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Köthe-Bochner spaces that are Hilbert spaces

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ABSTRACT. We are concerned with Köthe-Bochner spaces that are Hilbert spaces (resp. hilbertable spaces). It is shown that this is equivalent to the fact that, separately, L_ρ and X are Hilbert spaces (resp. hilbertable spaces). The complete characterization of the L_ρ spaces that are Hilbert spaces, given by the first-author, is used.

1. INTRODUCTION

The Köthe spaces (in a particular form) were primarily introduced by G. Köthe and O. Toeplitz under the name of "Stufenräume" in the seminal paper [14]. Subsequently, G. Köthe continued the study of these spaces in [13]. The study of the Köthe spaces L_ρ within the natural framework of general measurable functions is mainly due to A. C. Zaanen, W. A. J. Luxemburg and their school (see the doctoral thesis [20] of W. A. J. Luxemburg under the supervision of A. C. Zaanen and the series of papers [21] by A. C. Zaanen and W. A. J. Luxemburg). The spaces L_ρ are natural generalizations of Lebesgue spaces, Orlicz spaces, Lorentz spaces and other function spaces. It is worth noticing that the name "Köthe spaces" was given by J. Dieudonné (see [5]).

The Köthe-Bochner spaces $L_\rho(X)$ (where X is a Banach space) are the natural (vector valued) generalization of the (scalar valued) Köthe spaces L_ρ. Their theory is developing very fast, being of great actuality (see e.g. the important monograph [17]). Here are some examples of recent papers concerned with the theory of Köthe-Bochner spaces $L_\rho(X)$: Geometric properties of spaces $L_\rho(X)$ are studied, e.g. in [10] and [11]. Operators on spaces $L_\rho(X)$ are studied in [8] and [12]. In [15] interpolation and extrapolation is studied in connection with vector integration. The spaces $L_\rho(X)$ and vector integration, involving linear operators are studied in [16]. We lay stress upon the paper [1], where the representation of the spaces $L_\rho(X)$ as tensor products of L_ρ and X is studied (practically only for $L_\rho = L^1$).

After a preliminary part, presenting notions, notations and results which are used, the first part of the paper is dedicated to those spaces $L_\rho(X)$ that are Hilbert spaces (i.e. they are complete and their norm is generated by a scalar product). We prove that this happens if and only if, separately, L_ρ and X are Hilbert spaces. The proof relies heavily on the result in [2], where we proved that L_ρ is Hilbert if and only if L_ρ is a weighted L^2 space (for some uniquely determined weight function). Some examples are introduced.

In the second part, we study those spaces $L_\rho(X)$ which are hilbertable (i.e. they are equivalent to some Hilbert space). It is proved that (like in the case of Hilbert spaces) this happens if and only if, separately, L_ρ and X are hilbertable (under some conditions).

Both results are of the same type: $L_\rho(X)$ has property (P) if and only if, separately, L_ρ and X have property (P). This happens many times in the theory of $L_\rho(X)$ spaces. A nice example in this respect is Theorem 3.6.17 in [17] which asserts that $L_\rho(X)$ has the

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Radon-Nikodym property if and only if separately, L_ρ and X have the Radon-Nikodym property.

The basic theoretical notions used throughout the paper can be found in: [3] and [22] (for L_ρ spaces), [17] (for $L_\rho(X)$ spaces; our definition is a bit more general). For measure and integration theory, see [4], [6] and [9]. For functional analysis, see [7]. Related results can be found in [18].

2. Preliminary facts

Throughout the paper: $\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{R}_+ = [0, \infty)$, $\overline{\mathbb{R}}_+ = [0, \infty] = [0, \infty) \cup \{\infty\}$, $K = \mathbb{R}$ or $\mathbb{C}$. All sequences $(x_n)_n$ are indexed with $\mathbb{N}$. When writing $(x_n)_n \subset A$, this means that $x_n \in A$ for any $n \in \mathbb{N}$.

