1-\alpha)}} \leqq |f(z)| \leqq \frac{|z|}{(1 - \sqrt{|z|})^{2(1-\alpha)}} \qquad (z \in \mathbb{U}). \tag{3.42}$$

The equalities in (3.42) are attended for

$$f(z) = \frac{z}{(1 - \sqrt{z})^{2(1-\alpha)}}. \tag{3.43}$$

Proof. We note that there exists a function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$. This function $w(z)$ also satisfies

$$f(z) = \frac{w(z)}{(1 - \sqrt{w(z)})^{2(1-\alpha)}} \qquad (z \in \mathbb{U}). \tag{3.44}$$

If we write that $w(z) = |w(z)|e^{i\theta}$, then $f(z)$ gives us that

$$\begin{aligned} |f(z)| &= \frac{|w(z)|}{\left(1 - \sqrt{|w(z)|}e^{i\frac{\theta}{2}}\right)^{2(1-\alpha)}} \\ &= \frac{|w(z)|}{\left\{\left(1 - \sqrt{|w(z)|}cos\frac{\theta}{2}\right)^2 + |w(z)|sin^2\frac{\theta}{2}\right\}^{1-\alpha}} \\ &= \frac{|w(z)|}{\left(1 + |w(z)| - 2\sqrt{|w(z)|}cos\frac{\theta}{2}\right)^{1-\alpha}}. \end{aligned} \tag{3.45}$$

Applying the Schwarz lemma for $w(z)$, we say that $|w(z)| \leqq |z|$ $(z \in \mathbb{U})$. Therefore, we obtain that

$$\frac{|z|}{(1 + \sqrt{|z|})^{2(1-\alpha)}} \leqq |f(z)| \leqq \frac{|z|}{(1 - \sqrt{|z|})^{2(1-\alpha)}} \tag{3.46}$$

for $z \in \mathbb{U}$. This completes the proof of the theorem. □

Letting $\alpha = 0$ in Theorem 3.5, we have

Corollary 3.3. *If $f(z) \in \mathcal{P}^*(0)$, then*

$$\frac{|z|}{(1+\sqrt{|z|})^2} \leqq |f(z)| \leqq \frac{|z|}{(1-\sqrt{|z|})^2} \qquad (z \in \mathbb{U}). \tag{3.47}$$

The equalities in (3.47) are attended for

$$f(z) = \frac{z}{(1-\sqrt{z})^2}. \tag{3.48}$$

Further, letting $|z| \to 1$ in Theorem 3.5, we see

Corollary 3.4. *If $f(z) \in \mathcal{P}^*(\alpha)$, then*

$$|f(z)| \geqq \left(\frac{1}{4}\right)^{1-\alpha}. \tag{3.49}$$

The equality in (3.49) is attended for $f(z)$ given by (3.43) with $z = e^{i2\pi}$.

Noting that $f(z) \in \mathcal{Q}^*(\alpha)$ if and only if $zf'(z) \in \mathcal{P}^*(\alpha)$, we also have

Theorem 3.6. *If $f(z) \in \mathcal{Q}^*(\alpha)$, then*

$$\frac{1}{(1+\sqrt{|z|})^{2(1-\alpha)}} \leqq |f'(z)| \leqq \frac{1}{(1-\sqrt{|z|})^{2(1-\alpha)}} \qquad (z \in \mathbb{U}). \tag{3.50}$$

The equalities in (3.50) are attended for

$$f(z) = \int_0^z \frac{1}{(1-\sqrt{t})^{2(1-\alpha)}} dt. \tag{3.51}$$

Corollary 3.5. *If $f(z) \in \mathcal{Q}^*(0)$, then*

$$\frac{1}{(1+\sqrt{|z|})^2} \leqq |f'(z)| \leqq \frac{1}{(1-\sqrt{|z|})^2} \qquad (z \in \mathbb{U}). \tag{3.52}$$

The equalities in (3.52) are attended for

$$f(z) = \int_0^z \frac{1}{(1-\sqrt{t})^2} dt. \tag{3.53}$$

Corollary 3.6. *If $f(z) \in \mathcal{Q}^*(\alpha)$, then*

$$|f'(z)| \geqq \left(\frac{1}{4}\right)^{1-\alpha}. \tag{3.54}$$

The equality in (3.54) is attended for $f(z)$ given by (3.51) with $z = e^{i2\pi}$.

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