If X is a vector space (over K), we say that two norms $\| \, \|_1$ and $\| \, \|_2$ on X are *equivalent* if there exist two numbers $0 < a \leq b$ such that $a \| \, \|_1 \leq \| \, \|_2 \leq b \| \, \|_1$. This is equivalent to the fact that $\| \, \|_1$ and $\| \, \|_2$ generate the same topology on X.

A *measure space* is a triple $(T, \mathcal{T}, \mu)$, where T is a non empty set, $\mathcal{T} \subset \mathcal{P}(T) = \{A \mid A \subset T\}$ is a σ-algebra and $\mu : \mathcal{T} \to \overline{\mathbb{R}}_+$ is a non null σ-additive measure. We shall always assume that $(T, \mathcal{T}, \mu)$ is a *σ-finite measure space* (or μ is *σ-finite*). We shall also assume that μ is *complete*.

The set of all μ-measurable functions $u : T \to \overline{\mathbb{R}}_+$ will be denoted by $M_+(\mu)$.

For any $A \in \mathcal{T}$, we have $\varphi_A \in M_+(\mu)$, where φ_A is the characteristic (indicator) function of A.

A *μ-function norm* (or, simply, a *function norm*) is a function $\rho : M_+(\mu) \to \overline{\mathbb{R}}_+$ with the properties (for any $u, v \in M_+(\mu)$ and any $\alpha \in \mathbb{R}_+$):

(i) $\rho(u) = 0$ if and only if $u(t) = 0 \ \mu - a.e.$;

(ii) $\rho(u) \leq \rho(v)$, whenever $u \leq v$;

(iii) $\rho(u + v) \leq \rho(u) + \rho(v)$;

(iv) $\rho(\alpha u) = \alpha \rho(u)$ (with the convention $0 \cdot \infty = 0$).

The function norm ρ is called *saturated* if there exists a sequence $(T_n)_n \subset \mathcal{T}$ such that $T = \bigcup_{n=1}^{\infty} T_n$ and $\mu(T_n) < \infty$, $\rho(\varphi_{T_n}) < \infty$, for any $n \in \mathbb{N}$ (this definition is equivalent to the classical one, see [3] and [22]). We say that the function norm ρ has *the Riesz-Fischer property* (and write $\rho \ R - F$) if $\rho\left(\sum_{n=1}^{\infty} u_n\right) \leq \sum_{n=1}^{\infty} \rho(u_n)$ for any sequence $(u_n)_n \subset M_+(\mu)$.

A *weight function* is a function $w \in M_+(\mu)$ such that $\mu(\{t \in T \mid w(t) = 0\}) = \mu(\{t \in T \mid w(t) = \infty\}) = 0$. Hence, it is possible to consider that a weight function w takes its values in $(0, \infty)$, identifying $\mu - a.e.$ equal functions. Any weight function w defines *the Hilbert function norm* $\rho_2(\mu, w) : M_+(\mu) \to \overline{\mathbb{R}}_+$, defined via $\rho_2(\mu, w)(u) = \left(\int u^2 w d\mu\right)^{\frac{1}{2}}$, which is saturated.

Let X be a non null Banach space. A function $f : T \to X$ is called *μ-measurable* if there exists a sequence $(f_n)_n$ of μ-simple functions such that $f_n \underset{n}{\to} f \ \mu - a.e.$ Let $M_X(\mu) = \{f : T \to X \mid f \text{ is } \mu\text{-measurable}\}$. The vector space $M_X(\mu)$ has the property that, for any $f \in M_X(\mu)$, one has $|f| \in M_+(\mu)$, where $|f| : T \to \mathbb{R}_+$ is defined via $|f|(t) = \|f(t)\|$.

For any μ-function norm ρ we define the vector space

$$\mathcal{L}_\rho(X) = \{f \in M_X(\mu) \mid \rho|f| < \infty\}$$

(we wrote $\rho|f| \overset{def}{=} \rho(|f|)$). Then $\mathcal{L}_\rho(X)$ is seminormed with the seminorm $f \to \rho|f|$. The null space of this seminorm is

$$N_X(\mu) = \{f \in \mathcal{L}_\rho(X) \mid \rho|f| = 0\} = \{f \in M_X(\mu) \mid f(t) = 0 \ \mu - a.e.\}.$$

We define

$$L_\rho(X) \overset{def}{=} \mathcal{L}_\rho(X)/N_X(\mu)$$

and we see that $L_\rho(X)$ is a normed space (called *Köthe-Bochner space*) with norm $\tilde{f} \to \rho|f|$ for any representative $f \in \tilde{f}$. We can prove that $L_\rho(X)$ is Banach if and only if ρ $R-F$ (this is valid for any Banach space X, in particular for $X = K$).

In the particular case when $X = K$, we write L_ρ instead of $L_\rho(K)$ (we call L_ρ Köthe space). The Köthe spaces L_ρ generalize the Lebesgue spaces $L^p(\mu)$ (for $\rho = \| \ \|_p, 1 \le p \le \infty$).

3. Köthe-Bochner Spaces that Are Hilbert Spaces

The Köthe spaces L_ρ that are Hilbert spaces (i.e. L_ρ is complete and there exists a scalar product $(.,.)$ on L_ρ such that $\left\|\tilde{f}\right\| = \sqrt{(\tilde{f},\tilde{f})}$ for any $\tilde{f} \in L_\rho$) were characterized in [2]. Namely, one can prove that L_ρ is Hilbert if and only if there exists a weight function $w : T \to (0,\infty)$ such that $L_\rho = L^2(\mu, w)$, i.e. $\rho = \rho_2(\mu, w)$, which means that $\left\|\tilde{f}\right\| = \rho|f| = \left(\int |f|^2 w d\mu\right)^{\frac{1}{2}}$ for any $f \in \tilde{f} \in L_\rho$. The scalar product is given via $(\tilde{f},\tilde{g}) = \int f\overline{g}wd\mu$. The weight function w is uniquely $\mu - a.e.$ determined by the Köthe Hilbert space L_ρ.

Theorem 3.1. *Assume $(T, \mathcal{T}, \mu)$ is a σ-finite measure space, ρ is a μ-function norm which is saturated and X is a non null Banach space. The following assertions are equivalent:*

1. *The space $L_\rho(X)$ is a Hilbert space.*
2. *The spaces L_ρ and X are Hilbert spaces.*

Proof. A. We prove 1. $\Rightarrow$ 2. Namely, we shall generate a scalar product on L_ρ (resp. X) which generates the norm of L_ρ (resp. X). The scalar product of $\tilde{f}, \tilde{g}$ in $L_\rho(X)$ will be denoted by $(\tilde{f}|\tilde{g})$.

$a)$ Construction for L_ρ.

Fix arbitrarily $x \in X$ with $\|x\| = 1$. For any $\tilde{f}, \tilde{g}$ in L_ρ, define $(\tilde{f},\tilde{g}) \overset{def}{=} (\widetilde{fx}|\widetilde{gx})$, where $fx \in \mathcal{L}_\rho(X)$ is defined via $(fx)(t) = f(t)x$ $a.s.o.$ In this way, we defined a scalar product on L_ρ. We have $\rho|f| = \sqrt{(\tilde{f},\tilde{f})}$, consequently this scalar product $(.,.)$ generates the norm of L_ρ.

We do not forget that L_ρ is Banach, because $L_\rho(X)$ is Banach. So, L_ρ is a Hilbert space.

$b)$ Construction for X.

Fix arbitrarily $\tilde{f} \in L_\rho$ such that $\left\|\tilde{f}\right\| = \rho|f| = 1$. For any x, y in X, define $[x,y] \overset{def}{=} (\widetilde{fx}|\widetilde{fy})$. In this way, we defined a scalar product on X. Notice that, for any $x \in X$, one has $[x,x] = \|x\|^2$. This shows that the scalar product $[.,.]$ generates the norm of X.

B. We prove 2. $\Rightarrow$ 1. Namely, we see first that $L_\rho(X)$ is Banach, because L_ρ is Banach. The end of the proof will consist in showing that the norm of $L_\rho(X)$ satisfies the parallelogram identity.

Using the results in [2], we find a $\mu - a.e.$ unique weight function $w : T \to \mathbb{R}_+$ such that $\rho(u) = \left(\int u^2 w d\mu\right)^{\frac{1}{2}}$ for any $u \in M_+(\mu)$, because L_ρ is Hilbert. It follows that the norm $\left|\left\|\tilde{f}\right\|\right|$ of $\tilde{f} \in L_\rho(X)$ has the expression

$$\left|\left|\left|\tilde{f}\right|\right|\right| = \rho|f| = \left(\int |f|^2 w d\mu\right)^{\frac{1}{2}}$$

if $f \in \tilde{f}$. We show that $|||\ |||$ satisfies the parallelogram identity.

Let $\tilde{f}, \tilde{g}$ in $L_\rho(X)$. Because X is a Hilbert space, one has, for any $t \in T$,

$$(|f+g|(t))^2 + (|f-g|(t))^2 = 2((|f|(t))^2 + (|g|(t))^2),$$

$f \in \tilde{f}, g \in \tilde{g}$. Consequently:

$$\left|\left|\left|\tilde{f}+\tilde{g}\right|\right|\right|^2 + \left|\left|\left|\tilde{f}-\tilde{g}\right|\right|\right|^2 = \int |f+g|^2 w d\mu + \int |f-g|^2 w d\mu = \int (|f+g|^2 + |f-g|^2) w d\mu = 2\int (|f|^2 + |g|^2) w d\mu = 2\int |f|^2 w d\mu + 2\int |g|^2 w d\mu = 2\left(\left|\left|\left|\tilde{f}\right|\right|\right|^2 + |||\tilde{g}|||^2\right).$$

□

Remark 3.1. Maybe, it is instructive to insist upon the connection of the Hilbert structures of $L_\rho(X)$ and L_ρ. So, assume $L_\rho(X)$ is a Hilbert space. During the proof of Theorem 3.1 (part $A.a$)) we used the scalar product $(.|.)$ of $L_\rho(X)$ to construct a scalar product $(.,.)$ on L_ρ which generates the norm of L_ρ. This last scalar product is unique, being generated by an unique weight function.

In the sequel, we shall consider some special notations. Namely, if ρ is a μ-function norm, $x \in X$ and $f \in \mathcal{L}_\rho$, we shall define $fx \in \mathcal{L}_\rho(X)$ and $\widetilde{fx} \in L_\rho(X)$ as follows: $fx : T \to X$ is defined via $(fx)(t) = f(t)x$ and $\tilde{f}x = \widetilde{fx}$. For the examples, we shall be concerned with the particular case of the discrete measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), card)$, where $card : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}}_+$ is the counting measure. The only negligible set is ϕ. A function $f : \mathbb{N} \to H$ is identified with a sequence: $f \equiv (x_n)_n \subset H$, where $x_n = f(n)$ for any n. If X is a Banach space, any function $f : \mathbb{N} \to X$ is $card$-measurable. For any $card$-function norm ρ and any Banach space X one has $L_\rho(X) \equiv \mathcal{L}_\rho(X)$ (equivalence classes in L_ρ contain only one element). Assuming X itself is a Köthe space, $X = L_r$, for some $card$-function norm r, we have for any $f \in L_\rho(L_r) : f \equiv (f(m))_m$, where $f(m) \in L_r \equiv \mathcal{L}_r$, hence we can identify $f(m) \equiv (x_{mn})_n \subset K$. Consequently, any $f \in L_\rho(L_r)$ can be identified with an infinite scalar valued matrix: $f \equiv (x_{mn})_{m,n}$.

Example 3.1 (Form of $L_\rho(L_r)$ spaces of sequences which are Hilbert spaces)**.** Let us consider two $card$-function norms ρ and r (see Preliminary Part) such that $L_\rho(L_r)$ is a Hilbert space. According to Theorem 3.1, this is equivalent to the fact that L_ρ and L_r are Hilbert spaces. This means that $L_\rho = L^2(card, u)$ and $L_r = L^2(card, v)$ for some weight functions $u, v : \mathbb{N} \to \mathbb{R}_+$. Hence, $u \equiv (a_n)_n$ and $v \equiv (b_n)_n$, where $0 < a_n < \infty$, $0 < b_n < \infty$ for any n. It follows that $L^2(card, u) = \{(x_n)_n \subset K \mid \sum_{n=1}^{\infty} |x_n|^2 a_n < \infty\}$ equipped with the norm $x \to \|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2 a_n\right)^{\frac{1}{2}}$ and the scalar product $(x,y) = \sum_{n=1}^{\infty} x_n \overline{y}_n a_n$, where $x \equiv (x_n)_n$ and $y \equiv (y_n)_n$. For $L^2(card, v)$, replace $(a_n)_n$ with $(b_n)_n$.

An element $f \equiv (x_{mn})_{m,n} \in L_\rho(L_r)$ has the form $f(m) \equiv (x_{mn})_n$ for any m, hence $|f|(m) = \left(\sum_{n=1}^{\infty} |x_{mn}|^2 b_n\right)^{\frac{1}{2}}$. Then $\rho|f| = \left(\sum_{m=1}^{\infty} (|f|(m))^2 a_m\right)^{\frac{1}{2}} = \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^2 a_m b_n\right)^{\frac{1}{2}}$.

Consequently:

$$L_\rho(L_r) \equiv \{(x_{mn})_{m,n} \subset K \mid \sum_{m,n} a_m b_n |x_{mn}|^2 < \infty\}$$

equipped with the norm

$$\|(x_{mn})_{m,n}\| = \left(\sum_{m,n} a_m b_n |x_{mn}|^2\right)^{\frac{1}{2}}$$

and the scalar product

$$((x_{mn})_{m,n}, (y_{mn})_{m,n}) = \sum_{m,n} a_m b_n x_{mn} \overline{y}_{mn}.$$

Example 3.2. We exhibit two examples of spaces $L_\rho(L_r)$, where ρ and r are $card$-function norms such that either L_ρ or L_r is not Hilbert and the parallelogram identity is violated.

A. Take $\rho(u) = \sup_n u(n)$ and $r(u) = \left(\sum_{n=1}^{\infty} u(n)^2\right)^{\frac{1}{2}}$ for any $u \in M_+(card)$. Hence $L_\rho = l^\infty$, $L_r = l^2$ and l^∞ is "not good" (is not Hilbert). Also take $f : \mathbb{N} \to l^2$,

$$f(m) = (\tfrac{1}{1}, \tfrac{1}{2}, ..., \tfrac{1}{m}, 0, 0, ...)$$

($f \equiv (x_{mn})_{m,n}$, where $x_{mn} = \dfrac{1}{n}$, if $n \le m$ and $x_{mn} = 0$, if $n > m$).

Let also $g : \mathbb{N} \to l^2$,

$$g(m) = (0, 0, ..., 0, \tfrac{1}{m+1}, \tfrac{1}{m+2}, ...)$$

($g \equiv (y_{mn})_{m,n}$, where $y_{mn} = 0$, if $n \le m$ and $y_{mn} = \dfrac{1}{n}$, if $n > m$).

We have: $\|f+g\|^2 + \|f-g\|^2 = \dfrac{\pi^2}{3}$, whereas $2(\|f\|^2 + \|g\|^2) = 2\dfrac{\pi^2}{3} - 2 \neq \dfrac{\pi^2}{3}$.

B. Take $\rho(u) = \left(\sum_{n=1}^{\infty} u(n)^2\right)^{\frac{1}{2}}$ and $r(u) = \sup_n u(n)$, for any $u \in M_+(card)$. Hence $L_\rho = l^2$ and $L_r = l^\infty$ and, again, l^∞ is "not good". Also take $f : \mathbb{N} \to l^\infty$,

$$f(m) = (\tfrac{1}{m+1}, \tfrac{1}{m+2}, ..., \tfrac{1}{2m}, 0, 0, ...)$$

($f \equiv (x_{mn})_{m,n}$, where $x_{mn} = \dfrac{1}{m+n}$, if $n \le m$ and $x_{mn} = 0$, if $n > m$).

Let also $g : \mathbb{N} \to l^\infty$,

$$g(m) = (0, 0, ..., 0, \tfrac{1}{2m+1}, \tfrac{1}{2m+2}, ...)$$

($g \equiv (y_{mn})_{m,n}$ where $y_{mn} = 0$, if $n \le m$ and $y_{mn} = \dfrac{1}{m+n}$, if $n > m$).

We have: $\|f+g\|^2 + \|f-g\|^2 = \dfrac{\pi^2}{3} - 2$, whereas $2(\|f\|^2 + \|g\|^2) = \dfrac{7\pi^2}{12} - 4 \neq \dfrac{\pi^2}{3} - 2$.

4. Hilbertable Köthe-Bochner Spaces

In this paragraph, we shall pass from the isometric level, described in the preceding paragraph, to the isomorphic level.

Recall that, if X is a Banach space and $Y \subset X$ is a closed subspace, we say that Y is *complemented* if there exists another closed subspace $Z \subset X$ such that $X = Y \oplus Z$ (i.e. any $x \in X$ admits an unique decomposition $x = y + z$, with $y \in Y$ and $z \in Z$).

We shall say that a Banach space X is *complementable* if any closed subspace $Y \subset X$ is complemented and we shall say that X is *hereditarily complementable* if any closed subspace $Y \subset X$ is complementable. The two notions are equivalent.

Recall that a Banach space $(X, \| \ \|)$ is called *hilbertable* if there exists a norm $||| \ |||$ on X which is equivalent to $\| \ \|$ and such that $(X, ||| \ |||)$ is a Hilbert space. Because any Hilbert space is complementable, it follows that any hilbertable space is complementable. A remarkable result of J. Lindenstrauss and L. Tzafriri (see [19]) says that, conversely, any complementable space is hilbertable.

Theorem 4.2. *Assume $(T, \mathcal{T}, \mu)$ is a σ-finite measure space, ρ is a μ-function norm which is saturated and X is a non null Banach space. Consider the following assertions:*

1. *The space $L_\rho(X)$ is hilbertable.*
2. *The spaces L_ρ and X are hilbertable.*

Then 1. $\Rightarrow$ 2. The implication 2. $\Rightarrow$ 1. is valid in case L_ρ is strongly hilbertable, i.e. there exists a $\mu-$function norm ρ_1 such that the norms of L_ρ and L_{ρ_1} are equivalent and L_{ρ_1} is a Köthe Hilbert space.

Proof. A. We prove 1. $\Rightarrow$ 2. The proof is based upon the equivalence complementable $\Leftrightarrow$ hereditarily complementable and the equivalence hilbertable $\Leftrightarrow$ complementable (Theorem of J. Lindenstrauss and L. Tzafriri). Let us accept that $L_\rho(X)$ is hilbertable. This implies that L_ρ is Banach, because $L_\rho(X)$ is Banach.

$a)$ We prove that L_ρ is hilbertable.

To this end, we consider an element $x \in X$ with $\|x\| = 1$ and define

$$L_\rho x = \{\tilde{u}x \mid \tilde{u} \in L_\rho\}.$$

Then $L_\rho x$ is a closed subspace of $L_\rho(X)$.

Passing to the very proof, let $H \subset L_\rho$ be a closed subspace. It generates the closed subspace $Hx \subset L_\rho x$, $Hx = \{\tilde{u}x \mid \tilde{u} \in H\}$ (similar proof). Using the equivalence complementable $\Leftrightarrow$ hereditarily complementable and the fact that $L_\rho(X)$ is complementable, we find a closed subspace $Y \subset L_\rho x$ such that $L_\rho x = Hx \oplus Y$. Clearly Y has the form $Y = Gx$, where $G \subset L_\rho$ is a (vector) subspace. Because Y is closed and the vector subspaces Y and G are linearly isometric (for $y = \tilde{g}x \in Y, \|y\| = \|\tilde{g}\|$, first norm in $L_\rho(X)$, second norm in L_ρ), it follows that G is closed in L_ρ. The equality $L_\rho x = Hx \oplus Gx$ means $L_\rho = H \oplus G$.

$b)$ We prove that X is hilbertable.

Same idea. Let $\tilde{u} \in L_\rho$ with $\|\tilde{u}\| = 1$. We define

$$\tilde{u}X = \{\tilde{u}x \mid x \in X\}.$$

Then $\tilde{u}X$ is a closed subspace of $L_\rho(X)$.

Now, let $Y \subset X$ be a closed subspace. It follows that $\tilde{u}Y$ is a closed subspace of $\tilde{u}X$ (similar proof). Again, we find a closed subspace $V \subset \tilde{u}X$ such that $\tilde{u}X = \tilde{u}Y \oplus V$. Hence V has the form $\tilde{u}Z$, with $Z \subset Y$, Z(vector) subspace. Being linearly isometric with V, Z is closed. Clearly, the equality $\tilde{u}X = \tilde{u}Y \oplus \tilde{u}Z$ means $X = Y \oplus Z$.

B. We prove 2. $\Rightarrow$ 1. The proof will be computational. In any case, $L_\rho(X)$ is Banach, because L_ρ is Banach, being hilbertable.

Because X is hilbertable, there exists a Hilbert norm $\| \ \|_1$ on X and $0 < a \le b < \infty$ such that, for any $x \in X$

$$a \|x\|_1 \le \|x\| \le b \|x\|_1$$

where $\| \ \|$ is the norm of X. Hence, for any $f : T \to X$, one has

$$a|f|_1 \le |f| \le b|f|_1 \tag{4.1}$$

where, as usual, $|f| : T \to \mathbb{R}_+$ acts via $|f|(t) = \|f(t)\|$ and $|f|_1 : T \to \mathbb{R}_+$ acts via $|f|_1(t) = \|f(t)\|_1$.

Because L_ρ is strongly hilbertable, there exists a Hilbert function norm ρ_1 on $(T, \mathcal{T}, \mu)$ and $0 < \alpha \leq \beta < \infty$ such that, for any $u \in M_+(\mu)$

$$\alpha\rho_1(u) \leq \rho(u) \leq \beta\rho_1(u).$$

Namely, ρ_1 is generated by a weight function $w : T \to \mathbb{R}_+$, via

$$\rho_1(u) = \left(\int u^2 w d\mu\right)^{\frac{1}{2}}$$

and this implies, for any $u \in M_+(\mu)$:

$$\alpha\left(\int u^2 w d\mu\right)^{\frac{1}{2}} \leq \rho(u) \leq \beta\left(\int u^2 w d\mu\right)^{\frac{1}{2}} \tag{4.2}$$

On $L_\rho(X)$, we have the norm given via $\tilde{f} \to \left\|\tilde{f}\right\| = \rho|f|$, $f \in \tilde{f}$. The idea is to consider the Hilbert norm $\tilde{f} \to \left\|\tilde{f}\right\|_1 = \rho_1|f|_1$ (see the construction in the proof of Theorem 3.1) and to show that $\|\ \|$ and $\|\ \|_1$ are equivalent. Indeed: $\rho_1|f|_1 = \left(\int |f|_1^2 w d\mu\right)^{\frac{1}{2}}$ and, using (4.1) and (4.2), we get:

$$\alpha\left(\int a^2|f|_1^2 w d\mu\right)^{\frac{1}{2}} \leq \alpha\left(\int |f|^2 w d\mu\right)^{\frac{1}{2}} \leq \rho|f| \leq \beta\left(\int |f|^2 w d\mu\right)^{\frac{1}{2}} \leq \beta\left(\int b^2|f|_1^2 w d\mu\right)^{\frac{1}{2}}$$

which means

$$\alpha a\rho_1|f|_1 \leq \rho|f| \leq \beta b\rho_1|f|_1.$$

In other words, we have for any $\tilde{f} \in L_\rho$

$$\alpha a\left\|\tilde{f}\right\|_1 \leq \left\|\tilde{f}\right\| \leq \beta b\left\|\tilde{f}\right\|_1.$$

□

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